# NON-LINEAR NETWORK PROBLEMS*

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**1. Flow problems.** We shall be concerned with connected *networks*. These will be defined as finite *connected* graphs, on which the *boundary* is explicitly specified.

As a *graph*[1], such a network consists of a finite set $N$ of *nodes* (or vertices), $A_1, \cdots, A_n$, certain nodes being joined in pairs by a finite set $L$ of oriented *links* (or branches) $a_1, \cdots, a_r$. Thus the graph is specified by an incidence matrix of $n$ rows and $r$ columns, $\| \epsilon_{kj} \|$, where $\epsilon_{kj}$ is $+1$, $-1$, or 0 according to whether the node $A_k$ is the initial node, the final node, or not incident on the oriented link $a_j$. It will be assumed that each link $a_j$ joins exactly two nodes, hence we may write $a_j = A_{i(j)}A_{f(j)}$, where $A_{i(j)}$ is the initial node of the link $a_j$ and $A_{f(j)}$ is the final node of the link $a_j$. This implies that the incidence matrix has just two non-zero entries in each column (one being $+1$ and one $-1$). It will also be assumed that each node $A_k$ is incident on at least one link. This implies that each row of the incidence matrix has at least one non-zero entry.

Further, a subset $\partial N$ of $N$, called the *boundary*, is supposed to be specified. This subset $\partial N$ may or may not be empty. If $\partial N$ is not empty then the elements of $\partial N$ are called the *terminals* of the network. Finally, the network is supposed to be *connected*[2] in the usual sense that a graph is said to be connected.

We shall consider first a special class of network problems, which we shall call "flow problems". Whether they concern hydraulic networks or direct (electrical) current networks, flow problems involve two real valued functions: a *potential* function $u(A_k)$ defined on the nodes, and a *current* function $i(a_j)$ defined on the oriented links. In hydraulic networks $u(A_k)$ is the pressure head; in direct current problems, it represents the voltage.**

In network flow problems, leaks are neglected. One thus assumes, at each interior node $A_h$ in $N - \partial N$, *Kirchhoff's node law*

$$\sum_{j=1}^{r} \epsilon_{hj} i(a_j) = 0, \qquad h = 1, \cdots, n; \tag{1}$$

where, in view of the definition of the incidence matrix, the summation is effectively

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[1]See W. H. Ingram and C. M. Cramlet [12], J. L. Synge [20], and the books by O. Veblen [7] and D. König [9]. The basic ideas are due to G. Kirchhoff [1] and H. Poincaré [4, 5]. (Numbers in square brackets refer to the bibliography at the end of the paper).

[2]Actually, this assumption plays a very small rôle, but it simplifies the statement of various results.

**Professor W. Prager has kindly drawn our attention to the occurrence of similar flow problems in the mathematical study of the distribution of traffic over a network of roads, see [28].

taken only over the links incident on the node $A_h$. For physical equilibrium, the currents must also satisfy certain *equilibrium relations*

$$i(a_j) = c_j(\Delta u_j), \qquad j = 1, \cdots, r, \tag{2}$$

where $\Delta u_j = u(A_{i(j)}) - u(A_{f(j)})$, with $A_{i(j)}$ the initial and $A_{f(j)}$ the final node, respectively, of the oriented link $a_j$.

Physically, the *conductivity* functions $c_j(\Delta u)$ are usually increasing and continuous. In our theorems below, we shall usually assume one or both of these conditions. For reference, we write[3]

$$c_j(\Delta u) \text{ is a strictly increasing function of } \Delta u, \tag{2a}$$

$$c_j(\Delta u) \text{ is a continuous function of } \Delta u. \tag{2b}$$

Thus, in hydraulic problems, it is commonly assumed that

$$c_j(\Delta u) = K_j \cdot \text{sign}(\Delta u) \cdot | \Delta u |^{\alpha}, \qquad \text{where} \qquad K_j > 0, \qquad \alpha > 0. \tag{2c}$$

(For turbulent flow in pipes, $\alpha = 1.85$ is commonly accepted.) In direct current problems, a *linear* relation

$$i(a_j) = c_j \Delta u_j, \tag{2d}$$

is generally used. Since the general case will be considered in Sec. 4, we shall omit the physical condition $c_j > 0$, which corresponds to (2c) with $\alpha = 1$, and gives the classical case treated by Kelvin [2].

In summary, we will assume (1) at all "interior" nodes (i.e., the nodes of $N - \partial N$) and (2) on all links. At each node $A_h$ of the boundary $\partial N$, the total "influx" $\nu_h$ must clearly satisfy

$$\nu_h = \sum_j \epsilon_{hj} i(a_j). \tag{3}$$

Comparing this last equation with (1), we get the necessary condition

$$\sum \nu_h = 0, \tag{3'}$$

summed over $\partial N$. [This follows because

$$\sum_{h=1}^{n} \sum_{j=1}^{r} \epsilon_{hj} i(a_j) = 0,$$

and (1) then implies

$$\sum_{\partial N} \sum_{j=1}^{r} \epsilon_{hj} i(a_j) = 0,$$

which is (3').]

To obtain a "boundary value problem", some condition must be given at each *terminal* $A_h$ in $\partial N$; for example, one might assume

I. The potential $u(A_h)$ is given, or

II. The "total influx" $\nu_h$ at $A_h$ is given.

Because of the obvious analogy with potential theory, we shall refer to a boundary value problem in which a condition of Type I is given at each terminal as a "Dirichlet

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[3] The significance of (2a) and (2b) was first stressed by d'Auriac [14] and Duffin [17].

problem". Similarly, if the total influx is specified at each terminal we shall speak of a "Neumann problem". Problems involving both types have been treated in the literature[4]. Still more generally, one can consider "mixed" conditions, of the type (notice that II′ includes II as a special case)

II′. A functional relation $\nu_h = F_h(u)$ is given, where

$$F_h(u) \text{ is a non-increasing, continuous function of } u. \tag{4}$$

(In heat flow problems, this would correspond to a linear or non-linear "law of cooling".)

**2. Uniqueness theorem.** It is not hard to prove a general uniqueness theorem, involving boundary value problems with boundary conditions of Types I, II, or II′, which is adequate for most physical flow problems. To formulate it, let $u = u(A_k)$ and $i = i(a_j)$; and $u'$ and $i'$ represent two different solutions of the same boundary value problem. Consider the expression

$$\begin{aligned} D^* &= \sum_L (i_k - i_k')(\Delta u_k - \Delta u_k') \\ &= \sum_L [c_k(\Delta u_k) - c_k(\Delta u_k')][\Delta u_k - \Delta u_k']. \end{aligned} \tag{5}$$

In the linear case, clearly (5) simplifies to

$$D^* = \sum_L c_k(\Delta u_k - \Delta u_k')^2. \tag{5′}$$

In any case, the following result is immediate:

LEMMA 1. *If all the conductivity functions $c_k(\Delta u)$ satisfy (2a), then $D^* \geqq 0$. Strict inequality holds unless $u - u'$ is a constant.*

We now make a second evaluation of $D^*$. By (3),

$$\begin{aligned} \sum_L [c_k(\Delta u_k) - c_k(\Delta u_k')][\Delta u_k - \Delta u_k'] &= \sum_L \{[c_k(\Delta u_k) - c_k(\Delta u_k')][\sum_N \epsilon_{hk}(u(A_h) - u'(A_h))]\} \\ &= \sum_N (\nu_h - \nu_h')[u(A_h) - u'(A_h)]; \end{aligned}$$

where, for example,

$$\nu_h = \sum_L \epsilon_{hj} i(a_j)$$

is the "influx" corresponding to the potential $u$ at the node $A_h$ of $N$. It follows from (5) and (1) that

$$D^* = \sum_{\partial N} (\nu_h - \nu_h')[u(A_h) - u'(A_h)]. \tag{5+}$$

For boundary value problems involving only conditions of Types I or II at the terminals, one has $D^* = 0$. If conditions of Types I, II, or II′ occur, provided that (4) is assumed, clearly $D^* \leqq 0$. Comparing with Lemma 1, we get

THEOREM 1. *There is at most one solution to any boundary value problem defined by (1) and (2), with boundary conditions of Types I, II, or II′, provided that (2a)*

[4] D'Auriac [14] and Duffin [17] consider the Dirichlet and Neumann problems, plus a special "mixed" problem, where a Dirichlet condition is imposed at some boundary nodes and a Neumann condition at the remainder of the terminals. D'Auriac proves uniqueness and Duffin, existence and uniqueness theorems.

and (4) are assumed and that, in the Neumann problem, potential functions which differ only by a constant are considered to be identical.

**3. Dissipation function; variational principle for the Dirichlet problem.** If $i$ and $u$ are any two functions defined on the oriented links and the nodes, respectively, of a network, we may define the *dissipation function* as the sum

$$D = \sum_L i(a_k)\Delta u(a_k). \tag{6}$$

(The name "dissipation function" expresses the fact that, in the two physical problems mentioned in Sec. 1, the expression $D$ represents the rate of energy dissipation.) We shall now derive an alternative formula for $D$, analogous to (5+) for $D^*$. Since

$$\Delta u(a_k) = u(A_{i(k)}) - u(A_{f(k)}) = \sum_h \epsilon_{hk} u(A_h),$$

one has

$$\sum_L c_k(\Delta u_k)\Delta u_k = \sum_L \{c_k(\Delta u_k) \sum_N \epsilon_{hk} u(A_h)\} = \sum_N \nu_h u(A_h);$$

and thus

$$D = \sum_{\partial N} \nu_h u(A_h). \tag{6+}$$

The expression $D$, according to (6+), represents the rate of energy influx.

In the linear case, the dissipation function reduces to $\sum_L c_k(\Delta u_k)^2$, and it is classical[5] that this is minimized by the solution of the network problem over the class of potentials assuming the given terminal potentials. We shall now derive an analogous variational principle for the non-linear case. However, this will not, in general, involve the dissipation function.

To formulate the new variational principle, we suppose $u$ is given on $\partial N$, but is unknown on $N - \partial N$. For any assumed values of $u$ on $N - \partial N$, we can then satisfy (2) automatically by *defining* $i(a_k) = c_k(\Delta u_k)$ on each link $a_k$. It remains to satisfy (1), and for this we shall find a variational formulation. Namely, define the functions $C_k$ by

$$C_k(\Delta u) = \int_0^{\Delta u} c_k(x)\, dx,$$

so that the derivatives

$$C_k'(\Delta u) = \frac{dC_k}{d(\Delta u)} = c_k(\Delta u);$$

for simplicity, we shall assume condition (2b). (Duffin [17, pp. 965-967] uses this same device of auxiliary functions for what we call the Neumann problem.)

THEOREM 2. For given $u(A_h)$ on $\partial N$ (i.e. for the Dirichlet problem), assuming (2b), the first variation of

$$V(u) = \sum_L C_k(\Delta u_k) \tag{7}$$

is zero at each ("interior") node of $N - \partial N$ if and only if Kirchhoff's node law (1) holds at each interior node.

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[5]See W. Thomson [2]; J. C. Maxwell [3, vol. I, pp. 403-408].

*Proof*: (By the first variation is of course meant the following limit:

$$\delta V(u) = \frac{d}{d\epsilon} V(u + \epsilon \delta u) \Big|_{\epsilon=0},$$

where $\delta u$ is any potential function defined on $N$ but which vanishes on $\partial N$.) By direct computation, writing $a_k = A_{i(k)}A_{f(k)}$, one has

$$\begin{aligned}\delta V &= \sum_L C_k'(\Delta u_k)[\delta u(A_{i(k)}) - \delta u(A_{f(k)})] \\ &= \sum_L \{C_k'(\Delta u_k) \sum_N \epsilon_{hk} \delta u(A_h)\} \\ &= \sum_N \{\delta u(A_h) \sum_L \epsilon_{hk} i(a_k)\}.\end{aligned}$$

For $A_h$ in $\partial N$, the number $\delta u(A_h)$ is zero, while for $A_h$ in $N - \partial N$, the $\delta u(A_h)$ are arbitrary. The conclusion of Lemma 2 is now evident.

COROLLARY. If (2a), (2b) hold, then Kirchhoff's node law (1) holds if and only if $V(u)$, considered as a function of the arbitrary values of the potential at interior nodes, has an absolute minimum.

For if, regardless of (2a), the function $V(u)$ has even a local minimum, then $\delta V = 0$, and hence Kirchhoff's node law holds. While, on the other hand, if (2a) and (2b) hold, then $V(u)$ is a *convex* function of $u$, since it is a sum of convex functions either of the individual $u_h = u(A_h)$, or of pairs of these variables, as may be readily seen from (7). We leave the detailed verification of this to the reader. Hence, if Kirchhoff's law holds for some $u$, the convex function $V(u)$ must have an absolute minimum for this particular $u$.

*Remark.* In the case (2c) of an *exponential conductivity law*, with the same exponent $\alpha$ for all links in the network, the dissipation function $D$ is proportional to the function $V$, and therefore $D$ can be used in place of $V$ in the results of this section.

**4. Existence theory for the Dirichlet problem.** We shall now derive an existence theorem for the Dirichlet problem which is adequate for most physical applications. In order to avoid giving the impression that it is the "best possible", we shall preface it by giving a much stronger result for the linear case.

In the linear case (2d), a given trial potential function, when used to construct $i$ by means of $i(a_k) = c_k[u(A_{i(k)}) - u(A_{f(k)})]$, for each link $a_k = A_{i(k)}A_{f(k)}$, will satisfy Kirchhoff's node law (1); i.e., (see Sec. 3) will solve the Dirichlet problem, if and only if

$$\sum_k \epsilon_{hk} c_k [u(A_{i(k)}) - u(A_{f(k)})] = 0, \tag{8a}$$

for every $A_h$ in $N - \partial N$. This gives a system of $s$ linear equations in the $s$ unknowns $u(A_h) = u_h$, which may be more compactly written thus

$$\sum_{j=1}^{n} c_{hj} u(A_j) = b_h, \qquad h = 1, \cdots, s, \tag{8}$$

where the numbers $b_h$ are known. The matrix of coefficients $\| c_{hj} \|$ of (8), which is *symmetric* (as follows readily from (8a) and the definition of the incidence matrix $\| \epsilon_{hk} \|$) will be called the *conductivity matrix* of the network. It is well known[6] that for any system like (8), existence and uniqueness are equivalent to each other, and also to the

[6] G. Birkhoff and S. MacLane [25, chap. X].

condition that the determinant of the matrix of coefficients be different from zero. This gives the following result[7].

THEOREM. If (2d) holds, the Dirichlet problem is solvable, for a given network $N$ (with at least one interior node and at least one boundary node) for arbitrary values, if and only if det $\| c_{hi} \| \neq 0$. This condition is also necessary and sufficient for uniqueness.

We now see how special, in the linear case, is the condition (2a) requiring all conductivities to be positive. If this condition holds, then all the diagonal elements $c_{hh}$ are positive, while

$$c_{hh} \geqq \sum_{i \neq h} | c_{hi} |, \qquad \text{for} \qquad h = 1, \cdots, s,$$

with strict inequality holding if and only if the node $A_h$ is linked directly to a boundary node (which will certainly occur for at least one node, since neither $N - \partial N$ nor $\partial N$ is empty, and the network is connected). It follows[8] that, if, in addition the matrix $\| c_{hk} \|$ is not reducible to the form

$$\begin{Vmatrix} P & U \\ 0 & Q \end{Vmatrix}$$

by the same permutation of the order of the rows and columns, where the matrices $P$, $U$, $Q$, 0 are all square matrices, and 0 consists only of zeros, then det $\| c_{hk} \| \neq 0$. However, it is easy to see, again using the theorem mentioned in footnote 8, that if all the conductivities are positive, and the conductivity matrix of a *connected* network has the "exceptional" form just mentioned, then its determinant is still not zero. For if $\| c_{hk} \|$ is of this exceptional form then its determinant is the product of the determinants of $P$ and $Q$, each of which is again symmetric and "dominantly diagonal", and may be further reduced, in the same way that the original conductivity matrix was reduced, in case either of them is exceptional. (Notice that, in view of the symmetry of $\| c_{hk} \|$, it follows that the submatrix $U$ must consist only of zeros.) Continuing this reduction as far as possible until only non-exceptional symmetric matrices occur (only a finite number of steps are possible) one finds that the det $\| c_{hk} \|$ is the product of a finite number of determinants, each corresponding to a "dominantly diagonal" matrix which is not "exceptional", and that all elements not appearing in this product are zero. Since the given network is *connected*, at least one node in each subnetwork associated with these submatrices must be linked directly to a boundary node of the given network. Hence, by the theorem mentioned in footnote 8, the determinant of each subnetwork is not zero, and thus det $\| c_{hk} \|$ is not zero either. It is clear that this class of non-singular, dominantly diagonal symmetric matrices is but a very small subclass of the class of all non-singular symmetric matrices.

The existence theorem which we shall now prove for the (possibly) non-linear case corresponds, however, to the theorem (in the linear case) obtained from Theorem 2 upon making the superfluous additional assumption that the conductivity matrix $\| c_{hi} \|$ is "dominantly diagonal" in the sense just described above.

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[7]See C. Saltzer [27, p. 122], J. L. Synge [20, p. 127].

[8]See Theorem III of Olga Taussky [18, p. 673]. For an application to electrical networks, see M. Parodi [13].

By Theorem 2, *any* local minimum of $V(u)$ will provide a solution, in the non-linear or linear case. However, if every $C_k(\Delta u) \to +\infty$ as $|\Delta u| \to +\infty$, then $V(u)$ will be bounded below everywhere; and be arbitrarily large[9] outside any sufficiently large bounded "cube" in $(u_1, \cdots, u_s)$ space. Hence $V(u)$ will have an *absolute* minimum inside some such cube, by a theorem of Weierstrass on continuous functions. We conclude

THEOREM 3. If (2b) holds, and if, for all $k$,

$$\int_0^{\infty} c_k(x)\, dx = \int_0^{-\infty} c_k(x)\, dx = +\infty, \tag{9}$$

in the sense of improper Riemann integration, then the Dirichlet problem has a solution for arbitrary boundary values.

COROLLARY. If (2a) and (2b) both hold, then (9) may be replaced by the conditions

$$c_k(x) > 0, \qquad \text{for some} \qquad x > 0, \tag{9a}$$

and

$$c_k(x) < 0, \qquad \text{for some} \qquad x < 0. \tag{9b}$$

**5. Neumann problem.** The Neumann problem is *dual* to the Dirichlet problem, in the sense that the rôles of $u$ and $i$ are interchanged. To make the duality more marked, we note that, since any continuous, strictly increasing function $y = c_k(x)$ has a (unique) continuous, strictly increasing inverse function $x = r_k(y)$, conditions (2a), (2b) are self-dual. Accordingly, we shall replace (2) in Sec. 1 by

$$\Delta u_k = r_k[i(a_k)], \tag{10}$$

and refer to the $r_k$ as *resistance* functions. The condition that there exists a single-valued potential $u(A_h)$, such that $\Delta u_k = u(A_i) - u(A_j)$ whenever $a_k = A_iA_j$, is evidently *Kirchhoff's circuit law*

$$\sum_{\Gamma} r_k[i(a_k)] = 0, \tag{11}$$

for any sequence $\Gamma$ of oriented links forming a closed *cycle* (or circuit).

For a given influx $\nu$ on $\partial N$, satisfying the consistency conditions (3′), the most general current function $i$ which satisfies Kirchhoff's circuit law (11) is obtained by "adding", onto some fixed current function satisfying the same conditions, "cyclic" currents $\beta_1, \cdots, \beta_t$ around closed cycles $\Gamma_1, \cdots, \Gamma_t$ forming a *basis* for the closed cycles of the network. This fact is easily seen in the case of a planar network (graph), when the basic cycles may be taken as the (oriented) boundaries of the polygons into which the network subdivides the plane, and one has $r + 1 = n + t$. The general case is also classic[10] (i.e., $t = r - n + 1$ for a connected graph).

Thus, once an initial current distribution satisfying (3′) and (11) has been found, each $\beta = (\beta_1, \cdots, \beta_t)$ determines a unique current distribution on the set of links $L$, satisfying (3′) and (11), while (10) may be taken as *defining* $\Delta u$. (We treat (10) as a substitute for (2), recalling that (2) and (10) are equivalent if (2a) and (2b) hold.)

We now define $R_k(i) = \int_0^i r_k(y)\, dy$, for each link $a_k$, so that the derivative $R_k'(i) = dR_k/di = r_k(i)$, and assume for simplicity that the $r_k(y)$ are continuous.

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[9]This follows from a modification of an argument of Duffin [17, p. 965] which uses in an essential way the fact that the network is connected.

[10]Poincaré [4], Veblen [7, p. 9], Ingram and Cramlet [12, p. 137], and Synge [20, p. 123].

THEOREM 2′. For given consistent values [see (3′)] of $\nu_h$ on $\partial N$, the first variation of the function

$$W(\beta) = \sum_L R_k[i(a_k)] \tag{12}$$

vanishes identically if and only if the $r_k[i(a_k)]$ satisfy Kirchhoff's circuit law (11).

*Proof*: By direct computation,

$$\delta W = \sum_L R_k'[i(a_k)]\delta i(a_k) = \sum_B \delta\beta_i \sum_{\Gamma_i} r_k[i(a_k)], \tag{13}$$

where the last sum is taken over a basis $B$ of the closed cycles of the network, which consists of $\Gamma_1, \cdots, \Gamma_t$. This last sum is zero for arbitrary $\delta\beta$ if and only if the individual sum taken over each $\Gamma_i$ is zero, which is equivalent to (11).

COROLLARY. If (2a), (2b) hold, then Kirchhoff's circuit law (11) holds if and only if $W(\beta)$ has an absolute minimum.

The proof is similar to that of the corollary to Lemma 2, and may be omitted.

*Remark.* In the case (2c) of an exponential resistance law, with the same exponent $\alpha$ for all links in the network, it follows (see Sec. 3) that the dissipation function $D$ is proportional to the function $W$, and therefore $D$ can be used in place of $W$ in the above results. This is known in the linear case[11].

THEOREM 3′. The Neumann problem has a solution for any set of compatible boundary influxes [see (3′)], provided that (2b) holds and that

$$\int_0^\infty r_k(y)\,dy = \int_0^{-\infty} r_k(y)\,dy = +\infty. \tag{14}$$

The proof is identical with that of Theorem 3, and the analogue of the corollary to Theorem 3 also follows similarly.

**6. Mixed boundary value problem; relaxation methods; existence theorem.** Without striving for maximum generality, we shall prove an existence theorem which is adequate for most applications. The method of proof to be employed is constructive, in that in many concrete instances the performance of the "relaxation steps" used in the proof can actually be used in order to construct numerically a solution to a network boundary value problem.

Let us suppose that our boundary conditions are of Types I and II′. Since Kirchhoff's node law (1) really corresponds to a condition of Type II′, with $F_h(u) \equiv 0$, we can reformulate our problem as that of satisfying a condition of Type II′ at *all* those nodes $A_1, \cdots, A_m$ where the potential $u(A_h)$ *is not* prescribed; we shall denote this set (supposed to be not empty) of nodes by $M$. Further, we shall suppose that the set of nodes at which the potential *is* prescribed, which is $N - M$, contains at least one node. It is for this class of boundary value problems that an existence theorem will be proved, under the assumption that (2a) and (2b) hold and that

$$\lim_{x\to-\infty} c_k(x) = -\infty; \qquad \lim_{x\to+\infty} c_k(x) = +\infty. \tag{15}$$

Thus we shall exclude the case of "saturation currents".

(The requirement that the set $N - M$ be non-empty, seemingly—but only seemingly—excludes the Neumann problem from consideration. Because, granting, for the purposes

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[11]W. Thomson (Lord Kelvin) [2], and J. C. Maxwell [3].

of the present discussion, that an existence theorem has been proved for the above mentioned class of mixed problems, then an existence theorem for the Neumann problem readily follows from it. This can be seen by merely assigning arbitrarily the value of the potential at a fixed node of the network, as an additional boundary condition, besides the given Neumann conditions. By this obvious artifice, any Neumann problem can be turned into a mixed problem of the class described above, and hence has a solution for each arbitrarily assigned value of the potential at the chosen fixed node, i.e. a "one-parameter" family of solutions. In view of this, the Neumann problem need not be mentioned in the following discussion.)

Just as in Sec. 3, we can satisfy (2) by *fiat* for any choice of $u_1 = u(A_1), \cdots, u_m = u(A_m)$, merely by defining $i(a_j) = c_j(\Delta u_j)$ for each link $a_j$. We can then compute $\nu_h = \sum_L \epsilon_{hj} i(a_j)$, for each node $A_h$ in $M$, and define the *discrepancy* (or *residual*) function

$$\delta_h = \nu_h - F_h(u_h), \qquad h = 1, \cdots, m. \tag{16}$$

An existence theorem clearly asserts that $\delta(u) = 0$ for some $u = (u_1, \cdots, u_m)$.

LEMMA 4. The function $\delta = T(u)$ is one-to-one and continuous.

*Proof*: The continuity of $T$ follows from the fact that the functions $c_k$ are continuous by (2b), and the functions $F_h$ are also continuous, by (4). It remains to show that distinct $u$ determine distinct $\delta$. This follows readily from Theorem 1, but we shall go over the proof, to emphasize the rôle of the requirement that the set $N - M$, where the potential values are assigned, is not empty. To this end, consider, as in (5) and (5+) that

$$D^* = \sum_L (i_k - i_k')(\Delta u_k - \Delta u_k') \geqq 0,$$

by (2a). On the other hand

$$\begin{aligned} D^* &= \sum_L \{(i_k - i_k') \sum_N \epsilon_{hk}[u(A_h) - u'(A_h)]\} \\ &= \sum_N (\nu_h - \nu_h')[u(A_h) - u'(A_h)], \\ &= \sum_M (\nu_h - \nu_h')[u(A_h) - u'(A_h)] \end{aligned}$$

and if $T(u) = \delta = \delta' = T(u')$, then

$$D^* = \sum_M [F_h(u_h) - F'(u_h)][u(A_h) - u'(A_h)] \leqq 0,$$

by (4). Hence $D^* = 0$, and by Lemma 1, it follows that $u - u'$ is a constant. But this constant difference must be zero, since it is zero for each node in $N - M$, which is not empty.

Now, still assuming (2a), (2b) and (15), we pass on to a relaxation method. We shall consider the *residuals*[12] $\delta_h$ of a variable trial function $u(A_h)$, which are defined by (16). We first prove four lemmas involving the "order" relation.

LEMMA 5. As $u(A_h)$ is increased, all other values of $u$ being held fixed, $\delta_h$ increases, all "adjacent" $\delta_k$ decrease, and all other $\delta_j$ remain constant.

*Proof*: For if $A_k$ is "adjacent" to $A_h$, that is, there is either a link $A_hA_k$ or a link $A_kA_h$ in the network, then an increase in $u(A_h)$ increases [by (2a)] either $i(A_hA_k)$ or

[12] We shall conform to the terminology of R. V. Southwell [11], where possible.

$-i(A_kA_h)$, as the case may be; hence it increases $\nu_h$, decreases $\nu_k$, and leaves unchanged $\nu_j$ when $A_j$ is not adjacent to $A_h$. Also, by (4), an increase in $u_h = u(A_h)$ either decreases or leaves unchanged $F_h(u_h)$, and leaves unchanged all remaining $F_j(u_j)$, where $A_j \neq A_h$.

LEMMA 6. Consider a node $A_h$, and suppose that the values of $u$ at all adjacent nodes $A_k$ are increased, while the values of $u$ at $A_h$, and at all nodes not adjacent to $A_h$ are held fixed. Then $\delta_h$ decreases, while $\delta_k$, where $A_k$ is adjacent to $A_h$, increases.

The proof follows along similar lines to that of Lemma 5.

Now, consider $\delta_h$ [see (16)] as a function of the single real variable $u(A_h)$, all the other $u(A_k)$ being kept constant. From (2b), (15), and (4) it follows that $\delta_h$ is a strictly increasing continuous function of $u(A_h)$, which varies continuously from $-\infty$ to $+\infty$ as $u(A_h)$ does the same. Hence there is exactly one choice of $u(A_h)$ which will make $\delta_h[u(A_h)] = 0$ ("liquidate the residual" at $A_h$), provided that all the other $u(A_k)$ are kept constant. We define (exact) *point relaxation* at each node $A_h$ to consist of replacing the value $u(A_h)$ by this particular value which makes $\delta_h$ vanish at $A_h$, all the other values $u(A_k)$, for $A_k \neq A_h$, being kept constant.

LEMMA 7. Relaxation at a given node is isotone on trial solutions of the same problem (i.e. it preserves order).

*Proof*: The proof is by contradiction. Suppose that $u$ and $u'$ are such that $u(A_k) \geqq u'(A_k)$ for all $A_k$ in $N$, and let $v_h = v(A_h)$, and $v'_h = v'(A_h)$ denote, respectively, the functional values obtained from $u$ and $u'$ by point relaxation at the node $A_h$. Suppose, contrary to what we wish to prove, that $v_h < v'_h$. Now, starting with the "relaxed" function $u'_R$ (i.e. with the function whose value at each node $A_j \neq A_h$ is $u'(A_j)$, while at $A_h$ its value is $v'_h$) one can proceed in two steps to the "relaxed" function $u_R$, and obtain a contradiction, as follows. First, replace the values of $u'$ at all the nodes different from $A_h$ by the corresponding values of $u$ at these nodes, leaving the functional value unchanged at $A_h$ itself, and denote the resulting "hybrid" function $u'^*$. By Lemma 6, and the definition of point relaxation, one has that

$$0 = \delta_h(u'_R) \geqq \delta_h(u'^*). \tag{17}$$

Secondly replace the value of $u'^*$ at $A_h$, which is $v'_h$, by $v_h$, and leave the values of $u'^*$ at all nodes different from $A_h$ unchanged. The resulting function is precisely the "relaxed" function $u_R$. Since, by assumption, $v_h < v'_h$, it follows from Lemma 5 that

$$\delta_h(u'^*) > \delta_h(u_R). \tag{18}$$

But a comparison of inequalities (17) and (18) then shows that $\delta_h(u_R) < 0$, contradicting the fact that, since $v_h$ was obtained by point relaxation of $u$ at $A_h$, the number $\delta_h(u_R)$ must be zero. This completes the proof of Lemma 7.

In the proof of the following lemma and the theorem to follow we shall make use of two more additional assumptions, one concerning the conductivity functions $c_k$ and the other concerning the functions $F_h$ of (4). For convenience we write them as follows:

$$\text{For every } k, \text{ one has } c_k(0) = 0, \tag{19}$$

$$\text{For every function } F_h \text{ which does not vanish identically, there is a number } x_1 \text{ such that } F(x_1) \leqq 0 \text{ and a number } x_2 \text{ such that } F(x_2) \geqq 0. \tag{20}$$

In view of (4), it follows from (20) that whenever $F_h$ is not identically zero then it is $\geqq 0$ for all sufficiently negative $x$ and that it is $\leqq 0$ for all sufficiently positive $x$. As for

(19), it certainly holds in the important special cases (2c) and (2d), and it means intuitively, that "if the potential is constant then there is no flow of current".

LEMMA 8. Suppose that (19) and (20) hold, in addition to (2a), (2b) and (15). Let $u_0$ be an arbitrary trial function (i.e. having the prescribed values on $N - M$). Then there exist two other trial functions $v_0$ and $w_0$ such that

$$v_0(A_h) \leqq u_0(A_h) \leqq w_0(A_h),$$

and

$$\delta_h(v_0) \leqq 0 \leqq \delta_h(w_0), \qquad h = 1, \cdots, m.$$

*Proof*: It will suffice to show how to construct the trial function $v_0$ such that both

$$v_0(A_h) \leqq u_0(A_h) \tag{21}$$

and

$$\delta_h(v_0) \leqq 0, \tag{22}$$

for $h = 1, \cdots, m$, since the construction of $w_0$ is entirely analogous. The function $v_0$ will be defined in the following manner:

$$v_0(A_h) = \begin{cases} C, & \text{for} \quad A_h \text{ in } M \\ u_0(A_h), & \text{for} \quad A_h \text{ in } N - M, \end{cases} \tag{23}$$

where $C$ is a constant, which is to be chosen sufficiently negative so that the requirements (21) and (22) asked of $v_0$ are met. First of all, if $C \leqq \min_N u_0$ then (21) clearly holds. As for (22), notice that if $A_h$ in $M$ is *not* linked to any node of $N - M$, and $F_h \equiv 0$, then [by (19)] it follows from (23), for *any* choice of $C$ in (23), that $\delta_h(v_0) = \nu_h(v_0) = 0$, and (22) holds; however, if $F_h \not\equiv 0$, then still $\nu_h(v_0) = 0$, so that [by (20)] by choosing $C$ sufficiently negative it will be true that $\delta_h(v_0) = -F_h(v_0) \leqq 0$, and (22) will again hold. It remains to consider the case when $A_h$ in $M$ is linked to at least one node in $N - M$. From (3), in view of (19), it follows that, for *any* choice of $C$ in (23), *only* the links joining $A_h$ to a node of $N - M$ contribute essentially to the sum in $\nu_h(v_0)$, and from (2a), (2b), (4) it is seen that if $C = v_0(A_h)$ is sufficiently negative, then $\delta_h(v_0) = \nu_h(v_0) - F_h(v_0)$ will be $\leqq 0$, fulfilling (22). Thus, all in all, in order that $v_0$ defined by (23) fulfill the requirements (21), (22), one sees that $C$ must satisfy a *finite* number of conditions, all of which may be made to hold simultaneously, if only $C$ is chosen sufficiently negative. This completes the proof of Lemma 8.

We are now ready to prove our main result[13].

THEOREM 4. Suppose (2a), (2b), (15), (19), (20) hold. Let $u_0$ be any initial trial solution of a mixed network flow problem, and suppose $u_1, u_2, u_3, \cdots$ are obtained by successively "point-relaxing" the residuals of the initial trial solution $u_0$ at an infinite sequence of nodes of $M$, in such a way that each node $A_h$ in $M$ occurs infinitely often in the sequence of nodes. Then the sequence of trial functions $u_1, u_2, u_3, \cdots$ converges to the solution $z$ of the given problem, the uniqueness of which has already been established in Theorem 1.

*Proof*: First, by Lemma 8, there exist trial functions $v_0$ and $w_0$ such that both

$$v_0(A_h) \leqq u_0(A_h) \leqq w_0(A_h),$$

and

$$\delta_h(v_0) \leqq 0 \leqq \delta_h(w_0), \qquad h = 1, \cdots, m.$$

[13]This generalizes directly a result of J. B. Diaz and R. C. Roberts [22].

Let $v_1$, $u_1$, $w_1$ denote the functions obtained from $v_0$, $u_0$, $w_0$, respectively, by point relaxation at the first node of the preassigned sequence of nodes. By Lemma 7, the initial sandwich order is preserved, i.e.

$$v_1(A_h) \leqq u_1(A_h) \leqq w_1(A_h), \qquad h = 1, \cdots, m.$$

As a matter of fact, since

$$\delta_h(v_0) \leqq 0 \leqq \delta_h(w_0), \qquad h = 1, \cdots, m,$$

it actually follows that (see Lemma 5)

$$v_0(A_h) \leqq v_1(A_h) \leqq u_1(A_h) \leqq w_1(A_h) \leqq w_0(A_h), \qquad h = 1, \cdots, m$$

and that

$$\delta_h(v_1) \leqq 0 \leqq \delta_h(w_1), \qquad h = 1, \cdots, m.$$

Similar inequalities hold for any positive integer $n$, if we denote by $v_n$, $u_n$, $w_n$, respectively, the functions arising from $v_0$, $u_0$, $w_0$, respectively, after *successive* point relaxation at the first $n$ nodes of the preassigned sequence of nodes. Namely, we have

$$\begin{aligned} v_0(A_h) \leqq v_1(A_h) \leqq \cdots \leqq v_n(A_h) \leqq u_n(A_h) \leqq w_n(A_h) \leqq \cdots \\ \leqq w_1(A_h) \leqq w_0(A_h), \end{aligned} \tag{24}$$

and

$$\delta_h(v_n) \leqq 0 \leqq \delta_h(w_n), \qquad h = 1, \cdots, m.$$

Since, for each $A_h$ in $M$, the sequence of numbers $v_0(A_h)$, $v_1(A_h)$, $\cdots$, $v_n(A_h)$, $\cdots$ is non-decreasing and bounded above [e.g., by $w_0(A_h)$] it follows that the following limit exists

$$v(A_h) = \lim_{n\to\infty} v_n(A_h), \qquad h = 1, \cdots, m. \tag{25}$$

For $A_h$ on $N - M$ we have that $v(A_h)$ equals $u_0(A_h)$, which is exactly the value each function $v_n$ has at $A_h$. Thus, to show that the function $v$ is indeed a solution of the mixed problem, it only remains to show that

$$\delta_h(v) = 0, \qquad h = 1, \cdots, m. \tag{26}$$

To do this, consider $A_h$ in $M$. Since $A_h$ occurs infinitely often in the preassigned sequence of nodes employed in point relaxation, it follows that there is an infinite sequence of positive integers $n_1 < n_2 < n_3 \cdots$ such that

$$\delta_h(v_{n_k}) = 0, \qquad k = 1, 2, 3, \cdots.$$

But then from (25) and the continuity of $\delta$ (see Lemma 4), Eq. (26) follows.

By proceeding in a similar manner with the non-increasing, bounded below sequence of numbers $w_0(A_h)$, $w_1(A_h)$, $\cdots$, $w_n(A_h)$, $\cdots$ one obtains that the function $w$ defined by

$$w(A_h) = \lim_{n\to\infty} w_n(A_h), \tag{27}$$

for $A_h$ in $N$, is also a solution of the mixed problem. The uniqueness Theorem 1 then shows that $v = w$, *the* solution of the mixed problem and finally (24), (26), and (27) then show that $u_n$ also converges to *the* solution of the mixed problem.

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