A PRIORI BOUNDS FOR TEMPERATURE IN CIRCULATING FUEL REACTORS*

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1. Introduction. The neutron density, u, and temperature, v, in a circulating fuel reactor satisfy non-linear partial differential equations of the form

$$\begin{aligned} u_t &= u_{xx} + A(v)u, \\ v_t + cv_x &= u \qquad (0 \le x \le a,\ t \ge 0). \end{aligned} \tag{1.1}$$

These equations are in dimensionless form; t is a time-variable; x is a space-variable; and c is the positive, constant speed at which the fuel flows through the reactor. $A(v)$ is a given function of v. The neutron density is zero at the boundaries of the reactor:

$$u(0, t) = u(a, t) = 0. \tag{1.2}$$

The input fuel is kept at a constant temperature, say

$$v(0, t) = 0, \tag{1.3}$$

in an appropriately chosen scale. The initial state of the reactor is given by

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x), \tag{1.4}$$

where u_0, v_0 are known functions. A physical derivation of the above equations, with $A(v)$ = constant-v, appears in a forthcoming report by J. Fleck [5].

The purpose of this paper is to determine *a priori* bounds for norms of u and v. Under natural assumptions on the given functions A, u_0, v_0, we suppose that smooth solutions u, v exist to the mixed initial- and boundary-value problem posed by the preceding equations. We then derive numerical bounds, depending only on the given functions, for the maximum-norm of v and for an integral-norm of u. These bounds will hold in the infinite domain $0 \le x \le a$, $0 \le t < \infty$. Our results are found by the elementary sort of argument used by E. Hopf [1] and by L. Nirenberg [2] in their justifications for the maximum principles for elliptic and for parabolic equations.

To prove a preliminary result, we shall use the following form of the weak maximum principle for parabolic equations. Let $U(x, t)$ be a solution of the equation

$$U_t = U_{xx} + \phi(x, t)U \tag{1.5}$$

which is twice continuously differentiable in the closed rectangle

$$R_T: \quad 0 \le x \le a, \qquad 0 \le t \le T; \tag{1.6}$$

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let $\phi(x, t)$ be continuous and ≤ 0 in R_T ; assume that U takes positive values in R_T ; then U attains its maximum value on one of the boundary segments

$$x = 0, \qquad t = 0, \qquad x = a; \tag{1.7}$$

similarly, if U takes negative values in R_T , then U attains its minimum value on (1.7). This principle has long been known in various forms; see for example M. Picone [3]. In [2] Nirenberg proves the strong maximum principle, which asserts that, if U attains a positive maximum or a negative minimum at any point of R_T *not* in the set (1.7), then $U \equiv$ constant.

2. Assumptions on the given functions. We suppose that the given functions $u_0(x)$, $v_0(x)$ are twice continuously differentiable* functions of x satisfying the boundary conditions

$$u_0(0) = u_0(a) = 0, \qquad v_0(0) = 0. \tag{2.1}$$

Because u represents a density, we shall require

$$u_0(x) \geq 0 \qquad (0 \leq x \leq a). \tag{2.2}$$

Let

$$\alpha = \min v_0(x), \qquad \beta = \max v_0(x) \qquad (0 \leq x \leq a). \tag{2.3}$$

We suppose that $A(w)$ is a given, continuous function defined for all $w \geq \alpha$. Since $\alpha \leq v_0(0) = 0$, we may define

$$B(w) = \int_0^w A(\omega)\, d\omega \qquad (w \geq \alpha). \tag{2.4}$$

We now make the important assumption

$$\limsup_{w \to \infty} B(w) < \min(\gamma, 0), \tag{2.5}$$

where

$$\gamma = \min\left[-u_0(x) + cv_0'(x) + v_0''(x) + B(v_0(x))\right] \qquad (0 \leq x \leq a). \tag{2.6}$$

This assumption does not appear to be unnatural; for example, if $A(w) \equiv$ constant-w, we have $B(w) \to -\infty$ as $w \to \infty$, so that (2.5) holds regardless of the value of γ. Since $B(w)$ is a continuous function, with $B(0) = 0$, assumption (2.5) implies that there exists a unique, finite, non-negative number M such that

$$B(M) = \min(\gamma, 0), \qquad \text{and} \qquad B(w) < \min(\gamma, 0) \qquad \text{for} \qquad w > M. \tag{2.7}$$

3. Boundedness of temperature.

Theorem. Let $u(x, t)$, $v(x, t)$ be defined for $0 \leq x \leq a$, $0 \leq t < \infty$ as functions with continuous derivatives up to the third order. Let u, v satisfy the differential equations (1.1) and the boundary and initial conditions (1.2), (1.3), (1.4). Let the given functions u_0 , v_0 , A satisfy the conditions of Sec. 2. Then $v(x, t)$ is bounded; v satisfies the inequality

$$\alpha \leq v(x, t) \leq \max(\beta, M), \tag{3.1}$$

where α, β, M are the constants defined in (2.3) and (2.7).

*We will not attempt here or in the statement of the theorem to use the weakest possible assumptions of smoothness.

Proof. As a preliminary result, we prove that $u(x, t)$ is ≥ 0 for all $t \geq 0$. Consider the rectangle R_T defined by (1.6). Following Nirenberg, we define

$$\lambda = \lambda(T) = \max A[v(x, t)] \quad \text{for} \quad (x, t) \text{ in } R_T$$

and define

$$U(x, t) = e^{-\lambda t}u(x, t). \tag{3.2}$$

Then U satisfies Eq. (1.5) with

$$\phi(x, t) = A[v(x, t)] - \lambda.$$

Since ϕ is ≤ 0 in R_T, we may conclude from the maximum principle that U, and therefore u, assume negative values in R_T only if U assumes a negative minimum on one of the segments (1.7). But u, and therefore U, are ≥ 0 on (1.7) because of the boundary conditions (1.2) and the inequality (2.2). Therefore, u is ≥ 0 in R_T. Since T is arbitrary, $u(x, t)$ is ≥ 0 in R_∞, i.e. for all $t \geq 0$.

To prove the theorem it suffices to show that $v(x, t)$ satisfies the inequalities (3.1) in every finite, closed rectangle R_T. We consider R_T as the sum of four boundary segments and the interior, according to the following definitions:

$$N: \quad 0 < x < a, \quad t = T; \tag{3.3}$$

$$E: \quad x = a, \quad 0 < t \leq T; \tag{3.4}$$

$$S: \quad 0 \leq x \leq a, \quad t = 0; \tag{3.5}$$

$$W: \quad x = 0, \quad 0 < t \leq T; \tag{3.6}$$

$$I: \quad 0 < x < a, \quad 0 < t < T. \tag{3.7}$$

We also define sets R_T^- and R_T^+ as the intersections of R_T with the half-planes $ct - x \leq 0$ and $ct - x > 0$.

To find a lower bound for v, we draw from each point p in R_T the segment of positive slope $1/c$ extending to a point p' on $S + W$. The point p' lies on S or on W according as the point p lies in R_T^- or in R_T^+. Since $u \geq 0$, it follows by integration of the equation $v_t + cv_x = u$ that $v(p') \leq v(p)$. By (2.3) and (1.3) we have $v \geq \alpha$ on S and $v = 0$ on W. Therefore,

$$v \geq \alpha \text{ in } R_T^- \quad \text{and} \quad v \geq 0 \text{ in } R_T^+, \tag{3.8}$$

from which, since $\alpha \leq v_0(0) = 0$, we find the required lower bound

$$v \geq \alpha \text{ in } R_T. \tag{3.9}$$

Using the definition (2.4) of B, we may eliminate u from (1.1) to obtain the single equation for v

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)[-v_t + v_{xx} + B(v)] = 0. \tag{3.10}$$

This equation implies that there exists a continuous function f for which

$$v_t = v_{xx} + B(v) + f(ct - x). \tag{3.11}$$

To find an upper bound for v we shall need an upper bound for f. On S (3.11) becomes

$$f(-x) = v_t(x, 0) - v_{xx}(x, 0) - B[v(x, 0)],$$
$$= u_0(x) - cv_0'(x) - v_0''(x) - B[v_0(x)].$$

Therefore, by the definition (2.6),

$$f(y) \leq -\gamma \qquad (-a \leq y \leq 0). \tag{3.12}$$

On W (3.11) becomes

$$f(ct) = v_t(0, t) - v_{xx}(0, t) - B[v(0, t)]. \tag{3.13}$$

But $v = v_t = 0$ on W, and $B(0) = 0$. Since $u = 0$ on W, the second equation (1.1) gives $v_x = 0$ on W. But, by (3.8), $v(p) \geq 0$ in R_T^+, so that $v(p) \geq 0$ in a neighborhood of each point of W. Therefore,

$$v_{xx} \geq 0 \text{ on } W. \tag{3.14}$$

It now follows from (3.13) that

$$f(y) \leq 0 \qquad (y > 0). \tag{3.15}$$

Combining (3.12) and (3.15), we find

$$f(ct - x) \leq \begin{cases} -\gamma & \text{for} \quad (x, t) \text{ in } R_T^- \\ 0 & \text{for} \quad (x, t) \text{ in } R_T^+ . \end{cases} \tag{3.16}$$

Let P be a point at which v attains its maximum value in R_T. Then P lies in exactly one of the four continua W, S, $N + I$, E. If $P \in W$, then $v(P) = 0$, by the boundary condition (1.3). If $P \in S$, then $v(P) = \beta$, by (2.3).

Suppose $P \in N + I$. Since $v(P)$ is the maximum value of v, and since the set $N + I$ is open with respect to x and open from below with respect to t, we must have

$$v_x(P) = 0, \qquad v_{xx}(P) \leq 0, \qquad v_t(P) \geq 0. \tag{3.17}$$

Therefore, by (3.11) and (3.16),

$$B[v(P)] \geq -f \geq \min(\gamma, 0). \tag{3.18}$$

From (2.7) we now conclude

$$v(P) \leq M. \tag{3.19}$$

Finally, suppose $P \in E$. Since $v(P)$ is the maximum value of v, the definition (3.4) of E leads to the inequalities

$$v_x(P) \geq 0, \qquad v_t(P) \geq 0. \tag{3.20}$$

But $v_t + cv_x = u = 0$ on E, and c is > 0. Therefore, (3.20) yields

$$v_x(P) = 0, \qquad v_t(P) = 0. \tag{3.21}$$

From the maximum-property of $v(P)$ we may now conclude

$$v_{xx}(P) \leq 0. \tag{3.22}$$

From (3.21) and (3.22) follow (3.17) and, hence, the inequality $v(P) \leq M$. Summarizing the results for all of the four sets W, S, $N + I$, and E, we find $v(P) \leq \max(\beta, M)$ regardless of where the maximum value $v(P)$ is attained in R_T. This completes the proof of the theorem.

4. Boundedness of a norm of neutron density. By comparing the equations

$$u_t = u_{xx} + A(v)u, \qquad u_t^* = u_{xx}^* + \lambda u^*,$$

where λ is a constant $\geq A(v)$, one easily obtains (see Pólya and Szegö [4]) an inequality of the form

$$u(x, t) \leq K \exp(\lambda - \pi^2/a^2)t, \tag{4.1}$$

where K is a positive constant. According to (3.1) we shall certainly have $\lambda \geq A(v)$ if we define

$$\lambda = \max A(w) \qquad [\alpha \leq w \leq \max(\beta, M)].$$

Then (4.1) shows that the non-negative function u is bounded if the length a is sufficiently small, i.e. if $a^2 \leq \pi^2/\lambda$. But this restriction is too strong in cases of physical interest.

If a is unrestricted, the arguments of Sec. 3 evidently do not yield an upper bound for the neutron density $u(x, t)$. However, as an immediate consequence of the theorem, we can find a bound for a certain integral-norm of u. Define the boundaries

$$E_\infty: \ x = a, \ t > 0; \qquad S: \ 0 \leq x \leq a, \ t = 0; \qquad W_\infty: \ x = 0, \ t > 0.$$

From a point on E_∞, with ordinate t, we draw the segment L_t of slope $1/c$ to a point on $S + W_\infty$. We then define

$$n(t) = \int_{L_t} u \, ds; \tag{4.2}$$

this integral may be properly called a norm, since u is ≥ 0. By the second equation (1.1) we have

$$(1 + c^2)^{1/2} \frac{dv}{ds} = u \text{ on } L_t .$$

Therefore,

$$n(t) = (1 + c^2)^{1/2}[v(a, t) - v(a', t')],$$

where (a', t') equals $(a - ct, 0)$ or $(0, t - a/c)$ according as $ct - a \leq 0$ or $ct - a > 0$. Thus, by (2.3) and (1.3),

$$-v(a', t') \begin{cases} \leq -\alpha & \text{if} \quad ct - a \leq 0 \\ = 0 & \text{if} \quad ct - a > 0. \end{cases}$$

Since, by the theorem, $v(a, t) \leq \max(\beta, M)$, we have proved the following result:

Corollary. Let the conditions of the theorem be satisfied, and let the norm $n(t)$ be defined by (4.2). Then

$$(1 + c^2)^{-1/2} n(t) \leq \begin{cases} \max(\beta, M) - \alpha & \text{if} \quad ct - a \leq 0 \\ \max(\beta, M) & \text{if} \quad ct - a > 0. \end{cases}$$

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