ON UNIQUENESS IN THE THEORY OF PLASTICITY*

BY

D. C. DRUCKER

Brown University

Summary. The fundamental definitions of work-hardening and perfect plasticity have far reaching implications with respect to uniqueness of solution for elastic-plastic bodies. Satisfaction of the basic postulate, that in a cycle work cannot be extracted from the material and the system of forces acting upon it, guarantees an existing solution to be stable but not necessarily unique. Uniqueness follows for the usual linear relation between the increments or rates of stress and strain and also for combinations of such linear forms. Conversely, lack of uniqueness results for an elastic-perfectly plastic body when, for example, the maximum shearing stress criterion of yield is employed with the Mises flow rule.

Uniqueness [1][2][3][4][1]. Plastic stress-strain relations for work-hardening materials are strongly path dependent. It is, therefore, not reasonable to expect that a given set of final boundary values will give a unique solution for stress and strain in the interior of a body. A complete specification of the history of the applied surface tractions, body forces, and surface displacements will generally be required. It is simpler then to speak of a body under existing surface tractions T_i, body forces F_i, displacements u_i, stresses σ_{ij}, and strains ϵ_{ij}. The question is then whether the stress and strain rates, σ'_{ij} and ϵ'_{ij} are determined uniquely by the rates of change of the applied forces and displacements T'_i, F'_i, and u'_i.

The terminology of plasticity is developing continually and is much a matter of

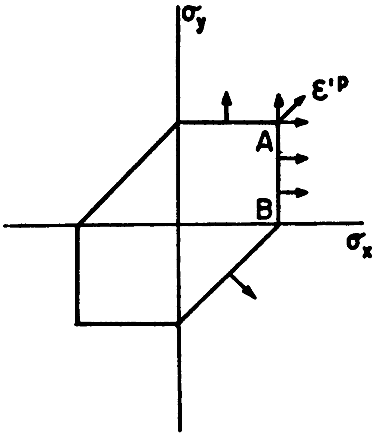

Fig. 1. Tresca criterion and associated flow rule.

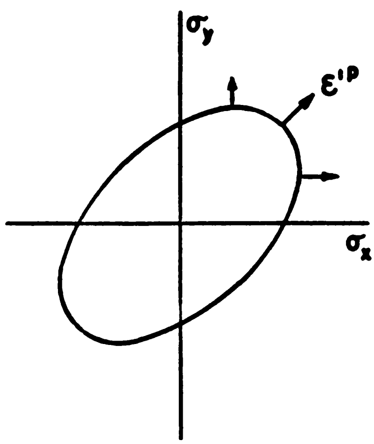

Fig. 2. Mises criterion and associated flow rule.

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[1]Numbers in brackets refer to the bibliography at the end of the paper.

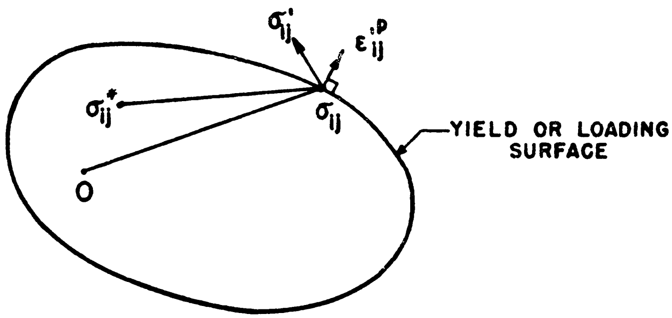

FIG. 3. Convexity and normality.

individual preference. A brief description therefore will be given of some of the terms and concepts employed here. The state of stress σ_{ij} at a material point of the body may be represented by a point σ_{ij} plotted in a stress space. Coordinates of the point are the components of stress. The vector from the origin to the stress point is called the stress vector. If, for example, the normal stresses σ_x and σ_y are the only non-zero components of stress, the familiar two-dimensional plots of Figs. 1 and 2 are useful. In general there are nine components of stress, six of which are independent because cross shears are equal $\sigma_{ij} = \sigma_{ji}$. The general stress space is nine-dimensional and is shown symbolically as in Fig. 3.

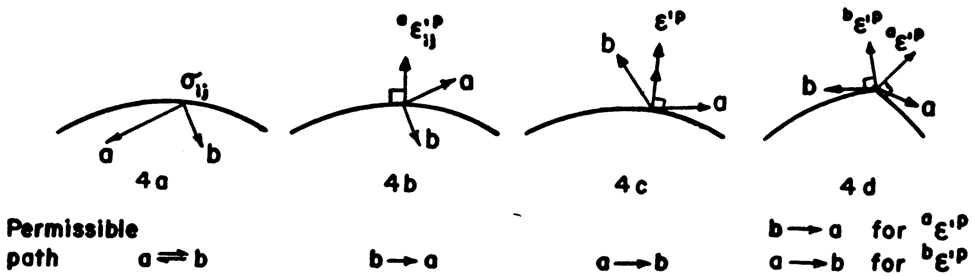

FIG. 4. Permissible paths.

When the stress at a material point changes, the stress point moves about in stress space. The region in which only elastic changes in strain occur is bounded by a surface called the yield surface. Figures 1 and 2 show the two most commonly assumed yield criteria. For a work-hardening material, a change in stress which causes the stress point to move outside the existing yield surface is termed loading. Both plastic and elastic deformation will be produced and a new yield or loading surface will be established. Subsequent yield surfaces need not resemble the original in appearance. Again, changes in stress which move the stress point about inside the region bounded by the new surface will cause elastic changes only in strain. Perfect plasticity is a limiting case of work-hardening. The yield surface is fixed in stress space as the material cannot carry stresses "above" yield. The stress point may not move outside the surface and plastic deformation will occur for points on the yield surface.

If strain coordinates are superposed on the corresponding stress coordinates, the strain rate may be exhibited as a free vector on the same diagrams. It is generally desirable to place the strain rate vector ϵ'_{ij} at the stress point σ_{ij} to emphasize that the strain rate is associated with both σ_{ij} and σ'_{ij}. For convenience and simplicity the plastic component of the strain rate, ϵ'^{p}_{ij}, alone will be plotted. The scalar product of the stress vector and the plastic strain rate vector $\sigma_{ij}\epsilon'^{p}_{ij}$ is rate of plastic work or energy dissipation.

Consideration of a homogeneous state of stress and strain in a plane and the Tresca yield criterion with its associated flow rule, Fig. 1, illustrates the well known fact that too much should not be expected in the way of uniqueness. At point A, many plastic strain rates ϵ'^{p} are associated with a given state of stress [5]. On side AB, many states of stress are associated with a given plastic strain rate. Any such lack of uniqueness in the small will show up in the study of uniqueness in general.

Uniqueness proofs ordinarily follow a standard pattern [1]-[4] which will be reviewed briefly before introducing a new point of view. Two solutions **a** and **b** are assumed; ${}^a\sigma'_{ij}$, ${}^a\epsilon'_{ij}$ and ${}^b\sigma'_{ij}$, ${}^b\epsilon'_{ij}$ corresponding to T'_i on the boundary A_T, to u'_i on the remaining boundary A_u, and to F'_i in the volume v. The theorem of virtual work is then employed (repeated indices denote summation):

$$\int_A T^*_i u_i \, dA + \int_v F^*_i u_i \, dv = \int_v \sigma^*_{ij} \epsilon_{ij} \, dv. \tag{1}$$

The starred quantities are related through equilibrium and the unstarred are compatible. There need be no relation between the two sets of quantities. The difference between the two assumed states **a** and **b**, therefore, can be substituted in Eq. (1) although ${}^a\sigma'_{ij} - {}^b\sigma'_{ij}$ may not produce ${}^a\epsilon'_{ij} - {}^b\epsilon'_{ij}$. Substitution gives

$$0 = \int_v [{}^a\sigma'_{ij} - {}^b\sigma'_{ij}][{}^a\epsilon'_{ij} - {}^b\epsilon'_{ij}] \, dv \tag{2}$$

because ${}^aT'_i = {}^bT'_i$ on A_T, ${}^au'_i = {}^bu'_i$ on A_u and ${}^aF'_i = {}^bF'_i$.

If it can be shown that the integrand in (2) is positive definite, uniqueness is proved to the extent at least of either σ'_{ij} or ϵ'_{ij} having but one possible value at each point of the body. In what follows, the term uniqueness will be used without qualification. As a first step, the strain rates are ordinarily decomposed into their elastic and plastic portions

$$\epsilon'_{ij} = \epsilon'^{e}_{ij} + \epsilon'^{p}_{ij} \tag{3}$$

and the integrand is written

$$[{}^a\sigma'_{ij} - {}^b\sigma'_{ij}][{}^a\epsilon'^{e}_{ij} - {}^b\epsilon'^{e}_{ij}] + [{}^a\sigma'_{ij} - {}^b\sigma'_{ij}][{}^a\epsilon'^{p}_{ij} - {}^b\epsilon'^{p}_{ij}]. \tag{4}$$

The first term is positive definite for both linear and nonlinear elasticity. If then the second term is positive or zero, uniqueness is established. For a rigid-plastic material the first term is identically zero and this sufficiency proof requires the second term to be positive when ${}^a\sigma'_{ij} \not\equiv {}^b\sigma'_{ij}$ or ${}^a\epsilon'_{ij} \not\equiv {}^b\epsilon'_{ij}$.

At this stage it seems appropriate to introduce the alternative point of view which avoids all mathematics for a wide class of stress-strain relations.

Mechanical-thermodynamic postulate. The fundamental definition of a work-hardening material which has been advanced previously [5][6] is that over a cycle no work can be extracted from the material and the system of forces acting upon it. An alternative and more precise set of statements refers to an external agency which applies and removes a set of forces to the already loaded body. The external agency must do positive work during the application of force. Over the cycle of application and removal of force the work done by the external agency must be positive if plastic deformation occurs and will be zero if elastic changes only take place. The definition covers the usual theories and any theory of plasticity based upon a work-hardening mechanism. Slip theory of Batdorf and Budiansky [7] which is based upon a work-hardening shear mechanism, or polycrystalline theory based upon single crystals which individually satisfy the work-hardening requirement [8] are good examples.

The definition for an ideally or perfectly plastic body is similar but the work done by the external agency may be zero when plastic deformation takes place.

It follows from consideration of a homogeneous state of stress that the yield and subsequent loading surfaces are convex and that the plastic strain rate vector is normal to such surfaces at a smooth point or lies between adjacent normals at a corner [5], Figs. 1, 2. These results may be stated in mathematical form, see Fig. 3, as

$$(\sigma_{ij} - \sigma_{ij}^*)\epsilon_{ij}'^p \geq 0, \tag{5}$$

$$\sigma_{ij}'\epsilon_{ij}'^p \geq 0 \quad \text{(work-hardening)},$$
$$= 0 \quad \text{(perfectly plastic)}. \tag{6}$$

The equality sign in (6) applies to a work-hardening material only for the trivial case of $\epsilon_{ij}'^p \equiv 0$.

Permissible stress paths and a uniqueness proof. Satisfaction of the basic thermodynamic postulate leads inevitably to convexity of the yield and subsequent loading surfaces and also to generalized normality of $\epsilon_{ij}'^p$. These results must be obtained no matter how elaborate or how simple the subsequent physical reasoning and mathematical development. The postulate might seem also to be powerful enough to insure uniqueness.

According to the concept of work-hardening or of elastic-perfectly plastic bodies, the external agency must do work to change one stress state in a body to another. In other words, constant external forces cannot, in themselves, provide the necessary addition to the sum of the stored and dissipated energy. This does give a certain type of uniqueness. If a state of stress is attained at a given system of loads, it is stable. Solutions for elastic-perfectly plastic bodies are therefore unique. However, uniqueness of solution for work-hardening bodies is more general. Stability does not necessarily rule out the possibility of two different states being reached as the external loads are changed by very small increments. There need not be any permissible path between the two states corresponding to no change in load. In a sense an insurmountable local energy barrier may exist.

If, however, the system is truly linear in the increments or rates of force, displacement, stress and strain a permissible path does exist. A combination of two alternative solutions would also be a solution. The stability and the uniqueness questions are the same. Thus the fundamental postulate insures uniqueness for linear and non-linear elasticity.

The incremental linearity of the usual plasticity theory and the extended incremental linearity obtained by combinations of linear forms [4][10] are obviously not linearity in the true sense. Plastic action is an irreversible process. Coefficients appearing in the linear expressions relating the rates of stress to the rates of strain depend upon whether the stress changes produce elastic-plastic strain or purely elastic changes. There will be variation from point to point of the body for this reason as well as for the more usual ones applicable to the plastic strain rates. Similarly the differences in strain rates between the two assumed solutions **a** and **b** will be related linearly to the differences in the stress rates. The coefficients will vary from point to point in the body depending now upon both solutions.

A pseudo-material may be defined with the same linear incremental stress-strain relations which apply to the difference between **a** and **b** but with the added physically improper assumption of reversibility. The question of uniqueness for the real material is then replaced by a question of stability of solution for the pseudo-material.

The work done by the external agency in going from **a** to **b** is the same as for **b** to **a** for this pseudo-material. A permissible path from **a** to **b** or **b** to **a** always does exist, in an extended sense, in the actual material Fig. 4. The work done by the external agency, therefore, is positive at each point. Two solutions cannot exist for the pseudo-material and uniqueness follows for the real material. It should always be kept in mind that uniqueness may be limited by conditions as in Fig. 1 and that rates of change of overall geometry are assumed negligible.

The preceding paragraphs are a proof, in words, that the fundamental postulate does insure uniqueness for the linear incremental theories of plasticity which enjoy such popularity and for the more recent type of theory which employs combinations of incrementally linear mechanisms [4][7][8][9][10].

A return to the more conventional uniqueness proof will, perhaps, help to clarify the considerations of irreversibility and permissible path. An infinitesimal time may be taken to elapse for each of the two assumed stress rates separately. The state of stress at a point in the body will change from the existing stress σ_{ij} to the stress points **a** or **b** depending upon which assumed alternative solution is followed. Some typical examples are drawn in Fig. 4. If **a** and **b** are both elastic changes, Fig. 4a, expression (4) is positive. If **b** is elastic and **a** is elastic-plastic, Fig. 4b, the second term of expression (4) is positive from (5) and (6) and the entire expression is positive. Reversing **a** and **b** does not alter this result. When **a** and **b** both represent elastic-plastic changes, Fig. 4c, and the incremental relation is linear

$$\epsilon_{ij}'^{p} = H_{ijkl}\sigma_{kl}' , \tag{7}$$

the difference between **a** and **b** will satisfy (6) and (4) will be positive definite. The coefficients H_{ijkl} are functions of stress and may depend upon the strain and the history of loading [12].

It is instructive here to look at the difference between states **a** and **b** from the point of view implied by the fundamental definition of work-hardening. Call the infinitesimal time unity so that the two assumed states of stress are $\sigma_{ij} + {}^a\sigma_{ij}'$ and $\sigma_{ij} + {}^b\sigma_{ij}'$. The corresponding strains are $\epsilon_{ij} + {}^a\epsilon_{ij}'$ and $\epsilon_{ij} + {}^b\epsilon_{ij}'$. It may be possible to go from strain state **b** to strain state **a** by going in a straight line in space from **b** to **a**. The step ${}^a\sigma_{ij}' - {}^b\sigma_{ij}'$ may be considered then as applied by an external agency which produces the corresponding strain ${}^a\epsilon_{ij}' - {}^b\epsilon_{ij}'$. The fundamental postulate then gives

$$[{}^a\sigma_{ij}' - {}^b\sigma_{ij}'][{}^a\epsilon_{ij}' - {}^b\epsilon_{ij}'] > 0 \tag{8}$$

and uniqueness is established. Figures 4a and 4b are evidently in this category as is 4c if the linearity relation (7) applies. Note, however, that all paths are not permissible. In Fig. 4b, for example, the path must go from the elastic to the elastic-plastic domain. It will generally be true that for some regions of the body the permissible path will be from **b** to **a** and for other regions from **a** to **b**.

If a corner exists as at A of Fig. 1, Fig. 4d, and the assumption is made that two or more independent linear mechanisms operate, uniqueness again follows in an obvious way by treating separately the strain components each mechanism produces. Here the path may be from **a** to **b** for one component and **b** to **a** for another.

As can be seen, the permissible paths are somewhat peculiar and linearity has a rather unusual meaning in incremental plasticity theory. The pseudo-material which is linear and reversible has no physical validity. Nevertheless, satisfaction of the fundamental work-hardening requirement plus incremental linearity in the ordinary or extended sense insures the development of a stress-strain relation which will give unique solutions.

If the stress-strain relation is incrementally non-linear [5], the fundamental definition assures stability of solution. However, inequality (8) does not necessarily hold because there is no stress path from **b** to **a** in the neighborhood of σ_{ij} which will take the strain state **b** to the strain state **a** or vice versa.

The absence in plasticity of the overall uniqueness which holds in elasticity is closely connected with the impossibility of such paths for the real or for a reversible pseudo-material. Given any finite change in T_i , for example from zero, it might be thought uniqueness could be shown as in elasticity through an inequality like (8) in which the primes are dropped. The point is that in general, and there are exceptions, it is not possible to change strain state **b** to state **a** by going from stress state **b** to **a** along a path for which the virtual work term $[{}^a\sigma_{ij} - {}^b\sigma_{ij}][{}^a\epsilon_{ij} - {}^b\epsilon_{ij}]$ has the same sign as the actual positive work done by the external agency.

Perfectly plastic bodies [11]. As has been stated, the problem of stability of solution and of uniqueness are essentially the same for elastic-perfectly plastic bodies. The definition of perfect plasticity does therefore, assure uniqueness. A conventional type of proof is sketched below for comparison.

The yield surface for ideally or perfectly plastic material is fixed in stress space. The state of stress cannot be outside the yield surface and plastic deformation occurs only when the stress point is on the surface. Normality holds at a smooth point and extended

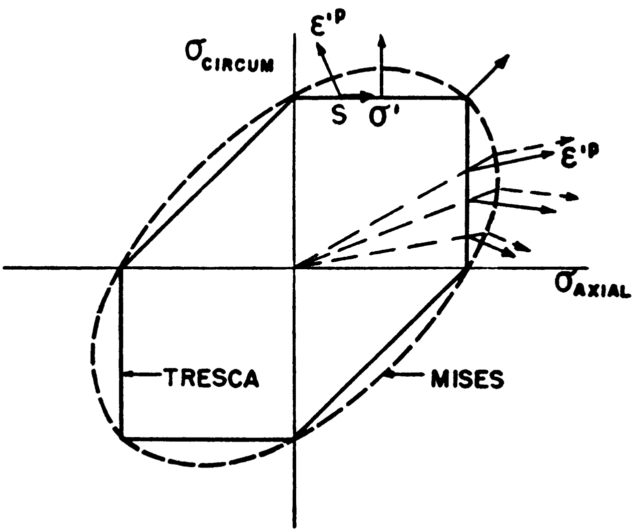

FIG. 5. Combination of one yield criterion and another flow rule.

normality applies at a corner. Considering a corner as representing the sum of independent mechanisms, the normality condition applies to each independent yield curve and there need be no distinction in the treatment of the two cases. Although the plastic term in expression (4) may be zero, the elastic is always positive unless ${}^a\sigma'_{ij} = {}^b\sigma'_{ij}$. Uniqueness, in this sense, is established for the elastic-plastic case but not for the rigid-plastic where the elastic term is identically zero at all times.

Tresca yield criterion and Mises flow rule. It has sometimes seemed convenient to adopt a maximum shearing stress criterion of yield and to couple it with the Mises flow rule, Fig. 5, in solving problems in perfect plasticity. The deviation of ϵ'^{p}_{ij} from the normal to the yield surface is the significant feature. It means that in a cycle work can be extracted from the material and the system of forces acting upon it. This violation of the fundamental postulate would seem to imply a lack of uniqueness which in turn casts some doubt on the validity of solutions obtained by such starting assumptions. In terms of the usual uniqueness proof, expression (4) can be made negative. Although the fundamental postulate or equivalently the positive definiteness of (4) provide sufficient conditions for uniqueness, their necessity does not follow. It might be true for all problems that a negative value of (4) or of work done by the external agency in one region of the body would be more than balanced by positive values elsewhere.

Proofs of necessity are difficult and often tied in with questions of existence of solutions. Such a proof will not be given. The following demonstration by means of a particular solution will serve to show lack of uniqueness for the material of Fig. 5 and make necessity quite plausible.

Consider a tube with a very thin wall subjected to interior pressure p and acted upon by an axial elastic spring under tension F and by a compressive force T which relieve some of the axial stress induced by the interior pressure, Fig. 6. The stress point

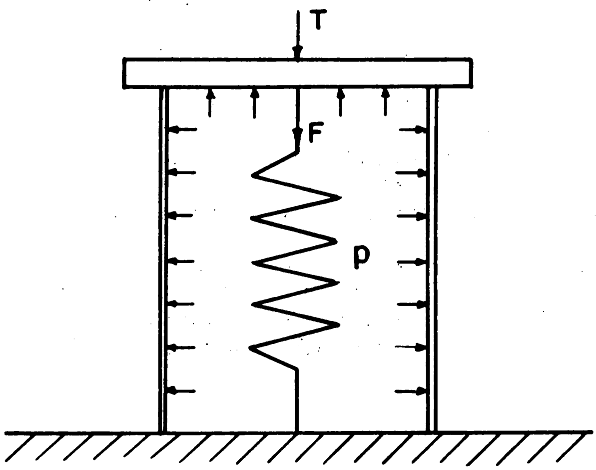

Fig. 6. Model to illustrate lack of uniqueness.

will be taken as point S, Fig. 5. Instability or lack of uniqueness shows up immediately. Consider a compressive surface traction rate T'. Equilibrium, compatibility and stress-strain relations are satisfied by a purely elastic decrease in height. Both the spring force and the axial tension in the tube wall would then decrease. If plastic deformation is assumed to occur, the situation is quite different. The plastic strain in the tube has a component of axial compression. This shortening of the tube can decrease the force in the spring to any desired extent and so increase the axial tension in the tube. It is easily seen that the lack of uniqueness is equivalent to an instability of solution. At $T = 0$, as plastic deformation takes place, the force in the spring continues to drop as the axial tension builds up in the tube until the strain rate vector has no axial component.

This lack of uniqueness does not occur if the fundamental definition of perfect plasticity is adhered to. The flow rule appropriate to the Tresca yield criterion gives a normal plastic strain rate vector which has no horizontal component at S. Work cannot be extracted; $\sigma'_{ij}\epsilon'^{p}_{ij}$ cannot be negative as in Fig. 5; T' produces the unique result that F and the axial stress in the tube decrease elastically.

It might well be argued that the example of Fig. 6 is not a good one because the change in tube diameter, which has been ignored, will increase the stress and there is an instability in this sense. However, in principle at least, the stresses could be set up in a flat sheet to obviate this trouble. It is the possibility of work extraction represented by the angle between the plastic strain rate vector and the normal to the yield surface which is responsible for the instability or lack of uniqueness.

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