where by definition

$$\zeta(\omega_1) = \eta_1 , \qquad \zeta(\omega_3) = \eta_3 .$$

(It is to be noted that $\zeta(z)$ is the Weierstrassian function associated with $\wp(z)$.)

This completes the analysis, for from Eqs. (4) and (10) it follows that

$$w_{xx} = \tfrac{1}{2}\nabla^2 w + F,$$

$$w_{yy} = \tfrac{1}{2}\nabla^2 w - F,$$

and, consequently, all of the expressions in Eqs. (3) can now be presented in closed form.

If we introduce Jacobi's theta functions, we can write

$$F = \frac{Pi}{8\pi^2 D}\left[\frac{\omega_3}{\omega_1}\frac{\theta_1'(\nu)}{\theta_1(\nu)} + \pi i(\nu - \nu^*)\right]\frac{\theta_1(\nu)\theta_2(\nu)\theta_3(\nu)\theta_4(\nu)\theta_1(\nu_0 + \nu_0^*)\theta_1(\nu_0 - \nu_0^*)}{\theta_1(\nu + \nu_0)\theta_1(\nu - \nu_0)\theta_1(\nu + \nu_0^*)\theta_1(\nu - \nu_0^*)}$$

$$+ \frac{Pi}{8\pi^2 D}\left[\frac{\omega_3}{\omega_1}\frac{\theta_1'(\nu_0)}{\theta_1(\nu_0)} + \pi i(\nu_0 - \nu_0^*)\right]\frac{\theta_1(\nu_0)\theta_2(\nu_0)\theta_3(\nu_0)\theta_4(\nu_0)\theta_1(\nu + \nu^*)\theta_1(\nu - \nu^*)}{\theta_1(\nu_0 + \nu)\theta_1(\nu_0 - \nu)\theta_1(\nu_0 + \nu^*)\theta_1(\nu_0 - \nu^*)}$$

$$+ \text{complex conjugate},$$

where

$$\nu = \frac{z}{2\omega_1}, \qquad \nu_0 = \frac{z_0}{2\omega_1}.$$

By some arguments involving the properties of the real and complex parts of analytic functions z of a complex variable z, it is possible to show that

$$w_{xy} + \frac{1}{4D}(yQ_x + xQ_y) = -\frac{1}{2}K_1(z, z_0, z^*, z_0^*) + K(z_0, z_0^*),$$

where K_1 is the harmonic conjugate of H_1. However, there seems to be no immediate way of evaluating the function K.

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GRAPHICAL DETERMINATION OF A DISCONTINUITY SURFACE BY WAVE REFLECTION*

By HEINZ G. HELFENSTEIN (*University of Alberta*)

1. Introduction. The reflection of waves, such as seismic reflection for prospecting has been discussed in many papers and books. The methods used in practice all share the simplification that the discontinuity surface is replaced in the neighborhood of the shot point by its tangent plane, and it is only this tangent plane which is determined. We shall discuss here a mathematically exact and fairly simple graphical method of

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determining points of the discontinuity surface from a limited number of well chosen points of the time-distance surface. The velocity of the waves in the surface layer is assumed to be constant and known.

2. Notation. Introduce a set of rectangular axes in space such that the surface of the earth is the x, y plane and the positive z-axis points vertically downwards. Assume the velocity s of seismic waves to be constant and known in the surface layer down to a discontinuity surface D. Waves originating from a shot at the origin are reflected at D.

To each point $F = (u, v, 0)$ on the surface of the earth there corresponds a value T of the time, viz. the time from the shot to the arrival of the reflected wave at F. Multiplying this time by the velocity s we obtain the total distance travelled by the wave. Speaking of the "time-distance surface" S we mean that $w = sT$ is plotted against u, v.

According to the law of reflection the ellipsoid of revolution which is the locus of the points whose sum of the distances from the two points (0, 0, 0) and $(u, v, 0)$ is equal to w is tangent to the surface D. In other words: D is the envelope of all these ellipsoids; the surfaces S and D are related by the contact transformation associating to each point (u, v, w) the above mentioned ellipsoid.

This transformation can of course be expressed by formulae connecting the equations of the surfaces and containing their partial derivatives. Since we can measure only a limited number of points of S, however, we try to avoid the computation of these partial derivatives. Our object is to indicate a construction by ruler and compass for points of D using only measurable data about S, in particular only a finite number of well chosen points of S.

3. Transformation of a line-element of S. We start with the following question: What information about D can be obtained from the knowledge of a single (non-vertical) line-element of the time-distance surface, given by the two close points $P = (u, v, w)$ and $P_1 = (u + \Delta u, v + \Delta v, w + \Delta w)$ on a line t in space?

With each of these points an ellipsoid of revolution E and E_1 is associated. The foci of E are $0 = (0, 0, 0)$ and $F = (u, v, 0)$; those of E_1 are 0 and $F_1 = (u + \Delta u, v + \Delta v, 0)$. In general, the surfaces E and E_1 will intersect in a fourth order curve C_4. Let I be an arbitrary point on C_4. Then by definition of the foci, we can write

$$I0 + IF = w,$$

$$I0 + IF_1 = w + \Delta w.$$

Subtracting yields

$$IF_1 - IF = \Delta w.$$

The last equation means that I is situated on one sheet of the hyperboloid of revolution H with foci F and F_1 and constant difference of distances from them equal to Δw. It is, more precisely, the sheet opening in the direction of descent of our line-element. The asymptotic cone belonging to this surface has its axis along the line FF_1, and its apex is the midpoint of this segment.

Denoting by Δs the distance FF_1, the opening of this cone is given by $\cos \alpha = \Delta w/\Delta s$. On the other hand, the slope of the line PP_1 with respect to the u, v-plane is given by $\tan \tau = \Delta w/\Delta s$.

Hence, for P_1 tending to P along the line t, the hyperboloid H degenerates in a right circular cone C whose apex is at F, whose axis is the projection of the line t on the u,

v-plane, and whose opening is found from

$$\cos \alpha = \tan \tau. \tag{1}$$

The above mentioned sheet of H containing I becomes that half of C which opens in the direction of descent of the line-element.

We showed that the intersection C_4 of E and E_1 is at the same time the intersection of E and H. These two surfaces intersect in a fourth order curve K_4 consisting in general of two loops, one on either side of F, corresponding to the two halves of the cone C. The loop in the direction of descent of the line-element is a *locus for the point* R on the discontinuity surface where the wave traveling to F was reflected.

Since both of the surfaces E and C are symmetrical with respect to the u, v-plane the same is true for their intersection K_4. Hence the projection of K_4 on the u, v-plane is a conic (more exactly, it consists of two arcs of a hyperbola).

If we know two line-elements through a point P on the time-distance surface (i.e., a tangent-plane) we are able to find the reflection point R by intersecting the two corresponding curves K_4 (both of which lie on E). All the curves K_4 originating from the ∞^1 line-elements which make up the tangent-plane of S intersect in the same points (there are 2 of these points, symmetrical with respect to the u, v-plane). This follows from the fact that the reflection point is uniquely determined by a point of S and its tangent plane.

4. Two special line-elements and the corresponding loci. For arbitrary line-elements this intersection of two fourth order curves is of course graphically too troublesome. We shall show, however, that in each tangent-plane of S there are two special line-elements yielding simple curves K_4.

(a) *First locus for R.* One of them is the line-element in radial direction: the axis of its corresponding cone C coincides with the axis of E, and K_4 therefore degenerates in a pair of circles the planes of which are perpendicular to the line OF. We can find them graphically in the following way.

First, from the slope of the radial line-element we find the angle according to (1) by an obvious construction (See Fig. 1).

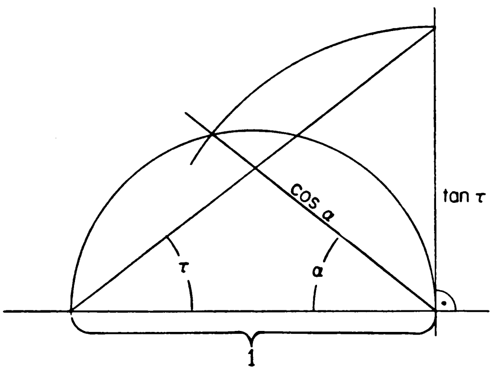

FIG. 1.

Then assume that OF is the direction of ascent of the time-distance surface in radial direction. The angle α is placed along the axis OF with its vertex at F and opening towards 0 (see Fig. 2). Along its other leg we mark off the distance $FQ_1 = w$. The mid-

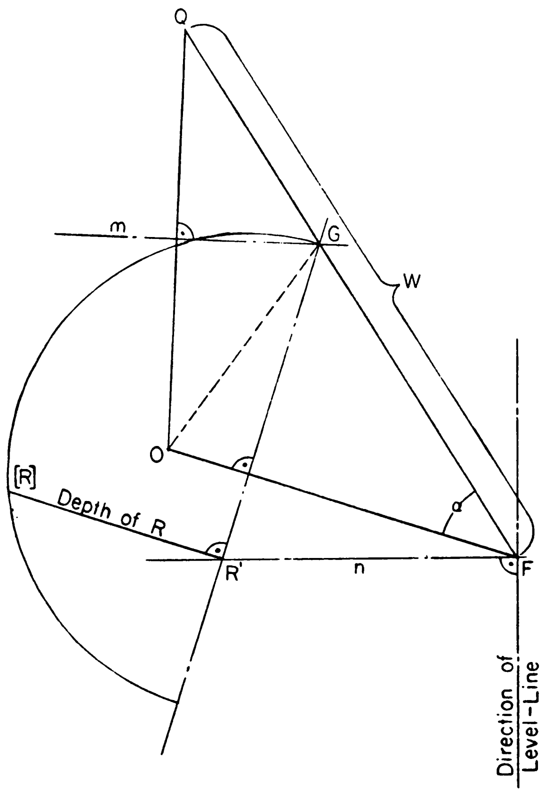

FIG. 2.

perpendicular of the segment OQ_1 intersects FQ_1 in a point C_1 for which

$$C_1O + C_1F = C_1Q_1 + C_1F = w.$$

Hence C_1 is a point of the ellipse in which the u, v-plane intersects E. As the line FQ_1 is also a generating line of the cone C (because of the angle α), C_1 is a point of K_4. For a time-distance surface descending in radial direction the distance w is measured off the line FQ_1 in the opposite direction and we get other points Q_2 and C_2. If C_1' and C_2' denote the symmetric points of C_1 and C_2 with respect to the line OF, the segments C_1C_1' and C_2C_2' are the projections of the two circles of which K_4 consists. C_1C_1' is the loop containing R if S is ascending in radial direction, otherwise it is C_2C_2'.

(b) *Second locus for R*. There is another choice of the line-element for which K_4 is still simpler. For the direction of the level-line of the time-distance surface we have $\tau = 0$. From (1) we conclude therefore that $\alpha = 90°$, which means that the corresponding cone C degenerates in a plane perpendicular to the projection of the level-line through F. K_4 becomes then an ellipse the points of which are counted twice, and the projection of which on the u, v-plane is a segment which is another locus for the projection of R. This gives the interesting property that the projection of the reflection point is always on the normal n to the level-line of the time-distance surface.

The depth of R can easily be found by means of the semi circles over C_1C_1' or C_2C_2'.

Our construction fails if C_1C_1' coincides with n. This can only happen if the level-line is itself radial. In this case we use for instance the line-element the projection of which is perpendicular to OF. Its angle τ is the same as the angle of the tangent plane of S. The axis of its corresponding cone C is perpendicular to OF. Hence we have to intersect this cone [its opening is again found from (1)] with the circle C_1C_1' which is done by turning down the plane of the latter.

As soon as we have a point of D we have also its tangent-plane: it coincides with the tangent-plane of the ellipsoid E.

5. Choice of the points on S. The accuracy of our construction depends largely on the accuracy with which the directions of the level-lines and the slopes of the radial sections can be found. If the discontinuity surface in a certain area A is to be found it is therefore advisable to place the recording instruments in several straight lines L_i radiating from the shot-point over A. From each line L_i we obtain the graph of a radial section of the time-distance surface by joining the measured values w on L_i by a smooth curve. From this curve the slopes τ of the tangents can be found graphically. In the map of the area A, the level-lines can be drawn from the measured data.

For each seismometer we obtain one reflection point with its tangent-plane. However, for each intersection of a level-line with one of the lines L_i which occurs in a "new" point (where no instrument was placed) we get another point of the discontinuity surface.

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A NOTE ON NUMERICAL DIFFERENTIATION*

By JOHN W. MILES (*University of California, Los Angeles*)

Summary. Given the matrix $\mathbf{f} = \{f_i\}$, representing $f(x)$ at the set of points $\{x_i\}$, the mth derivatives of $f(x)$ at these points are expressed in terms of *all* of the f_i according to $\mathbf{f}^{(m)} = \mathbf{C}^{-1}\mathbf{A}^m\mathbf{C}\mathbf{f}$, where $\mathbf{A}$ is the sum of the skew matrix $[(x_i - x_j)^{-1}]$ and the diagonal matrix formed by summing the terms in the corresponding rows of this skew matrix, and $\mathbf{C}$ is the diagonal matrix having as its elements the products of the elements in the corresponding rows of the skew matrix.

1. Introduction. Let f be the column matrix

$$\mathbf{f} = \{f_i\} = \{f(x_i)\}. \tag{1}$$

We require a square matrix $\mathbf{D}$ such that**

$$\mathbf{f}^{(m)} = \left\{ \frac{d^m f(x)}{dx^m} \bigg|_{x=x_i} \right\} = \mathbf{D}^m \mathbf{f}. \tag{2}$$

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**The representation of derivatives in matrix form, as in (2), also has been considered by J. Kuntzman in a paper presented at the International Mathematical Congress in Amsterdam (Sept. 1954), but no details have been published. It appears, from private correspondence with Prof. Kuntzman, that the results of Eq. (8) *et seq.* in the present paper are probably new.