ON A GENERALIZATION OF TAYLOR'S VIRTUAL MASS RELATION FOR RANKINE BODIES*

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1. Introduction. In 1928 G. I. Taylor [1] published a relation which expressed certain added mass coefficients of a moving body in terms of the singularity distributions which may be considered to generate the flow field about the body. This relation had the great advantage that, for Rankine bodies (i.e. for bodies generated by prescribed singularity distributions) these added masses could be computed directly from this relation without the necessity for the evaluation of the usually cumbersome expressions for the added masses as surface integrals in terms of potential functions. A derivation of Taylor's formula has also been given by Lamb [2].

In 1950, in his *Hydrodynamics* [3], Garrett Birkhoff casually included a generalization of Taylor's result. Recently, in the course of a seminar on added masses at the Taylor Model Basin, a new derivation of Birkhoff's generalization was obtained by the present writer which, furthermore, resulted in a simple and elegant interpretation of a term which Birkhoff had left as a surface integral. These results will now be presented.

2. Kinetic energy of a rigid body. Consider a rectangular Cartesian coordinate system (x_1, x_2, x_3) fixed to the body with origin at a point 0. The motion of the body can be described in terms of the vector velocity of translation $\mathbf{U}$ of 0 and rotation of the body with vector angular velocity $\mathbf{w}$ about 0. Let us denote the components of $\mathbf{U}$ by u_1, u_2, u_3 and the components of $\mathbf{w}$ by u_4, u_5, u_6. Let σ denote the mass density of the body and $\mathbf{r}$ the position vector of a point of the body with respect to 0.

In vector notation the kinetic energy T_1 of the body may be written in the form

$$2T_1 = \int \sigma(\mathbf{U} + \mathbf{w} \times \mathbf{r})^2 \, d\tau, \tag{1}$$

where $d\tau$ is an element of volume of the body and the integration is taken over the volume of the body. By expressing (1) in terms of the components of the vectors, the kinetic energy can readily be expressed as a quadratic form in the velocity components [4]

$$2T_1 = \sum_{i=1}^{6} \sum_{j=1}^{6} M_{ij} u_i u_j ,$$

where

$$\begin{aligned} M_{ij} &= M_{ji} , \\ M_{11} &= M_{22} = M_{33} = M, \text{ the mass of the body,} \\ M_{23} &= M_{31} = M_{12} = M_{14} = M_{25} = M_{36} = 0, \\ M_{26} &= -M_{35} = Mx_1' \, ; M_{34} = -M_{16} = Mx_2' \, ; \\ &\quad M_{15} = -M_{24} = Mx_3' , \end{aligned} \tag{2}$$

*Received March 28, 1955.

where x_1', x_2', x_3' are the components of the center of mass of the body, M_{44}, M_{55}, M_{66} are the moments of inertia of the body about the x_1, x_2, x_3 axes respectively, M_{56}, M_{64}, M_{45} are the negatives of the cross products of inertia about the x_1, x_2, x_3 axes respectively, *viz.*

$$M_{56} = -\int \sigma x_2 x_3 \, d\tau, \text{ etc.}$$

3. Kinetic energy of the fluid. It will be supposed that the body is moving through an inviscid, incompressible fluid, extending to infinity in all directions and at rest at infinity, and that the motion in the fluid is entirely due to that of the body. Then there exists a single-valued velocity potential ϕ which satisfies Laplace's equation

$$\nabla^2 \phi = 0 \tag{3}$$

and the boundary condition on the surface of the body

$$-\frac{d\phi}{dn} = (\mathbf{U} + \mathbf{w} \times \mathbf{r}) \cdot \mathbf{n}, \tag{4}$$

where $\mathbf{n}$ is the unit vector normal to the surface of the body, directed into the fluid. Let n_1, n_2, n_3 denote the direction cosines of the normal. Then the boundary condition (4) may also be written in the form

$$-\frac{d\phi}{dn} = \sum_{i=1}^{6} n_i u_i \,, \tag{5}$$

where

$$n_4 = x_2 n_3 - x_3 n_2 \,, \quad n_5 = x_3 n_1 - x_1 n_3 \,, \quad n_6 = x_1 n_2 - x_2 n_1 \,, \tag{6}$$

i.e. n_4, n_5, n_6 are the components of the vector $\mathbf{r} \times \mathbf{n}$.

Because of the linearity of Laplace's equation and the boundary conditions, we may now make the Kirchhoff assumption that

$$\phi = \sum_{i=1}^{6} u_i \phi_i \,. \tag{7}$$

Here ϕ_i is the velocity potential when the body has only the one component of motion, u_i, and that of unit magnitude. From (5) and (7) we obtain the boundary conditions

$$-\frac{d\phi_i}{dn} = n_i \,. \tag{8}$$

Let T_2 denote the kinetic energy of the fluid and ρ the density of the fluid, assumed to be uniform. Then we have

$$2T_2 = -\rho \iint \phi \frac{d\phi}{dn} \, dS, \tag{9}$$

where the integration is taken over the surface of the body. From (5) and (7) then

$$2T_2 = \sum_{i=1}^{6} \sum_{j=1}^{6} A_{ij} u_i u_j \,, \tag{10}$$

where

$$A_{ij} = \rho \iint \phi_i n_j \, dS \tag{11}$$

are the added masses of the fluid. It may also be shown that

$$A_{ij} = A_{ji} . \tag{12}$$

The foregoing brief review of the theory of added masses is essentially a summary of the treatment in Lamb's *Hydrodynamics* [5].

4. Taylor's formula and Birkhoff's generalization. Taylor's formula may be written in the form

$$A_{ii} + M = 4\pi\rho\mu_{ii} , \qquad i = 1, 2, \text{ or } 3,$$

where μ_{ii} is the ith component of the total dipole moment of the sources and doublets which give the flow about a body translating in the ith direction, and M is the mass of the displaced fluid. Birkhoff's generalization [3], in our present nomenclature, becomes,

$$A_{ij} - \rho \iint x_i \frac{d\phi_j}{dn} \, dS = 4\pi\rho\mu_{ij} , \qquad i = 1, 2, \text{ or } 3; j = 1, 2, \cdots , 6,$$

where μ_{ij} is the ith component of the dipole moment of a body corresponding to a j-component of motion of translation or rotation, and the surface integral is taken over the surface of the body.

Birkhoff derives his generalization by applying Green's reciprocal theorem to $x_i(d\phi_j/dn) - \phi_j(dx_i/dn)$ for a region bounded internally by the body and externally by the surface of a large sphere. Lamb [5] derives Taylor's formula by equating asymptotic expressions for the potential, given first in terms of the added mass, and then in terms of the dipole moment. It is shown in the following sections that Lamb's method can also yield Birkhoff's generalization, and furthermore that the term containing the surface integral is simply M_{ij} , the corresponding component of the mass tensor (2), for $\sigma = \rho$.

5. Asymptotic value of potential at a great distance from body. Let **r** denote the position vector of a point on the surface of the body with coordinates x_1 , x_2 , x_3 , and **R** the position vector of a point P at a great distance from the body, with coordinates ξ_1 , ξ_2 , ξ_3 . Let R be the magnitude of **R** and R_1 the magnitude of the vector **R** − **r**, i.e. the distance between P and a point on the surface.

The potential at P is given in terms of the values of the potential and its normal derivative on the surface by the well-known formula [6]

$$4\pi\phi_P = \iint \left[\phi \frac{d}{dn}\left(\frac{1}{R_1}\right) - \frac{1}{R_1}\frac{d\phi}{dn} \right] dS. \tag{13}$$

Let us expand $1/R_1$ in a Taylor series. We obtain

$$\frac{1}{R_1} = \frac{1}{R} + \frac{x_1\xi_1 + x_2\xi_2 + x_3\xi_3}{R^3} + \cdots \tag{14}$$

and, from (14)

$$\frac{d}{dn}\left(\frac{1}{R_1}\right) = \left(n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} + n_3 \frac{\partial}{\partial x_3}\right) \frac{1}{R_1} = \frac{n_1\xi_1 + n_2\xi_2 + n_3\xi_3}{R^3} + \cdots \tag{15}$$

Then, from (7), (11), and (15)

$$\iint \phi \frac{d}{dn}\left(\frac{1}{R_1}\right) dS \simeq \frac{1}{R^3} \iint \sum_{i=1}^{6} u_i\phi_i \sum_{j=1}^{3} n_j\xi_j \, dS$$

or

$$\iint \phi \frac{d}{dn}\left(\frac{1}{R_1}\right) dS \simeq \frac{1}{\rho R^3} \sum_{i=1}^{6} \sum_{j=1}^{3} A_{ij}u_i\xi_j \, . \tag{16}$$

Also, from (14)

$$\iint -\frac{1}{R_1}\frac{d\phi}{dn} dS \simeq \iint -\left(\frac{1}{R} + \frac{x_1\xi_1 + x_2\xi_2 + x_3\xi_3}{R^3}\right)\frac{d\phi}{dn} dS.$$

But $\iint -(d\phi/dn)dS = 0$ since the total flux of velocity through a closed stream surface is zero. Hence, from (5),

$$\iint -\frac{1}{R_1}\frac{d\phi}{dn} dS \simeq \frac{1}{R^3} \iint \sum_{i=1}^{6} u_in_i \sum_{j=1}^{3} x_j\xi_j \, dS \simeq \frac{1}{R^3} \sum_{i=1}^{6} \sum_{j=1}^{3} u_i\xi_j \iint n_ix_j \, dS. \tag{17}$$

But for $i = 1, 2$, or 3, Gauss' transformation gives

$$\iint n_ix_j \, dS = \iiint \frac{\partial x_j}{\partial x_i} d\tau = \delta_{ij}V, \tag{18}$$

where δ_{ij} is the Kronecker delta and V is the volume of the body. When $i = 4$, however, we obtain from (6)

$$\iint n_4x_j \, dS = \iint (x_2x_jn_3 - x_3x_jn_2) \, dS,$$

$$= \iiint (x_2\delta_{j3} - x_3\delta_{j2}) \, d\tau = \begin{cases} 0 & \text{if} \quad j = 1 \\ -x_3'V & \quad j = 2, \\ x_2'V & \quad j = 3 \end{cases} \tag{19}$$

where x_1', x_2', x_3' are the components of the position vector of the centroid of the volume of the body. Comparison of (18) and (19) with the values of M_{ij} tabulated following Eq. (2) shows that, when the body is of neutral density ($\sigma = \rho$),

$$\rho \iint n_ix_j \, dS = M_{ij} \, , \qquad i = 1, 2, \cdots, 6, j = 1, 2, 3. \tag{20}$$

Hence, substituting the results in (16), (17), and (20) into (13), we obtain

$$4\pi\rho R^3\phi_P \simeq \sum_{i=1}^{6} \sum_{j=1}^{3} (A_{ij} + M_{ij})u_i\xi_j \, . \tag{21}$$

Now suppose that the flow due to unit value of the ith component of the motion is generated by a set of sources C_{mi} and a set of vector doublets $\mathbf{d}_{ni}$ with components d_{ni1}, d_{ni2}, d_{ni3}. Then the potential at a point P due to this system of sources and doublets is

$$\phi_P = \sum_{i=1}^{6} u_i \left(\sum_m \frac{C_{mi}}{R_{mi}} + \sum_n \frac{\mathbf{d}_{ni} \cdot \mathbf{R}_{ni}}{R_{ni}^3} \right), \tag{22}$$

where R_{mi} is the distance from the mth source to P, $\mathbf{R}_{ni}$ the position vector of P relative to the nth doublet and R_{ni} the magnitude of $\mathbf{R}_{ni}$. But, from (14)

$$\frac{1}{R_{ni}} \simeq \frac{1}{R} + \frac{x_{m1}\xi_1 + x_{m2}\xi_2 + x_{m3}\xi_3}{R^3},$$

where x_{m1}, x_{m2}, x_{m3} are the coordinates of the mth source. Also

$$\sum_m \frac{C_{mi}}{R} = \frac{1}{R} \sum_m C_{mi} = 0$$

since the totality of sources generating a closed body is zero. Furthermore the last term in (22) becomes asymptotically

$$\frac{\mathbf{d}_{ni} \cdot \mathbf{R}_{ni}}{R_{ni}^3} \simeq \frac{\mathbf{d}_{ni} \cdot \mathbf{R}}{R^3} = \sum_{j=1}^{3} \frac{d_{nij}\xi_j}{R^3}.$$

Hence we obtain from (22)

$$R^3\phi_P \simeq \sum_{i=1}^{6} \sum_{j=1}^{3} \left(\sum_m C_{mi}x_{mj} + \sum_n d_{nij}\right) u_i \xi_j . \tag{23}$$

6. Relations between added masses and singularity distributions. Finally, comparing the asymptotic values of the potential in (21) and (23) we obtain the desired relation between the added masses and the source and doublet distribution

$$A_{ij} + M_{ij} = 4\pi\rho\left(\sum_m C_{mi}x_{mj} + \sum_n d_{nij}\right) \tag{24}$$

$i = 1, 2, \cdots 6, j = 1, 2, 3.$

It should be noted that Eq. (24) does not give relations for all the added masses. The six terms due to pure rotation, such as A_{44}, A_{45}, are not given. Attempts to find such relations, either by taking additional terms in the asymptotic series for the potential or by using Birkhoff's method [3], have given numerous results which resemble but are not quite the ones sought. Thus, although a relation for $A_{44} = \iint \phi_4(x_2n_3 - x_3n_2)dS$ has not been found, it can be shown that

$$\iint \phi_4(x_2n_3 + x_3n_2)\, dS + \iiint (x_2^2 - x_3^2)\, d\tau$$
$$= 4\pi\left[\sum_m C_{m4}x_{m2}x_{m3} + \sum_n (d_{n42}x_{n3} + d_{n43}x_{n2})\right], \tag{25}$$

where the volume integral is taken over the volume of the body and x_{n1}, x_{n2}, x_{n3} are the coordinates of the nth doublet. It is interesting to compare the term $M_{44} = \iiint (x_2^2 + x_3^2)d\tau$, which would occur in Eq. (24) if it could be applied for $j = 4$, with the corresponding term in (25). Also it has been found that, although a relation for $A_{45} = \iint \phi_5(x_2n_3 - x_3n_2)dS$ has not been found, we have

$$\iint \phi_5(x_2n_3 + x_3n_2)\, dS - \iiint x_1x_2\, d\tau$$
$$= 4\pi\left[\sum_m C_{m5}x_{m2}x_{m3} + \sum_n (d_{n52}xn_3 + d_{n53}x_{n2})\right]. \tag{26}$$

In this case the term $M_{45} = -\iiint x_1 x_2 d\tau$ occurs explicitly. These relations suggest certain guesses as to how (24) may be generalized, but we will refrain from mentioning them until some evidence for their validity is adduced.

References

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