ERRORS IN ASYMPTOTIC SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS*

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In a recent paper [1] R. L. Evans has described a general method for estimating errors in asymptotic expansions of solutions of ordinary linear differential equations, the basis of which is as follows.

Let

$$y(x) \sim \sum_{\alpha=0}^{\infty} c(\alpha)x^{\rho-\alpha} \tag{4†}$$

be the formal asymptotic expansion for large x of a solution $y(x)$ of the differential equation

$$L(y) \equiv \sum_{\nu=0}^{n} p_\nu(x)y^{(n-\nu)} = 0, \tag{3}$$

in which

$$p_\nu(x) = \sum_{\mu=-m(\nu)}^{n(\nu)} b_{\nu,\mu}\, x^{-\mu} \qquad [\nu = 0, 1, \cdots, n \text{ and each integer } n(\nu) \text{ is finite}].$$

The error in question is the difference between $y(x)$ and the sum of the first N terms in its asymptotic expansion, and is given by

$$v(x) = y(x) - u(x), \tag{6}$$

where

$$u(x) = \sum_{\alpha=0}^{N-1} c(\alpha)x^{\rho-\alpha}. \tag{5}$$

It satisfies a linear differential equation of the form

$$M(v) - L(v) = L(u), \tag{11}$$

where M is a linear differential operator of order n whose coefficients can be found from the $p_\nu(x)$, and whose term in the nth derivative cancels in (11) with that of $L(v)$. Accordingly (11) is of order $n - 1$. In particular, if $n = 2$ equation (11) is of the first order and yields immediately an indefinite integral for $v(x)$ from which bounds for $v(x)$ can be obtained.

The purpose of this note is to point out that Evans' result is incorrect. No non-trivial differential equation of the form (11) can exist. If it did, the relation $y(x) = v(x) + u(x)$, obtained from (6), would enable us to assert that the general solution of any nth order differential equation of the form (3) can always be made to depend on the solution of a non-homogeneous differential equation of order $n - 1$. In particular, we would always be able to express the general solution of any second-order equation of the form (3) as a finite combination of indefinite integrals of elementary functions.

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†The notation and equation numbers are the same as in [1].

This conclusion is borne out by an example given in [1] in which the method is applied to the equation

$$y'' + \left(-2 + \frac{1}{x}\right)y' - \left(\frac{1}{x} + \frac{n^2}{x^2}\right)y = 0,$$

with solutions $e^x I_n(x)$, $e^x K_n(x)$. It leads to the result

$$y = \sum_{\alpha=0}^{N-1} c(\alpha)x^{-\frac{1}{2}-\alpha} + (\text{constant}) \times x^{-N/2} e^x \int_x^\infty \xi^{-(N+1)/2} e^{-\xi}\, d\xi,$$

where the constant depends on N but not on x. This suggests that the derivatives of $x^{N/2}I_n(x)$ and $x^{N/2}K_n(x)$ can be expressed as finite combinations of elementary functions, which is incorrect.

If the analysis given in [1] is examined, it is found that the relations given between Eqs. (9) and (10) do not follow from (9) and (6). It would appear that the correct forms of (9) and these relations are given by

$$\sigma = \rho - N, \qquad v(x) \sim \sum_{\alpha=0}^{\infty} C(\alpha)x^{\sigma-\alpha},$$

in which event (7) transforms into itself with c, ρ replaced by C, σ respectively; in particular the equation between (10) and (10a) is to be replaced by

$$\begin{aligned}\{A_1(\sigma_a - \alpha - 2) + A_3\}C_a(\alpha + 2) + A_5C_a(\alpha)&\\ + \{(\sigma_a - \alpha - 1)(\sigma_a - \alpha - 2) + A_2(\sigma_a - \alpha - 1) + A_4\}C_a(\alpha + 1) = 0.& \qquad \text{(A)}\end{aligned}$$

In constructing the differential equation for $v(x)$ which "corresponds" to (A) it must be remembered that $C(-1)$, $C(-2)$, $\cdots$, $C(-N)$ are non-zero, accordingly this equation is non-homogeneous. Its correct form is in fact given by

$$L(v) = -L(u),$$

a result which is otherwise immediately obtainable from (3) and (6). The proposed scheme is accordingly nugatory.

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Reference

1. R. L. Evans, *Errors in asymptotic solutions of linear ordinary differential equations*, Quart. Appl. Math. 12, 295 (1954)