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CONCERNING THE EIGENVALUES OF A DIFFERENTIAL EQUATION IN CONVECTIVE HEAT TRANSFER*

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In a recent paper by Batchelor [1] there occurs the differential equation

$$d^2f/dy^2 + \lambda y(\tfrac{1}{2} - y)f = 0. \tag{1}$$

It is required to determine the values of λ for which (1) possesses a non-trivial solution, subject to the conditions

$$f(0) = f(\tfrac{1}{2}) = 0. \tag{2}$$

An approximate value of λ was obtained in the above cited paper by expanding "f" in a power series in y. Unfortunately, such an expansion is too slowly convergent to give a good estimate of λ with only a limited number of terms. However, if the function f is expanded as a power series in the eigenvalue λ, a much better value can be obtained with an equivalent number of terms. The means of obtaining such a series is quite simple and in fact a well known method, although its application to eigenvalue problems does not appear to have received the attention which would seem warranted in view of its utility as a means of obtaining numerical results.

A more convenient form of the equation for this method is obtained by setting $y = (t + 1)/4$. Denoting derivatives with respect to "t" by primes, the new equation is

$$f'' + \alpha^2(1 - t^2)f = 0, \tag{3}$$

where $\alpha^2 = \lambda/(16)^2$, and the boundary conditions are $f(\pm 1) = 0$. It may be noted that the general solution of Eq. (3) may be given in terms of the confluent hypergeometric function [2]. However, because of the limited tabulation of this function, this form of the solution is of little value in the present eigenvalue problem. Therefore, proceeding according to the previously mentioned plan of obtaining a formal expansion as a power series in α^2, we assume that $f(t)$ may be written in the form

$$f(t) = f_0(t) + \alpha^2 f_1(t) + \alpha^4 f_2(t) + \cdots + . \tag{4}$$

If this series is inserted in the differential equation (3), and the coefficients of the various powers of α^2 set equal to zero, there results the following series of equations:

$$f_0''(t) = 0, \qquad f_1''(t) = (t^2 - 1)f_0\,, \qquad f_2''(t) = (t^2 - 1)f_1\,, \qquad \text{etc.} \tag{5}$$

Solving this system in succession gives the desired coefficients in (8) as functions of "t". The boundary conditions require that $f(1) = 0$, which results in the characteristic equation in the form

$$f_0(1) + \alpha^2 f_1(1) + \alpha^4 f_2(1) + \cdots + = 0. \tag{6}$$

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If the series is terminated with the nth term, then (6) is merely a polynomial equation in α^2 whose roots are approximates to the eigenvalues.

The solutions of (5) for the case where f is even in t are given below

$$f_0(t) = 1,$$

$$f_1(t) = (-.5t^2 + .8333333t^4) \times 10^{-1},$$

$$f_2(t) = (4.1666667t^4 - 1.9444444t^6 + .1488095t^8) \times 10^{-2},$$

$$f_3(t) = (-1.3888889t^6 + 1.0912697t^8 - .2325837t^{10} + .0112735t^{12}) \times 10^{-3},$$

$$f_4(t) = (.2480158t^8 - .2755732t^{10} + .1002917t^{12} - .0133987t^{14} + .0004697t^{16}) \times 10^{-4},$$

$$f_5(t) = (-.275573t^{10} + .0412367t^{12} - .0206519t^{14} + .0047371t^{16} - .0004532t^{18} + .0000124t^{20}) \times 10^{-5},$$

$$f_6(t) = (.0020877t^{12} - .0037799t^{14} + .0025787^{16} - .0008297t^{18} + .0001366t^{20} - .0000101t^{22} + .00000002t^{24}) \times 10^{-6}.$$

Hence

$$f_0(1) = 1,$$

$$f_1(1) = -4.166667 \times 10^{-1}, \quad f_2(1) = 2.3710318 \times 10^{-2}, \quad f_3(1) = -.5189294 \times 10^{-3},$$

$$f_4(1) = .0598053 \times 10^{-4}, \quad f_5(1) = -.0042472 \times 10^{-5}, \quad f_6(1) = .0002044 \times 10^{-6}.$$

If only terms through α^6 are retained the equation

$$\beta^3 - 4.1666667\beta^2 + 2.3710318\beta - .5189294 = 0 \tag{7}$$

is obtained, where for convenience we have set $10/\alpha^2 = \beta$. The largest root of (7) is

$$\beta_1 = 3.528,$$

and the corresponding values of α^2 and λ are

$$\alpha_1^2 = 2.834, \qquad \lambda_1 = 725.6.$$

Listed below are the corresponding results obtained when 5, 6, and 7 terms are carried in the series:

Five terms:

$$\beta^4 - 4.1666667\beta^3 + 2.3710318\beta^2 - .5189294\beta + .0528053 = 0,$$
$$\beta_1 = 3.53638, \qquad \alpha_1^2 = 2.82775, \qquad \lambda_1 = 723.9. \tag{8}$$

Six terms:

$$\beta^5 - 4.1666667\beta^4 + 2.3710318\beta^3 - .5189294\beta^2 + .0598053\beta - .0042472 = 0,$$
$$\beta_1 = 3.536265, \qquad \alpha_1^2 = 2.827762, \qquad \lambda_1 = 723.91. \tag{9}$$

Seven terms:

$$\beta^6 - 4.1666667\beta^5 + 2.3710318\beta^4 - .5189294\beta^3 + .0598053\beta^2 - .0042472 + .0002046 = 0,$$
$$\beta_1 = 3.5363645, \qquad \alpha_1^2 = 2.827763, \qquad \lambda_1 = 723.907. \tag{10}$$

From the above results it may be safely concluded that the first eigenvalue of Eq. (1) is $\lambda_1 = 723.907$, correct to at least six figures.

It is theoretically possible to obtain not only the first, but also the higher, eigenvalues by this method. However, meaningful results can only be obtained for the higher ones if a very high degree of accuracy in the coefficients exists. With the present figures, the second highest roots of Eqs. (9) and (10) are respectively, .276 and .313. The latter value is probably correct to at least two places; the corresponding value of α_2^2 is 32.0.

It may be added that this same equation has been treated by Purday [3], also using a series in the independent variable. The results obtained there are

$$\alpha_1^2 = 2.83, \qquad \alpha_2^2 = 32$$

which agree with the values of the present analysis.

References

(1) G. K. Batchelor, *Heat transfer by free convection across a closed cavity between vertical boundaries at different temperatures*, Quart. Appl. Math. 12, 3 (1954)
(2) E. Kamke, *Differential equations*, Chelsea, p. 400, Eq. 2.12, 1948
(3) H. F. P. Purday, *An introduction to the mechanics of viscous flow*, Dover, p. 149, 1949

A USEFUL INTEGRAL FORMULA FOR THE INITIAL REDUCTION OF THE TRANSPORT EQUATION*

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A theorem pertaining to spherical harmonics** states that if an angle β is determined by two directions, whose coordinates on the surface of a unit sphere are (θ, α) and (θ', α'), and $F(\beta)$ is a function expandable in a series of spherical harmonics in argument $(\cos \beta)$,

$$F(\beta) = \sum_l \frac{2l+1}{4\pi} k_l P_l(\cos \beta), \tag{1}$$

then

$$\int_{4\pi} F(\beta) Y_l(\theta, \alpha)\, d\omega = k_l Y_l(\theta', \alpha'). \tag{2}$$

$Y(\theta, \alpha)$ may either be spherical surface harmonic, a tesseral harmonic, or a spherical harmonic.

The theorem has been noted because of its applicability in diffusion theory, for problems which militate use of the transport equation.

The transport equation for various types of scattering (e.g., neutron, gamma, light) may in general, be written,

$$\frac{1}{\mu}\, \mathrm{s} \cdot \nabla N(\mathbf{r}, \mathrm{s}) = -N + \frac{1}{4\pi} \int_{4\pi} p(\beta) N(\mathbf{r}, \mathrm{s}')\, d\omega' + S. \tag{3}$$

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**Hobson, *Spherical and Ellipsoidal Harmonics*, Cambridge, 1931, p. 146.