GENERAL THEORY OF ELASTIC STABILITY*

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Summary. Some general topics in elastic stability are discussed. In particular, attention is given to the relationship between adjacent-equilibrium-position and energy techniques, to the effects of non-linearity, and to the sensitivity of certain stability problems to the character of the loading.

1. Introduction. In analyzing the stability of an equilibrium state of a particular elastic system, those terms which arise from the equilibrium condition eventually cancel; consequently a number of writers have found it desirable to discuss stability in a general manner, removing these terms once and for all, and directing their attention towards the remaining terms. Usually, a certain equilibrium state is postulated, and one of two criteria is then used to determine the stability of this state. The first criterion states that a structure is unstable if an adjacent equilibrium state exists, whereas the second requires for instability that the over-all potential energy not be a relative minimum.

In setting up these criteria in analytical form, it is recognized that some sort of non-linearity is essential, in order for example to evade the uniqueness theorem of linear elasticity. Such non-linearity may arise either from the geometry of the situation (large displacements or non-linearized boundary conditions) or from the inclusion of higher order terms in the stress-strain law. Although the above uniqueness theorem reason may not be valid (on the grounds that the usual proof of this theorem contemplates the same body configuration for the two supposedly different stress-strain states to be proven identical, and so is not applicable to stability problems in any event), it is nevertheless clear that because of the cancellation of equilibrium terms it is well to include all important higher order terms. The method of incorporating these terms varies widely, as will be seen from a study of treatments by Bryan [1], Southwell [2], Biezeno and Hencky [3], Trefftz [4], Biot [5], Neuber [6], Prager [7], Goodier and Plass [8], and others. Most of the assumptions concerning non-linearity made in these treatments seem rather artificial—for example, Trefftz and Goodier obtain non-linearities by regarding as fundamental a curvilinear coordinate system which moves with the material fibers of the body, and there seems to be little basis for this method. Similarly, Prager uses techniques of superposition which are questionable when dealing with non-linear effects.**

The role of non-linearity has been further complicated as a result of a paper by Goodier [9], in which it is stated that the correct equations for the torsional buckling of a bar can apparently not be obtained by conventional energy techniques. Goodier gives a rather complicated analysis of this problem, using a Trefftz-type method, and his incorporation of non-linearities again appears to be quite arbitrary. In the problem of shell buckling, as discussed for example by Kármán and Tsien [10], it has been suggested

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**Nevertheless, the treatment of Prager is probably the clearest available. He uses the adjacent equilibrium position technique of Biezeno and Hencky, but obtains their results in a much more compact manner. In addition, the effects of inelasticity and thermal gradients are considered, and the final eigenvalue problem is thrown into the form of a variational principle.

that some sort of non-linearity may be partly responsible for the discrepancy between experiment and theory (besides the known effect of initial irregularity).

On the other hand, it is known that the effect of non-linearity in the stress-strain laws governing the usual structural metals is only of the same order of magnitude as the uncertainties in the ordinary elastic constants, and it does not seem physically reasonable that such effects should materially influence practical stability problems. In the present analysis, the stability problem is first analyzed without approximation; an engineering approximation is then obtained by consistently neglecting terms of a certain order of magnitude. Specifically, an arbitrary elastic body in equilibrium under certain loading is considered. An arbitrary virtual displacement is assumed, and the discrepancy between the work done by the loading and the increase in internal energy is calculated by means of the exact stress-strain relationship of non-linear elasticity; the most convenient form of this law is that due to Murnaghan [11]. It is decided that a positive discrepancy is the necessary and sufficient condition for instability, and the analytical consequences of this are worked out. The appropriate engineering approximation is made in the final result, and it is found that the result is different from and simpler than those usually obtained. As a special case, the problem of Goodier [9] is considered and it is found that the correct equations are obtained in a straightforward manner.

Recent discussion by Pflüger [12] and Ziegler [13] have directed attention towards certain fundamental problems in stability. Ziegler has shown by exemplification with non-conservative systems that the result of examining for stability the equations of motion of a perturbed mechanical system do not necessarily coincide with the familiar energy or adjacent-equilibrium-position techniques. Since the equations-of-motion method must be regarded as basic, Ziegler's work gives rise to some doubts as to the usefulness of the other methods. However, for the case of conservative systems (to which we restrict ourselves for the present; plastic buckling will be discussed elsewhere), the equations-of-motion method and the energy method are equivalent (a proof will be found in Whittaker [14]), and so the energy method may be used with confidence. However, the adjacent-equilibrium method and the energy method are certainly not equivalent even for conservative systems. Consider for example an (always elastic) column compressed beyond the buckling load but restrained from buckling; if the constraints are removed the column will buckle despite the absence of an adjacent equilibrium position.

But we can perhaps obtain an equivalence by altering the problem somewhat. Consider an elastic system which is in stable equilibrium under certain loading. As the loading is increased in some manner, the system following an equilibrium path, a point of instability (by the energy criterion) may be reached; does an adjacent equilibrium position exist at this critical point? This is a question of considerable practical significance. For example, in the problem of "tin-canning"—i.e., the problem of stability of a fairly flat shell under lateral pressure, where at a certain critical load the shell tends to snap suddenly through into an entirely different equilibrium position—there seems to be no adjacent equilibrium position (in the conventional Euler-column sense) corresponding to the critical load. Can an adjacent-equilibrium-technique then (as used in practice) give the correct answer to such problems?

It will be shown that the two techniques are indeed identical for the altered problem of the last paragraph (so that, for example, the correct answer in tin-canning problems is that conventionally obtained, and one must look elsewhere for the discrepancy between theory and experiment). In particular, it will be found that the differential equations of first variation of the general energy principle are precisely the same as the conditions for existence of an adjacent equilibrium position, these conditions being again calculated without approximation.

Another stability topic of considerable interest relates to the precise character of the loading applied to an elastic system. Such effects are not considered in previous general stability analyses, yet have been shown to be important by Tsien [15]. A particularly subtle example is given here in which, for any perturbation, the first-order work done by two alternative types of loading is the same, yet the buckling loads are widely different. Since the problem is a very practical one (buckling of a long cylinder under external pressure), it is clear that the character of the loading must be included in any general stability criterion. The appropriate analysis will be given for the two types of loading of greatest importance, viz., dead loading and pressure loading.

2. Analytical condition or instability with dead loading. Consider an arbitrary elastic body which is initially free from stress (state I). By the application of load or of heat, the body alters position and shape (and achieves state II). The material particle initially at the point (a_1, a_2, a_3) has now moved to (x_1, x_2, x_3), where subscripts are used to distinguish between the usual three fixed Cartesian axes. The ith component of the displacement vector is given by

$$v_i = x_i - a_i \tag{1}$$

and the Lagrangian strain tensor is

$$\eta_{ij} = \tfrac{1}{2}[\partial v_i/\partial a_j + \partial v_j/\partial a_i + (\partial v_s/\partial a_i)(\partial v_s/\partial a_j)], \tag{2}$$

where the summation convention is used (here and in the future) for repeated subscripts.

If U is the internal energy per unit mass, a symmetric function of the nine η_{ij}, of the absolute temperature T (or entropy S), and of position, then by Murnaghan's treatment [11] the Eulerian stress τ_{ij} in state II is given by

$$\tau_{ij} = \rho(\partial U/\partial \eta_{pq})_s(\partial x_i/\partial a_p)(\partial x_j/\partial a_q), \tag{3}$$

where ρ is the density in state II and the partial differentiation of U is to be carried out at constant entropy. It is now required to analyze the stability of the body in its deformed state II. From Sec. 1, the body will be considered stable if for each infinitesimal displacement (compatible with the boundary conditions) the work that would be done by the surface and body forces does not exceed that absorbed as an increase in internal energy.* If this condition is not met, then for some virtual displacement excess energy would be available for use as kinetic energy, and the appropriate displacement will increase in magnitude.

The body force per unit mass, F_i, will be assumed constant (e.g., gravitation). The surface loading, T_i per unit area in state II, is considered to be produced by fixed loads which vary neither in total magnitude nor direction during the trial displacement. Thus, under such "dead" loading, the material particles constituting a portion of the surface in state II will always be subject to the same total surface vector force, irrespective of their orientation or total area, throughout the trial displacement. Consequently, the

*In the equivalent potential energy form, this is the usual energetic stability criterion. We use the above form because of its additional generality; as will be shown elsewhere, it can then be applied to certain non-conservative systems also.

work done by the body and surface forces in a displacement u_i from state II would be, exactly,

$$W = \int \rho F_i u_i \, dV + \int T_i u_i \, dS,$$

where the volume and surface integrals are calculated for state II. Altering to volume integrals and using the equations of equilibrium gives

$$W = \int \tau_{ij}(\partial u_i/\partial x_j) \, dV$$

which, upon substitution from Eq. (3), becomes

$$W = \int (\partial U/\partial \eta_{pq})(\partial x_i/\partial a_p)(\partial u_i/\partial a_q)\rho \, dV. \tag{4}$$

The increase in internal energy is, exactly,

$$\Omega = \int (U' - U)\rho \, dV, \tag{5}$$

where U' denotes the internal energy per unit mass following the displacement u_i, and depends on the temperature of that state as well as on u_i. Note that the volume integral is still calculated for state II (this is allowable because the element of mass, $\rho \, dV$, is invariant).

Consequently, the general condition for stability is that, for each allowable u_i,

$$\int [(\partial U/\partial \eta_{pq})_s(\partial x_i/\partial a_p)(\partial u_i/\partial a_q) - \{U' - U\}]\rho \, dV \leq 0 \tag{6}$$

(in particular the integral vanishes for $u_i = 0$; if it vanishes for some $u_i \neq 0$ but is non-positive for all u_i, the state will be called neutrally stable.)

It is now necessary to calculate U'. Because buckling is usually rapid, it is reasonable to require the displacement u_i to be of an adiabatic character, and we will make this assumption. Had we at this point insisted on an isothermal motion, an entirely analogous calculation (best carried out by use of the Helmholtz free energy function instead of U) could have been made, and the same final results would be obtained in the sequel except that the isothermal rather than the adiabatic elastic constants would appear. Experimentally, the difference between these constants is negligible; then, using the fact that in general the motion u_i of the body would be somewhere between adiabatic and isothermal, it is seen that the particular thermal assumption at this point makes little difference. In any event, we consider for definiteness an adiabatic motion, so that in the power series expansion of U, viz.,

$$U' - U = (\partial U/\partial \eta_{pq}) \, \delta\eta_{pq} + \tfrac{1}{2}(\partial^2 U/\partial \eta_{pq} \, \partial \eta_{ij}) \, \delta\eta_{ij} \, \delta\eta_{pq} + \cdots \tag{7}$$

all partial derivatives are to be calculated for constant entropy and for state II. Using

$$\delta\eta_{ij} = \tfrac{1}{2}[(\partial x_r/\partial a_i)(\partial u_r/\partial a_j) + (\partial x_r/\partial a_j)(\partial u_r/\partial a_i) + (\partial u_r/\partial a_i)(\partial u_r/\partial a_j)] \tag{8}$$

in Eq. (7), and substituting the result into Eq. (6), gives as the condition for stability that

$$\int \rho \, dV[(\partial U/\partial \eta_{ij})(\partial u_r/\partial a_i)(\partial u_r/\partial a_j) + (\partial^2 U/\partial \eta_{ij} \, \partial \eta_{pq}) \, \partial \eta_{ij} \, \delta\eta_{pq} + \cdots]$$

be greater than zero for all non-zero permissible u_i. Alternatively, use of Eq. (3) allows the condition to be written as

$$\int dV[\tau_{pq}(\partial u_r/\partial x_p)(\partial u_r/\partial x_q) + \rho(\partial^2 U/\partial \eta_{ij}\, \partial \eta_{pq})\, \delta\eta_{ij}\, \delta\eta_{pq} + \cdots] > 0. \tag{9}$$

Except for pathological cases, only second-order terms need be considered, and the criterion for stability becomes

$$\int dV[\tau_{pq}(\partial u_r/\partial x_p)(\partial u_r/\partial x_q) + \rho(\partial^2 U/\partial \eta_{ij}\, \partial \eta_{pq})(\partial x_r/\partial a_i)(\partial x_s/\partial a_p)(\partial u_r/\partial a_j)(\partial u_s/\partial a_q)] > 0. \tag{10}$$

Here, all quantities are calculated for state II. For adiabatic virtual displacements, this criterion is exact, and must be used wherever non-linearity of the stress-strain law is essential.

3. Engineering approximation. Isotropic media. The second term in Eq. (10) may be calculated by means of a power series expansion in η_{ij} in terms of the various derivatives of U evaluated for state I. Using the subscript "0" to indicate state I,

$$\partial^2 U/\partial \eta_{ij}\, \partial \eta_{pq} = (\partial^2 U/\partial \eta_{ij}\, \partial \eta_{pq})_0 + (\partial^3 U/\partial \eta_{ij}\, \partial \eta_{pq}\, \partial \eta_{rs})_0 \eta_{rs} + \cdots . \tag{11}$$

For structural metals, the magnitude of the second term in Eq. (11) is generally smaller than the uncertainty in the experimental value of the first term (the first term represents the usual elastic constants, and the second and following terms represent non-linear elastic effects). Consequently, it is reasonable to approximate the second term coefficient by

$$\rho_0\, (\partial^2 U/\partial \eta_{ij}\, \partial \eta_{pq})_0\, ,$$

where the density in state II has also been replaced by the density in state I. This term is recognized as the conventional (adiabatic) elastic coefficient and will be denoted by c^0_{ijpq}. Then the second term may be written

$$[c^0_{ijpq}(\partial x_r/\partial a_i)(\partial x_t/\partial a_j)(\partial x_s/\partial a_p)(\partial x_m/\partial a_q)](\partial u_r/\partial x_t)(\partial u_s/\partial x_m). \tag{12}$$

Now the deformation (although not the displacement) between states I and II is assumed small; this means that the partial derivatives inside the square bracket of (12) represent, within the approximation being made, a pure rotation. But the quantities c^0_{ijpq} form a Cartesian tensor, so that the quantity in square brackets reduces simply to the elastic coefficients for the orientation of state II, i.e., to c_{rtsm}. Thus the stability condition, Eq. (10), becomes

$$\int dV[\tau_{pq}(\partial u_r/\partial x_p)(\partial u_r/\partial x_q) + c_{rtsm} e_{rt} e_{sm}] > 0, \tag{13}$$

where $e_{ij} = \frac{1}{2}[(\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)]$ and where the symmetry property of c_{rtsm} has been used. For isotropic media,

$$c_{rtsm} = G\left[\delta_{rs}\delta_{tm} + \delta_{ts}\delta_{rm} + \frac{2\sigma}{1 - 2\sigma}\,\delta_{rt}\delta_{sm}\right], \tag{14}$$

where G is the shear modulus and σ is Poisson's ratio. Use of this relation gives the stability criterion as

$$\int dV\left[\tau_{pq}(\partial u_r/\partial x_p)(\partial u_r/\partial x_q) + 2G\left\{e_{sm}e_{sm} + \frac{\sigma}{1 - 2\sigma}\, e_{tt}e_{mm}\right\}\right] > 0. \tag{15}$$

The only terms which have been neglected in the derivation of Eq. (15) are those which are of higher order in powers of η_{ij} .

4. Euler column. Before proceeding with the general theory, it is worthwhile to consider a simple example. Let a slender column of length L and cross-sectional area A be placed so that its neutral axis coincides with the x_3 axis, and so that it may buckle in the $x_1 - x_3$ plane only, the ends being restrained from lateral motion. A total load P is applied to the end of the column, producing a stress of $\tau_{33} = -(P/A)$, all other $\tau_{ij} = 0$. We now use Eq. (15), and see how large P must be, for certain trial displacements, before the left-hand side of Eq. (15) becomes negative; such a situation would correspond to buckling. Since the trial displacement will usually not be the exactly best ones for this purpose, the buckling load obtained in this manner will always be too high; this remark clearly holds in general also and is not restricted to the Euler column case (see Ref. [8]). In fact, the buckling load P_c is given by

$$(P_c/2GA) = \min_{\{u_i\}} \frac{\int \{e_{sm}e_{sm} + (\sigma/1 - 2\sigma)e_{tt}e_{mm}\}\, dV}{\int (\partial u_r/\partial x_3)(\partial u_r/\partial x_3)\, dV}. \tag{16}$$

Whenever P exceeds P_c , Eq. (15) shows that the column is unstable.

Let us choose a trial displacement, being guided in our choice by the tendency of plane cross sections to remain plane and perpendicular to the neutral axis during bending:

$$\begin{aligned} u_1 &= u(x_3), \\ u_2 &= 0, \\ u_3 &= -x_1 u'(x_3), \end{aligned} \tag{17}$$

where $u(x_3)$ is an arbitrary function of x_3 which vanishes at $x_3 = 0$, L, and where $u'(x_3)$ is its derivative. A straightforward calculation using Eq. (15) gives that

$$-(P/A)\left[A \int_0^L (u')^2 + I \int_0^L (u'')^2\right] + 2G\left(\frac{1-\sigma}{1-2\sigma}\right) I \int_0^L (u'')^2 < 0 \tag{18}$$

for instability, where I is the appropriate moment of inertia of the cross section. Because $(P/A) \ll G$, the second term in the square bracket is omitted and we obtain

$$\begin{aligned} P_c &= 2GI\left(\frac{1-\sigma}{1-2\sigma}\right) \min \frac{\int_0^L (u'')^2}{\int_0^L (u')^2} \\ &= 2GI\left(\frac{1-\sigma}{1-2\sigma}\right)(\pi/L)^2. \end{aligned} \tag{19}$$

This answer is too high, because

$$2G\left(\frac{1-\sigma}{1-2\sigma}\right) > E$$

so that the displacement (17) is deficient in some respect. The deficiency lies in the fact that u_1 and u_2 do not contain terms allowing for lateral expansion of the column during bending. Actually, instead of setting $e_{11} = e_{22} = 0$ in Eq. (15), we should more correctly have set $e_{11} = e_{22} = -\sigma e_{33}$. A simple calculation shows that the incorporation of such terms does not materially alter the first term in Eq. (15). A suitable altered displacement would in fact be

$$u_1 = u + \frac{\sigma}{2}(x_1^2 - x_2^2)u'',$$
$$u_2 = \sigma x_1 x_2 u'', \tag{20}$$
$$u_3 = -x_1 u'.$$

If this displacement is inserted into Eq. (15), and the magnitude of various terms examined (which is most easily done by assuming that u is not far removed from that u used to minimize Eq. (19), viz., sin $(\pi x/L)$), it is found that the first term of Eq. (15) is essentially unaltered, whereas the second becomes (very closely) Ee_{33}^2. Then a similar calculation to that of Eq. (19) gives

$$P_c = EI(\pi/L)^2.$$

It will be noted in Eq. (15) that the second term is the familiar strain energy term, so that the first term must in a sense represent work done by the loading. It is therefore not surprising that minor transverse alterations in the u_i (these alterations incidentally vanishing on the neutral axis) do not affect to any extent the value of the first term. This is a rather useful point to note, because it means that simple displacements of the type (17) may be used in many column problems, provided only that e_{11} and e_{22} are set equal to $(-\sigma e_{33})$ in Eq. (15).

5. Flexural buckling. Goodier [9] has examined the use of energy techniques in the flexural buckling of a twisted bar. Since his analysis is geometrically complicated and physically questionable it is worthwhile to show that the correct equations are obtained by use of Eq. (15) in a routine manner. Since the only question is as to whether or not certain terms occur, it is only necessary to consider a simple special case—that of a circular cylinder. If the central line coincides with the x_3-axis and if the angle of twist per unit length is θ, the stresses are

$$\tau_{13} = -G\theta x_2, \qquad \tau_{23} = G\theta x_1.$$

Assume a displacement of the form

$$u_1 = u - \beta x_2,$$
$$u_2 = v + \beta x_1,$$
$$u_3 = -x_1 u' - x_2 v',$$

where u, v, β are functions of x_3. (If the cylinder were non-circular, a term involving the warping function multiplied by β' should be adjoined to u_i.) Then using the technique of Sec. 4, condition (15) requires for stability

$$2G\theta \int_0^L (-u'v''I_1 + v'u''I_2) + GI_0 \int_0^L (\beta')^2 + E \int_0^L [I_2(u'')^2 + I_1(v'')^2] > 0$$

and minimizing* this expression yields

$$EI_2 u''' + Mv'' = 0, \qquad EI_1 v''' - Mu'' = 0, \qquad \beta'' = 0.$$

*That minimization is the appropriate procedure will be shown subsequently.

With the appropriate boundary conditions, these coincide with the final results of Goodier. (Note that I_1 and I_2 are defined in an opposite way to that of Goodier. Here I_1 is defined as being about the x_1-axis, i.e., $\int x_2^2\, dA$.)

6. Curvilinear coordinates. Buckling of a cylinder under dead load. Very often, the appropriate coordinate system is not Cartesian; in such cases it is useful to have available a more general formulation of Eq. (15). Let the differential element of distance be given by

$$ds^2 = h_1^2\, dy_1^2 + h_2^2\, dy_2^2 + h_3^2\, dy_3^2\ ,$$

where h_1, h_2, h_3 are functions of the three curvilinear coordinates y_1, y_2, y_3. Then denoting by τ_{ij} the curvilinear stress components and by u_i the curvilinear virtual displacement component (i.e., in the parametric direction of y_i), the first term of Eq. (15) may be shown by direct calculation to become

$$\sum_{p,q,r,m} \left[\frac{\tau_{pq}}{h_p h_q}\left\{u_{r,p}u_{r,q} + \frac{u_p u_q}{h_r^2} h_{p,r}h_{q,r} + \left(\frac{u_r u_m}{h_r h_m}\delta_{pq}h_{p,m}h_{p,r} + \frac{2u_m}{h_m}h_{q,m}u_{q,p} - \frac{2u_q}{h_r}u_{r,p}h_{q,r} - \frac{2u_m u_p}{h_m h_q}h_{p,q}h_{q,m}\right)\right\}\right], \tag{21}$$

where a comma indicates differentiation with respect to the appropriate y_i—thus $u_{r,p}$ means $(\partial u_r/\partial y_p)$.

The form of the second term is unaltered, but e_{ij} must now be interpreted as a curvilinear strain component, perhaps most conveniently given by

$$e_{ij} = \frac{1}{2}\sum_s \left[\frac{h_j}{h_i}\frac{\partial}{\partial y_i}\left(\frac{u_j}{h_j}\right) + \frac{h_i}{h_j}\frac{\partial}{\partial y_j}\left(\frac{u_i}{h_i}\right) + 2\delta_{ij}\frac{u_s}{h_s h_j}\frac{\partial h_i}{\partial y_s}\right]. \tag{22}$$

Consider for example a long thin circular shell of mean radius R, under the action of an external pressure P of the present dead-loading (see Sec. 3) type. As cylindrical coordinates, choose $y_1 = x$ along the axis of the shell, $y_2 = \theta$, the polar angle, and $y_3 = r$, the radial distance from the central axis. Then

$$ds^2 = dy_1^2 + y_3^2\, dy_2^2 + dy_3^2\ .$$

Choose

$$u_1 = 0,$$

$$u_2 = v + \frac{z}{R}(v - w'),$$

$$u_3 = w,$$

where v, w are functions of θ representing the motion of the central surface of the shell, and $z = r - R$. From shell theory, it is known that displacements of this type are suitable for calculating all strains except e_{13}, e_{23}, e_{33}. The former two are conventionally negligible, and the latter is usually calculated by assuming the induced τ_{33} stresses to be much smaller than the bending stresses in the shell. Then the e_{ij} to be used in Eq. (15) are

$$e_{11} = e_{12} = e_{13} = e_{23} = 0,$$

$$e_{22} = \frac{v'}{R} + \frac{w}{R} - \frac{z}{R^2}(w + w''),$$

$$e_{33} = -\frac{\sigma}{1-\sigma}e_{22}\ ,$$

where a minor approximation has been made. Using Eq. (21), the stability condition (15) becomes that

$$\int dV\left\{\left(-\frac{PR}{tr^2}\right)[(u_{2,2} + u_3)^2 + (u_2 - u_{3,2})^2] + \frac{E}{1-\sigma^2}\,[e_{22}^2]\right\} > 0,$$

where t is the thickness of the shell. Because $(PR)/t \ll E$, the first term may be submerged in the last to give, approximately,

$$\int dV\left\{-\frac{P}{tR}\,[(v - w') + \frac{z}{R}(v - w')]^2 + \frac{E}{1-\sigma^2}\left[\frac{v'}{R} + \frac{w}{R} - \frac{z}{R^2}(w + w'')\right]^2\right\} > 0. \quad (23)$$

Taking a unit length of cylinder and integrating over the cross-sectional area gives

$$P_c = \frac{E}{(1-\sigma^2)R}\min\frac{t\int_0^{2\pi}(v' + w)^2 + (t^3/12R^2)\int_0^{2\pi}(w + w'')^2}{\int_0^{2\pi}(v - w')^2}$$

which is obtained (very closely) by setting $v' = -w$ and $v = \sin 2\theta(v = \sin\theta$ would correspond to rigid body motion). Then

$$P_c = \frac{4EI}{(1-\sigma^2)R^3}, \quad (24)$$

where $I = t^3/12$. Since the usual result is (3/4) of this, it is clear that the assumption of dead loading has materially altered the critical load. Load-type sensitivity has been remarked for this problem by Stevens [16] and more generally by Tsien [15]; we consider it here to exemplify the manner in which the stability criterion will be generalized.

7. Pressure loading. We return now to the general theory of Sec. 2, and examine the effects of different types of loading. Firstly, it is clear that forces exerted by fixed constraints (pin joints, etc.) are in general included in the theory of Sec. 2, for even though such forces may alter in direction and magnitude during a trial displacement, the appropriate component of u_i at this point of application is zero. If secondly, however, some of the surface tractions are not of the dead-loading type, then additional terms must in general be adjoined to Eq. (15). We consider here only the practically most important such forces, viz., pressure-type forces, for which the force applied to a given portion of the surface of the body varies in such a manner as to remain always perpendicular to that portion and so as always to maintain the same magnitude per unit area. Further, the system is still assumed conservative, so that the total work done by these pressure forces is independent of the path. If then a pressure P acts on a portion S_P of the surface, the work done in the trial displacement u_i can be calculated by allowing the intermediate displacement to grow at a constant rate—i.e., if t is time, let the displacement at time t be $(u_i t)$ and calculate the work done from $t = 0$ to $t = 1$. This work, w_1, is

$$W_1 = \int_0^1 dt \int (dS_P)_t\left[\frac{d}{dt}(u_i t)\cdot(-P)\cdot(n_i)_t\right],$$

where the subscript t denotes evaluation at time t. But

$$(n_i)_t(dS_P)_t = \frac{1}{2}e_{ijk}e_{rpq}\frac{\partial(x_j + u_j t)}{\partial x_p}\frac{\partial(x_k + u_k t)}{\partial x_q}n_r\,dS_P$$

so that

$$W_1 = \int_0^1 dt \int dS_P\left[\frac{1}{2} e_{ijk}e_{rpq}\{\delta_{ip} + u_{i,p}t\}\{\delta_{kq} + u_{k,q}t\}n_r\right](-Pu_i),$$
$$= -P \int dS_P\left[n_iu_i + \frac{1}{2}(n_iu_{k,k}u_i - n_ku_{k,i}u_i) + \frac{1}{6} e_{ijk}e_{rpq}u_{j,p}u_{k,q}u_i\right]. \tag{25}$$

In this exact expression, the first term would already have been included if P had been treated as a dead load; consequently the additional work done is that due to the remaining terms. Again we omit terms of third order in u_i [see Eq. (10)], and remembering that a factor of -2 was incorporated into the derivation of Eq. (10), the term that must be adjoined to Eq. (10) is

$$\int dS_P P[n_iu_{k,k}u_i - n_ku_{k,i}u_i]. \tag{26}$$

Considering again the problem of Sec. 6, the term to be added to Eq. (23) is easily seen to be

$$P \int_0^{2\pi} (w^2 + wv' - vw' + v^2)\, d\theta$$

and instead of Eq. (24) we obtain

$$P_c = \frac{3EI}{(1 - \sigma^2)R^3} \tag{27}$$

which is the conventional result.

8. Adjacent-equilibrium-position method. It is now proposed to set up in analytical form the condition that an adjacent equilibrium position should exist. Using the notation of Sec. 2, and denoting quantities in the perturbed state by primes, the stresses following the virtual displacement u_i will be

$$\tau'_{ij} = \rho'\left(\frac{\partial U}{\partial \eta_{pq}}\right)' \frac{\partial x'_i}{\partial a_p} \frac{\partial x'_j}{\partial a_q}, \tag{28}$$

where $x'_i = x_i + u_i$. The partial derivatives of U will as before be calculated at constant entropy. Expanding the energy term in a power series gives

$$\left(\frac{\partial U}{\partial \eta_{pq}}\right)' = \frac{\partial U}{\partial \eta_{pq}} + \frac{\partial^2 U}{\partial \eta_{pq}\, \partial \eta_{rs}} \delta\eta_{rs} + \frac{1}{2} \frac{\partial^3 U}{\partial \eta_{pq}\, \partial \eta_{rs}\, \partial \eta_{lm}} \delta\eta_{rs}\, \delta\eta_{lm} + \cdots, \tag{29}$$

where the partial derivatives on the right-hand side of Eq. (29) are evaluated for state II, and where $\delta\eta_{ij}$ is given by Eq. (8). Inserting this result into Eq. (28), replacing x'_i by $x_i + u_i$, using Eq. (3), and neglecting all higher order terms (a process which by virtue of Secs. 2 and 3 is considered legitimate) gives eventually

$$\tau'_{ij} = \frac{\rho'}{\rho} \tau_{ij} + \tau_{is}u_{j,s} + \tau_{rj}u_{i,r} + \rho \frac{\partial^2 U}{\partial \eta_{pq}\, \partial \eta_{rs}} \left(\frac{\partial x_i}{\partial a_p} \frac{\partial x_j}{\partial a_q} \frac{\partial x_t}{\partial a_s} \frac{\partial x_m}{\partial a_r} u_{t,m}\right), \tag{30}$$

where a comma indicates partial differentiation with respect to x_i. Then using

$$\frac{\partial}{\partial x'_i} = [\delta_{si} - u_{s,i}] \frac{\partial}{\partial x_s}$$

(within higher order terms) the equation

$$\frac{\partial}{\partial x'_j}(\tau'_{ij}) + \rho' F_i = 0$$

becomes

$$\tau_{rj,j}u_{i,r} + \tau_{rj}u_{i,rj} + \left[\rho \frac{\partial^2 U}{\partial \eta_{pq}\,\partial \eta_{rs}} \frac{\partial x_i}{\partial a_p} \frac{\partial x_j}{\partial a_q} \frac{\partial x_t}{\partial a_s} \frac{\partial x_m}{\partial a_r} u_{t,m}\right]_{,j} = 0. \tag{31}$$

In addition to Eqs. (31), a boundary condition must be satisfied. For the case of dead loading, the condition is that

$$T'_i\, dS' = T_i\, dS$$

and using the fact that

$$n'_i\, dS' = (\rho/\rho') \frac{\partial x_r}{\partial x'_i} n_r\, dS$$

gives as this condition

$$\left(\tau_{pq}u_{r,p} + \rho \frac{\partial^2 U}{\partial \eta_{ij}\,\partial \eta_{pf}} \frac{\partial x_r}{\partial a_p} \frac{\partial x_s}{\partial a_i} \frac{\partial x_q}{\partial a_f} \frac{\partial x_l}{\partial a_j} u_{s,l}\right) n_q = 0. \tag{32}$$

Similarly, the boundary condition for that part of the surface where pressure forces act is

$$T'_i\, dS' = -Pn'_i\, dS',$$

whence it is found that Eq. (32) should be altered for this portion of the surface by adding to the left-hand side the term

$$P(\delta_{qr}u_{s,s} - u_{q,r})n_q\,. \tag{33}$$

9. Relation between the two methods. It has been remarked in Sec. 1 that the methods of Sec. 2 and Sec. 8 can at best be equivalent only for special situations, such as at points where an originally stable structure first becomes unstable. Consider therefore a structure which follows some stable equilibrium path as the load alters. The path will remain stable as long as the second order variation in potential energy is positive definite (vanishing only for zero displacement). Consequently, trouble can occur only at points where this second order variation [essentially the left-hand side of Eq. (10)] vanishes for non-zero displacements. Such a situation of neutral stability will in practice be followed by unstable equilibrium states as the load is further increased (see Poincaré [17]); we therefore investigate the condition under which the left-hand side of Eq. (10) first vanishes for non-zero displacements. Since it always vanishes for zero displacements, this condition is equivalent to requiring the minimum of Eq. (10) to be attained at non-zero u_i as well as at zero u_i, and this eigenvalue problem is that obtained by setting the first variation of Eq. (10) equal to zero.

The result of doing this is easily seen to be the same as Eq. (31), with the natural boundary condition (32). If pressure forces act on a portion of the surface, the result of a variation of Eq. (26) must be adjoined to the natural boundary conditions. This is not quite straightforward, because the condition that the pressure loading be conservative has not been explicitly stated (without such accessory conditions, the exact pressure work would depend on the path). The appropriate condition turns out to be that on the boundary of that portion of the surface where pressure forces act,

$$\int e_{rpq} u_p \, \delta u_q \, dx_r = 0 \tag{34}$$

for all variations δu_q . This condition would for example certainly be satisfied if the pressure surface completely enclosed the body, for then the path length would vanish. Alternatively the restraints on u_i may ensure satisfaction of Eq. (34)—as in the case of a hemispherical shell set on a rigid frictionless plane surface and subjected to external pressure.

Using Stokes' theorem on Eq. (34) gives

$$\int dS_P[(u_k \, \delta u_s)_{,s} - (u_s \, \delta u_k)_{,s}]n_k = 0$$

and if this is used in calculating the variation of Eq. (26), Eq. (33) is indeed obtained. It thus follows that at points where Eq. (10) first attains the minimum value of zero for non-zero u_i , an adjacent equilibrium position exists, and in this sense the two methods are equivalent.

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