parameters α from the minimum principle

$$\text{Min}_{\alpha_1,\cdots\alpha_N} \sum_{A=1}^{M} \left[\int \sigma_{ij}^{A} \epsilon_{ij} \, dV - \int u_i \sigma_{ij}^{A} n_j \, dS \right]^2. \tag{25}$$

Alternatively, the parameters α could be determined from a minimax principle, such as

$$\text{Min}_{\alpha_1,\cdots\alpha_N} \left\{ \text{Max}_{\beta_1,\cdots\beta_M} \left[\int \left(\sum_{A=1}^{M} \beta_A \sigma_{ij}^{A} \right) \epsilon_{ij} \, dV - \int u_i \left(\sum_{A=1}^{M} \beta_A \sigma_{ij}^{A} \right) n_j \, dS \right]^2 \right\}, \tag{26}$$

where the parameters β must satisfy $\sum_{A=1}^{M} \beta_A^2 = 1$.

A NOTE ON LAMINAR AXIALLY SYMMETRIC JETS*

By MARK BERAN (*Wellesley, Mass.*)

Summary. It is shown that there is no stream function of the form $\psi = rf(\theta)$, that is compatible with the complete Navier-Stokes equations, which represents a jet issuing from a small circular hole in an axially symmetric cone.

The asymptotic velocity field of a laminar viscous jet is generally accepted to have a stream function of the form $\psi = rf(\theta)$, corresponding to self-similar flow (Schlichting [1], Squire [2], and Yatseev [3]). The authors referred to have based their discussion on the fact that this assumption of self-similarity is compatible with both the boundary layer equations, and with the full Navier-Stokes equations.

The purpose of this note is to establish a serious shortcoming of such models. It is shown that there is no continuously differentiable velocity field associated with a stream function of the form $\psi = rf(\theta)$, which satisfies the Navier-Stokes equations and also adheres to a conical wall $\theta = \alpha > 0$.

Specifically, if $\psi = rf(\theta)$, then the velocity components in the r and θ directions are respectively [4]

$$u_r = \left[\frac{1}{r \sin \theta} \right] \frac{df}{d\theta}, \tag{1}$$

$$u_\theta = \left[\frac{-1}{r \sin \theta} \right] f. \tag{2}$$

The Navier-Stokes equations are equivalent to [5]

$$f^2 = 4\nu \cos \theta f - 2\nu \sin \theta \frac{df}{d\theta} - 2(c_1 \cos^2 \theta + c_2 \cos \theta + c_3) \tag{3}$$

for suitable constants c_1, c_2, c_3. We shall show that there is no solution of (3) which (*i*) makes u_r and u_θ continuous for $r > 0$, and (*ii*) satisfies $u_r(\alpha) = u_\theta(\alpha) = 0$, for $0 < \alpha \leq \pi$.

To show this, we also consider the differentiated form of (3), which is

$$\frac{-f}{\sin \theta} \frac{df}{d\theta} = 2f - 2 \sin \theta \frac{d}{d\theta} \left[\frac{1}{\sin \theta} \frac{df}{d\theta} \right] - (2c_1 \cos \theta + c_2). \tag{4}$$

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The boundary conditions implied by conditions (*i*) and (*ii*) are:

For Eq. (4): (a) $u_\theta = 0$, u_r finite, when $\theta = 0$.

(b) $u_r = u_\theta = 0$, when $\theta = \alpha$.

For Eq. (5): (c) $u_\theta = 0$, u_r finite, $\partial u_r/\partial\theta$ finite, when $\theta = 0$.*

These yield respectively the equations:

$$c_1 + c_2 + c_3 = 0, \tag{5}$$

$$c_1 \cos^2 \alpha + c_2 \cos \alpha + c_3 = 0, \tag{6}$$

$$2c_1 + c_2 = 0. \tag{7}$$

These have $c_1 = c_2 = c_3 = 0$ as their only solution.

As Squire [5] has shown, the general solution of Eq. (4) with $c_1 = c_2 = c_3 = 0$ is:

$$f = \frac{2\nu \sin^2 \theta}{a + 1 - \cos \theta}, \tag{8}$$

where a is an arbitrary constant.

Referring again to the boundary condition $u_r(\alpha) = u_\theta(\alpha) = 0$, we see that there is no finite value of $a(a = \infty$ yields a satisfactory but trivial solution) which satisfies this boundary condition, no matter what value of α is chosen.

Thus we have shown that there is no non-trivial solution of the form $\psi = rf(\theta)$ that is compatible with the Navier-Stokes equations and the boundary conditions (*i*) and (*ii*).

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*It is assumed that when these boundary conditions are substituted in Eqs. (4) and (5) that the limit $\theta \to 0$ is taken, since $\theta = 0$ is a singular point in the spherical polar coordinate system.

HEAT CONDUCTION IN SEMI-INFINITE SOLID IN CONTACT WITH LINEARLY INCREASING MASS OF FLUID*

By C. C. CHAO and J. H. WEINER (*Columbia University*)

Introduction. Problems of transient heat conduction in which the surface of a solid is in contact with a well-stirred fluid have been the subject of numerous investigations.[1] In all previous cases studied, the mass of the fluid has been considered constant. However, it is sometimes of interest to know the temperature in the solid and fluid

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[1]A review of previous work is found in [1], pp. 16-17.