Equation (3) states that the spatial rate of change of the distribution function $N(\mathbf{r}, \mathbf{s})$ at the position $\mathbf{r}$, in the direction of the unit vector $\mathbf{s}$, is due to three effects which appear on the right side of the equation. They are, in order of their appearance, first, radiation lost through absorption, and scattering out of the direction $\mathbf{s}$; secondly radiation which is scattered from all 4π solid angle into the direction $\mathbf{s}$. $p(\beta)$ is the angular differential cross section for the specific process involved. Thirdly there is the source contribution S. It is assumed in the above equation that a scatter is not accompanied by a change in wave length. μ is the total narrow beam absorption coefficient.

When the solution of (3) is through expansion processes, Eq. (2) becomes a useful integral formula for reducing the integral of (3) to its equivalent summation.

For instance, consider the problem which permits the unit vector $\mathbf{s}$ to be replaced by the ordinate angle θ (e.g., spherical symmetry and infinite plane source). In this case the distribution function may be expanded as

$$N(\mathbf{r}, \mathbf{s}') = \sum_l \frac{2l+1}{4\pi} a_l(\mathbf{r}) P_l(\cos\theta'). \tag{4}$$

Substitution of this expansion into the kernel of (3) will transform the original integral into a series of integrals, each term of which is similar to expression (2). From this we may write for the integral of (3),

$$\sum_l \frac{2l+1}{4\pi} a_l(\mathbf{r}) k_l P_l(\cos\theta), \tag{5}$$

k_l is the Legendre coefficient of the expansion of $p(\beta)$.

Following this with the replacement of $N(\mathbf{r}, \mathbf{s})$ by its equivalent expansion (4) into (3), will reduce the original integro-differential transport equation to a system of differential equations involving the sequence $\{a_l(\mathbf{r})\}$, knowledge of which completely determines $N(\mathbf{r}, \mathbf{s})$.

For more complicated geometries, where expansions in tesseral, or spherical surface harmonics are called for, the formula may be used in like manner, reducing the integral term of the transport equation to its corresponding summation, whence usually, a reduced system of equations is easily derivable.

TWO REMARKS ON HEISENBERG'S THEORY OF ISOTROPIC TURBULENCE*

By WILLIAM H. REID[1] (*Trinity College, Cambridge, England*)

1. Introduction. The exact dynamical equation for the rate of change with time of the energy spectrum function $E(k)$ in isotropic turbulence may be written in the form

$$\partial E(k)/\partial t = T(k) - 2\nu k^2 E(k), \tag{1}$$

where $T(k)$ is the transfer function usually denoted by this symbol. The incompleteness of Eq. (1) is well known and, in the past, several so-called "physical transfer theories" have been proposed in which a further relationship between $T(k)$ and $E(k)$ is postulated;

*Received May 9, 1955.

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the specific form of the relationship then follows from the particular mechanism of energy-transfer considered and from certain general dimensional considerations.

The theory which has attracted the greatest attention is the one due to Heisenberg [3] and is based on the concept of an eddy viscosity; in this theory $T(k)$ and $E(k)$ are related by an equation which may be written in the form

$$T(k) = -2\kappa \frac{d}{dk} \int_k^\infty [E(k')/k'^3]^{1/2} \, dk' \int_0^k k''^2 E(k'') \, dk''. \tag{2}$$

In this equation, κ is, by hypothesis, a constant of order unity. A number of writers have attempted to derive the value of κ under widely differing conditions and, in particular, existing results show a large variation of κ with the Reynolds number of the turbulence.

One of the purposes of the present note, therefore, is to show that these results are partially in error and that in fact κ, or more precisely the ratio S/κ, where S is the skewness factor of $-\partial u_1/\partial x_1$ is practically independent of the Reynolds number and thus to remove one cause for criticism of Heisenberg's theory.

2. The behavior of S/κ for small values of the Reynolds number. Since S is directly related to the second moment of the transfer function, one must first derive the explicit solution for $T(k)$. For small values of the Reynolds number, this can easily be done by expressing the solution as a power series in the Reynolds number. Thus, by assuming the usual type of initial period similarity

$$E(k, t) = \kappa^{-2}\nu^{3/2}t^{-1/2}F(x) \quad \text{and} \quad T(k, t) = \kappa^{-2}\nu^{3/2}t^{-3/2}U(x), \tag{3}$$

where $x = (\nu k^2 t)^{1/2}$, Eqs. (1) and (2) become

$$xF'(x) + (4x^2 - 1)F(x) = 2U(x) \tag{4}$$

and

$$U(x) = -2 \frac{d}{dx} \int_x^\infty [F(x')/x'^3]^{1/2} \, dx' \int_0^x x''^2 F(x'') \, dx''. \tag{5}$$

From Eq. (5) it is clear that for small values of the Reynolds number $U(x)$ is of higher order that $F(x)$ and since R_λ, the Reynolds number of the turbulence usually denoted by this symbol, always occurs in the combination κR_λ, we may write

$$F(x) = \sum_{n=2}^\infty F_n(x)(\kappa R_\lambda)^n \quad \text{and} \quad U(x) = \sum_{n=3}^\infty U_n(x)(\kappa R_\lambda)^n. \tag{6}$$

By substituting these series into Eqs. (4) and (5) and equating like powers of κR_λ, one obtains a sequence of equations for the determination of the functions $F_n(x)$ and $U_n(x)$. In particular, the first approximation to each of these functions can be readily obtained in the form

$$F_2(x) = \tfrac{3}{5} x e^{-2x^2} \tag{7}$$

and

$$U_3(x) = \frac{1}{8}\left(\frac{3}{5}\right)^{3/2} \frac{d}{dx} \,\mathrm{Ei}\,(-x^2)[1 - (1 + 2x^2)e^{-2x^2}], \tag{8}$$

where $-\mathrm{Ei}(-x)$ is the exponential integral function defined by

$$-\mathrm{Ei}\,(-x) = \int_x^\infty \frac{e^{-t}}{t} \, dt.$$

These functions are shown graphically in Fig. 1. In passing, it may be noted that $F_2(x)$, unlike $U_3(x)$, is independent of the assumed form of the transfer theory.

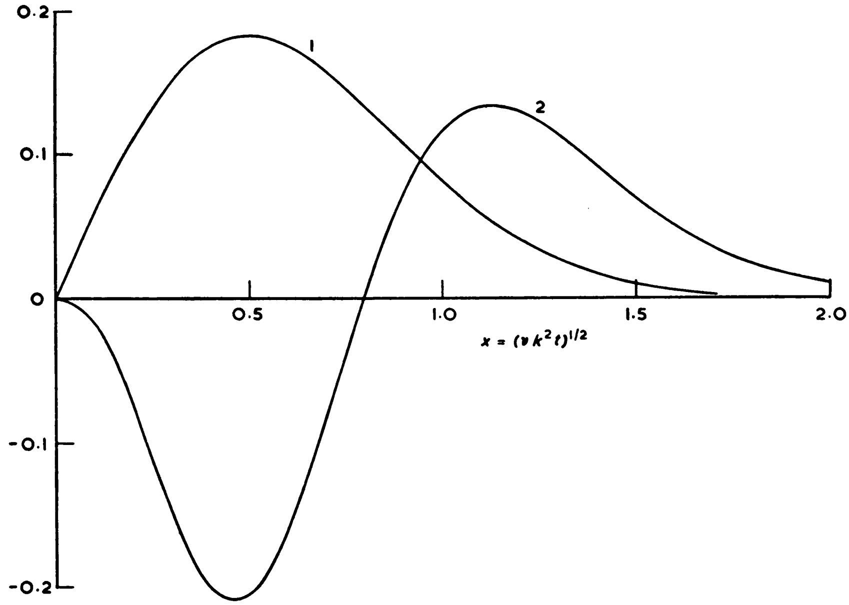

FIG. 1. Curve 1: The 'zero Reynolds number' energy spectrum, $F_2(x)$; independent of transfer theory. Curve 2: The first approximation to the transfer function, $10U_3(x)$; Heisenberg's transfer theory.

For small values of the Reynolds number, the ratio S/κ is then given by

$$\frac{S}{\kappa} = \frac{400}{7} \int_0^\infty x^2 U_3(x)\, dx + O(\kappa R_\lambda). \tag{9}$$

For the transfer function (8), the integral occurring in this equation has the value

$$\int_0^\infty x^2 U_3(x)\, dx = \frac{3(15)^{1/2}}{200} \left(\frac{4}{3} - \log 3\right)$$

and hence

$$S/\kappa \to 0.78 \qquad \text{as} \qquad R_\lambda \to 0. \tag{10}$$

Experimentally, we know that $S \doteqdot 0.48$ at the smallest Reynolds number for which experiments have been made [2, Fig. 6.3] and this makes $\kappa \doteqdot 0.62$. The numerical coefficient in Eq. (10) differs considerably, however, from the one given by Proudman [6]. Presumably, his result was obtained by expanding $U_3(x)$ in powers of x; in view of the somewhat peculiar behaviour of this function, the present method of using the exact form for $U_3(x)$ in closed form is obviously more reliable.

3. The behavior of S/κ for large values of the Reynolds number. For sufficiently large Reynolds numbers for which a universal equilibrium exists, the relation

$$S = \frac{3}{7}(30)^{1/2}\nu \frac{\int_0^\infty k^4 E(k)\, dk}{\left[\int_0^\infty k^2 E(k)\, dk\right]^{3/2}} \tag{11}$$

is exact; under these same conditions, one may therefore use Bass' equilibrium spectrum to evaluate the behaviour of the two integrals occurring in this equation. This is the procedure used by Lee [5] who found that, as R_λ *approaches* infinity,

$$S/\kappa \to 1.52. \tag{12}$$

In the course of the present investigation, however, it was found that when R_λ *equals* infinity, the behaviour of S/κ is singular with a value which differs from the one given by Eq. (12) and this fact, in itself, may not be without interest.

Thus, when R_λ *equals* infinity, the ratio S/κ can be determined by comparing the asymptotic form of the spectrum given by Kolmogoroff's universal equilibrium theory with the corresponding form given by Heisenberg's theory. For sufficiently small values of r, Kolmogoroff's prediction for the double correlation function (see, for example, [1, p. 85])

$$2\langle u^2\rangle[1 - f(r)] = (4/5S)^{2/3}(\epsilon r)^{2/3}, \tag{13}$$

where $\langle u^2\rangle$ is the mean square value of one component of the velocity, ϵ is the rate of viscous dissipation and $f(r)$ is the double velocity correlation coefficient usually denoted by this symbol, leads to the equilibrium spectrum

$$E(k) = \frac{55}{81}\cdot\frac{1}{(1/3)!}\left(\frac{4}{5S}\right)^{2/3}\epsilon^{2/3}k^{-5/3}, \tag{14}$$

and this is to be compared with Heisenberg's form of the equilibrium spectrum

$$E(k) = (8/9\kappa)^{2/3}\epsilon^{2/3}k^{-5/3}. \tag{15}$$

By equating the coefficient of $\epsilon^{2/3}k^{-5/3}$ in these two expressions, one obtains

$$\frac{S}{\kappa} = \frac{1}{810}\left[\frac{55}{(1/3)!}\right]^{3/2} \doteqdot 0.60 \tag{16}$$

and this then is the value of S/κ when R_λ *equals* infinity.

From a physical point of view, the correct limit is of course the one given by Eq. (12), and when this value is used in conjunction with the experimentally determined value of S for large values of the Reynolds number, about 0.30 [1, Fig. 6.3], we obtain the value $\kappa \doteqdot 0.20$.

4. Conclusions. The implication of this discussion then is that the ratio S/κ varies between 0.78 and 1.52 as the Reynolds number varies from zero to infinity and, from the nature of the theory, it is reasonable to suppose this variation to be monotonic. Furthermore, when the experimentally observed variation of S with the Reynolds number is taken into account, it is found that κ then varies between about 0.62 and 0.20, which neatly bracket the value (0.45 ± 0.05) suggested by Proudman [6] from his study of the correlation functions. It thus appears that the variation of κ with the Reynolds number is relatively small and cannot be said to constitute a serious criticism of Heisenberg's transfer theory.

Acknowledgement. I am indebted to Dr. G. K. Batchelor for his helpful discussion of the present work.

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NOTE ON LINEAR PROGRAMMING*

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Statement of theorem. Consider a set of m linear equations in n unknowns x_i ;

$$a_{\alpha i} x_i = b_\alpha , \tag{1}$$

where Greek indices range from 1 to m and Latin from 1 to n. The usual summation convention on repeated indices is used; $a_{\alpha i}$ and b_α are constants. Linear programming is a method of obtaining a solution (if it exists) of Eq. (1) satisfying in addition the requirements

$$x_i \geq 0, \qquad \text{all } i, \tag{2}$$

$$c_i x_i = \text{minimum}, \tag{3}$$

where the c_i are constants. The fundamental theorem used in the simplex method of Dantzig (1) is that if one solution exists, then an equivalent solution can be found in which not more than m of the x_i are non-zero; further, those columns of the matrix $(a_{\alpha i})$ which correspond to such non-zero (x_i) will be linearly independent. The usual proof of this theorem (e.g. Ref. (2)) involves tedious geometrical considerations in n-dimensional space; it therefore seems worth-while to point out that a simple direct proof exists.

Proof of theorem. Suppose (x_i') satisfies conditions (1), (2), (3). Some—perhaps all—of these (x_i') will be non-zero; say for example that x_2' , x_3' , x_6' , are alone not zero. If firstly the corresponding columns $(a_{\alpha 2}$, $a_{\alpha 3}$, $a_{\alpha 6})$ were linearly dependent, then a set of three constants K_2 , K_3 , K_6 (not all zero) would exist such that

$$A(K_2 a_{\alpha 2} + K_3 a_{\alpha 3} + K_6 a_{\alpha 6}) = 0, \qquad \text{all } \alpha \tag{4}$$

for any arbitrary constant A. Then because of Eq. (4), the new set (x_i'') defined by

$$\begin{aligned} x_i'' &= x_i' - AK_i \qquad \text{for } i = 2, 3, 6, \\ x_i'' &= 0 \qquad\qquad\quad \text{for other } i \end{aligned} \tag{5}$$

satisfies Eq. (1) and, for sufficiently small A, also Eq. (2). Clearly however $c_i x_i'' < c_i x_i'$ for appropriate A, unless

$$c_2 K_2 + c_3 K_3 + c_6 K_6 = 0. \tag{6}$$

*Received June 4, 1955.