EXPRESSION OF WAVE FUNCTIONS OVER A HALF SPACE IN TERMS OF THEIR BOUNDARY VALUES*

BY

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1. Introduction. We consider the expression of a solution u of the wave equation

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{1.1}$$

over the half space $z > 0$, in terms of its boundary values over the boundary $z = 0$:

$$u(P, t) = u(x, y, z, t) = \frac{1}{2\pi}\left[\int [u]\, d\omega + \frac{1}{c}\left[\frac{\partial u}{\partial t}\right] R\, d\omega\right], \tag{1.2}$$

and the alternative expression of u over $z > 0$ in terms of the boundary values of $(\partial u/\partial z)$ given by:

$$u = -\frac{1}{2\pi}\int\left[\frac{\partial u}{\partial z}\right]\frac{dx'\,dy'}{R}. \tag{1.3}$$

Here the integration is carried out over the plane $z = 0$, $d\omega$ denoting the element of solid angle subtended at P by the plane element $dx'dy'$ at P', while

$$[u],\ [\partial u/\partial t],\ [\partial u/\partial z] \tag{1.4}$$

denote the retarded values of these functions at P', that is their values at P' not at the time t, but at the time $t' = t - R/c$, where R is the distance PP'. The "retarded time"

$$t' = t - R/c \tag{1.5}$$

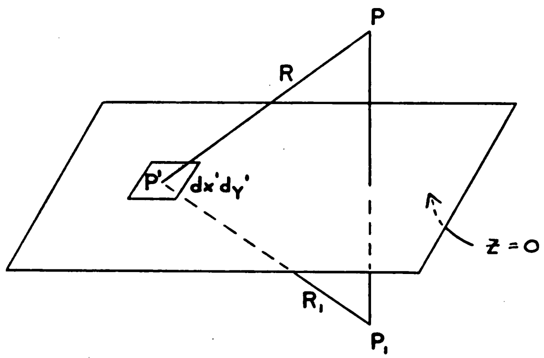

FIG. 1.

is the time t' such that a spherical wavelet, starting from P' at t' and with a radius expanding with a velocity c, would reach the point P at the time t, at which the left member of Eq. (1.2) is evaluated.

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In the version of this paper first submitted to this journal, the authors derived several proofs of these equations, one based on Fourier integral time-resolution, another on a slight modification of the method used by Kirchhoff in expressing solutions of the non-homogeneous wave equation

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -4\pi\rho(x, y, z, t) \tag{1.6}$$

in terms of the "source" distribution ρ, as a sum of the retarded potentials of the latter, as follows (see for instance [1]):

$$u = \int \frac{[\rho]}{R}\, dv. \tag{1.7}$$

After this paper was revised in accordance with the recommendations of the referee, it was pointed out to the authors that Eq. (1.2), (1.3) appear in reference [2], where a hint regarding their proof is also given. It, therefore, seemed pertinent to omit proofs of (1.2), (1.3), and confine ourselves to several applications of these equations and their n-dimensional counterparts.

The boundary value expressions under discussion, it should be pointed out, differ from the more commonly considered solution of the Cauchy problem for (1.1), in which u is expressed for $t > 0$ in terms of the values of u, $\partial u/\partial t$ at the time $t = 0$.

It is to be emphasized that the expressions S_1 of the wave functions in terms of their boundary values considered here are unique *only* if, as is customary in physical applications, the *retarded* potentials are used. A similar, but *different* expression S_2, utilizing the *advanced* potentials also exists. Thus a solution of (1.6) exists of the form (1.7) but with $[\rho]/R$ replaced by the *advanced potential* $\rho(t + R/c)/R$. A linear combination of these solutions of the form $\lambda S_1 + (1 - \lambda)S_2$ for any constant λ also furnishes a possible solution.

In the special case where u is independent of time, Eq. (1.1) reduces to the Laplace equation, and Eqs. (1.2), (1.3) simplify to

$$u(P, t) = \frac{1}{2\pi}\int u\, d\omega, \tag{1.8}$$

$$u(P, t) = -\frac{1}{2\pi}\int \frac{\partial u}{\partial z}\frac{dx'\, dy'}{R}. \tag{1.9}$$

Both of these equations are familiar from potential theory.

2. Examples. A. *Axially symmetric product solutions of the wave equation.* As a first example we consider the axially symmetric wave function

$$u = J_0(\lambda r) \exp[-(\gamma^2 - k^2)^{1/2} z] \exp(i\alpha t), \qquad r^2 = x^2 + y^2, \tag{2.1}$$

for $\gamma > k$. Applying Eq. (1.3) for a point P on the positive z-axis, there results, upon introduction of polar coordinates in the plane of integration and cancellation of the factor $\exp(i\alpha t)$,

$$\frac{\exp[-(\gamma^2 - k^2)^{1/2} z]}{(\gamma^2 - k^2)^{1/2}} = \int_0^\infty \frac{\exp(-ikR) J_0(\gamma r) r\, dr}{R}. \tag{2.2}$$

Since (see Fig. 2)

$$R^2 = r^2 + z^2, \qquad R\, dR = r\, dr, \tag{2.3}$$

one obtains

$$\frac{\exp[-(\gamma^2 - k^2)^{1/2}z]}{(\gamma^2 - k^2)^{1/2}} = \int_z^\infty \exp(ikR) J_0[\gamma(R^2 - z^2)^{1/2}]\, dR. \tag{2.4}$$

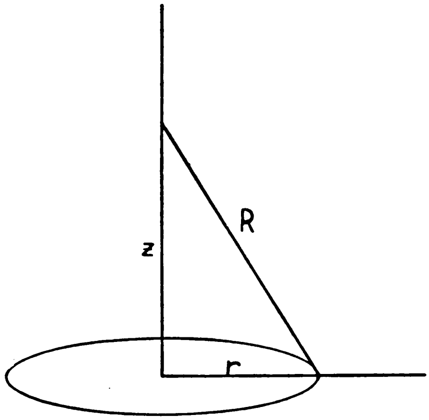

FIG. 2.

In a similar way one obtains for the same wave function (2.1), by applying either Eq. (1.2), the following integral

$$\begin{aligned} \exp[-(\gamma^2 - k^2)^{1/2}z] &= \int_0^\infty J_0(\gamma r) \exp(-ikR)(1 + ikR)zrR^{-3}\, dr \\ &= z \int_0^\infty \exp(-ikR)(1 + ikR) J_0[\gamma(R^2 - z^2)^{1/2}]\, dR/R^2. \end{aligned} \tag{2.5}$$

The integral (2.2) is essentially equivalent to (see Sommerfeld in [3], Watson [4])

$$\frac{\exp[ik(\gamma^2 + z^2)^{1/2}]}{(\gamma^2 + z^2)^{1/2}} = \int_0^\infty \lambda J_0(\lambda\gamma) \exp[-(\lambda^2 - k^2)^{1/2}z](\lambda^2 - k^2)^{-1/2}\, d\lambda. \tag{2.6}$$

Indeed, by carrying out the following substitutions in (2.6):

$$ik \to -z, \qquad \lambda \to r, \qquad r \to \gamma, \qquad z \to ik, \tag{2.7}$$

one transforms (2.6) into (2.5).

Further integrals can be obtained from the wave function (2.1) by choosing P in Eqs. (1.2), (1.3) not on the axis of symmetry. These integrals remain two-dimensional since the integrand is not axially symmetric about a line through P normal to the plane $z = 0$. However, by the application of the Neuman expansion the integral can be reduced to the axially symmetric case.

B. General axially symmetric wave functions. As our second example we consider any wave function $u(r, z, t)$ in $z > 0$ which is axially symmetric about the z-axis, that is, a solution of

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{2.8}$$

Eq. (1.3) yields, upon replacing r by $(R^2 - z^2)^{1/2}$, the following expression of u for a point on the positive z-axis in terms of $u_z = \partial u/\partial z$ along $z = 0$:

$$u(0, z, t) = -\int_z^\infty u_z[(R^2 - z^2)^{1/2}, 0, t - R/c]\, dR. \tag{2.9}$$

Eq. (1.2) yields similarly

$$\begin{aligned} u(0, z, t) = z \int_z^\infty u[(R^2 - z^2)^{1/2}, 0, t - R/c]R^{-1}\, dR \\ + (z/c) \int_z^\infty u[(R^2 - z^2)^{1/2}, 0, t - R/c]\, dR. \end{aligned} \tag{2.10}$$

C. Plane waves. As a third example consider an arbitrary *plane wave* solution of the form

$$u = f(t - z/c) \tag{2.11}$$

where f satisfies the conditions

$$\begin{aligned} f(s) &= 0 \quad \text{for} \quad s < 0, \\ f'(0) &= 0. \end{aligned} \tag{2.12}$$

Since the field is plane, it can be considered to be symmetric about the z-axis. Application of Eq. (1.3) for

$$t - z/c > 0, \qquad z > 0 \tag{2.13}$$

yields, upon introducing polar coordinates as in examples A and B,

$$\begin{aligned} f(t - z/c) &= -(1/c) \int_0^{r_1} f'(t - R/c)rR^{-1}\, dr \\ &= -(1/c) \int_z^{ct} f'(t - R/c)\, dR, \end{aligned} \tag{2.14}$$

where

$$r_1 = [(ct)^2 - z^2]^{1/2}.$$

That this is an identity follows immediately upon carrying out the integration.

Similar application of Eq. (1.2) to the plane wave function (2.11) satisfying (2.12) yields for z, t satisfying (2.13)

$$\begin{aligned} f(t - z/c) &= z \int_0^{r_1} [f(t - R/c)R^{-3} + f'(t - R/c)c^{-1}R^{-2}]r\, dr, \\ &= z \int_z^{ct} f(t - R/c)R^{-2}\, dR + zc^{-1} \int_0^{ct} f'(t - R/c)R^{-1}\, dR, \end{aligned} \tag{2.15}$$

where r_1 is as in (2.14). That this again is an identity follows by integrating the last integral by parts and utilizing (2.12).

It will be noted that the wave function (2.11) does *not* include a simple standing plane wave such as

$$u = \sin \alpha t \sin kz. \tag{2.16}$$

Indeed for this example the right-hand member of Eq. (1.2) vanishes and Eq. (1.2) obviously fails to hold. The function (2.16), if resolved into travelling waves, includes a plane wave travelling in direction of *negative z*. The reader will discover in supplying the proof of theorem (1.2) that the integrals over $\sum$, a hemisphere of large radius R_Σ, which must converge to zero as $R_\Sigma \to \infty$ to yield (1.2), fail to do so for the plane wave travelling in the direction of negative z.

D. Spherical wave. As our next example we consider a spherical wave originating from a point P_2, $z = -a$, $a > 0$, on the negative z-axis (see Fig. 3):

$$u = \frac{f(t - R_2/c)}{R_2}, \tag{2.17}$$

where R_2 is the distance from P_2. Again we assume that f satisfies the conditions (2.12).

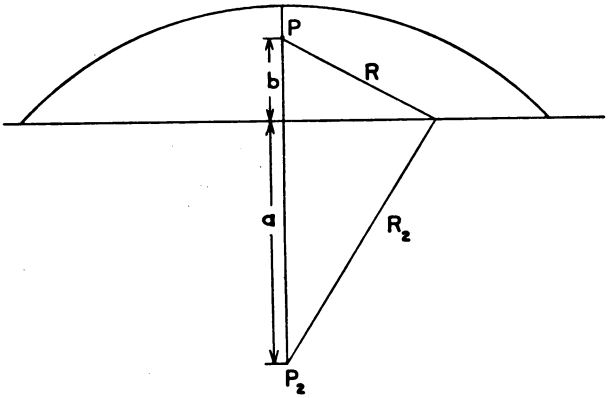

FIG. 3.

Eq. (1.2) now yields for a point P at $z = b$ on the positive z-axis for

$$t - (a + b)/c > 0, \tag{2.18}$$

upon recalling Eq. (2.10), the following relation

$$\frac{f[t - (a + b)/c]}{(a + b)} = b \int_b^{R'} \frac{f[t - (R_2 + R)/c]\, dR}{R_2 R^2} - \frac{b}{c} \int_b^{R'} \frac{f'[t - (R_2 + R)/c]\, dR}{R_2 R}. \tag{2.19}$$

The upper limit R' of the integrations is given by

$$R' + R_2 = ct. \tag{2.20}$$

Upon utilizing the relations

$$R^2 = b^2 + r^2, \qquad R_2^2 = a^2 + r^2 \tag{2.21}$$

one obtains from (2.19)

$$\frac{f[t - (a + b)/c]}{(a + b)} = b \int_a^{R_2'} f[t - (R_2 + R)/c][R_2^2 - (a^2 + b^2)]^{-3/2}\, dR_2 - (b/c) \int_a^{R_2'} f'[t - (R_2 + R)/c][R_2^2 - (a^2 + b^2)]\, dR_2, \tag{2.22}$$

where

$$R_2' = (a^2 - b^2)/2ct + ct/2. \tag{2.23}$$

That (2.22) is an identity follows by integrating the second integral by parts and utilizing the initial conditions (2.12).

3. Extension to n-dimensions. We now consider the extension of the above results to n-dimensional (Euclidean) spaces E_n and to solutions of the "wave equation"

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{3.1}$$

The aim is to express u over the region $x_n > 0$ in terms of its boundary values over $x_n = 0$.

The situation is quite different depending upon whether n is odd or even. For *odd* n an extension of Eq. (1.2) *is* possible and $u(P, t)$ can be expressed in terms of retarded values of u, $\partial u/\partial t$, $\cdots$, $\partial^{(n+1)/2} u/\partial t^{(n+1)/2}$ along the flat $x_n = 0$ as follows:

$$u(P, t) = \frac{2}{\Omega_n}\int \sum_{m=0}^{(n-1)/2} \frac{C_{nm}R^m}{c^m}\left[\frac{\partial^m u}{\partial t^m}\right] d\omega_n . \tag{3.2}$$

Here C_{nm} ; $m = 0, 1, \cdots$, are proper constants, the integration is carried out along $x_n = 0$, $d\omega_n$ denoting the element of "solid angle" subtended by $dx_1 \cdots dx_{n-1}$ at P' in $x_n = 0$ at the point P:

$$d\omega_n = \frac{x_n}{R^n}\, dx_1' \cdots dx_{n-1}' , \tag{3.3}$$

while $[u]$, $[\partial u/\partial t]$, $\cdots$, as for $n = 3$, are the values of u at P' at the time $t - R/c$, where $R = PP'$; finally

$$\Omega_n = \int d\omega_n = n[\Gamma(1/2)]^n/\Gamma[(n/2) + 1] \tag{3.4}$$

is the complete solid angle, that is the $(n - 1)$-dimensional content of the "surface" of the n-dimensional unit sphere:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1. \tag{3.5}$$

Similarly, for odd n, a generalization of Eq. (1.3) is possible, and is given by

$$u(P, t) = \frac{2}{\Omega_n}\int \cdots \int \sum_{m=0}^{(n-3)/2} \frac{R^m D_{nm}}{c^m}\left[\frac{\partial^{m+1} u}{\partial x_n' \, \partial t^m}\right] dx_1' \cdots dx_{n-1}' , \tag{3.6}$$

where again D_{nm} are proper constants.

For *even* n the situation is quite different, and the expression of u in terms of its boundary values is more complicated. This is but one instance of the many differences between solutions of the wave equation in even and odd number of dimensions. In the present case, one may ascribe this difference to the fact that the Bessel functions which are involved in the proof of Eqs. (3.2) or (3.6) turn out to be of integral order for even n, but of an order that differs from an integer by a half for odd n. The latter Bessel functions, as is well known, can be expressed in terms of exponentials and a product of a finite number of negative fractional powers of the argument; no such simple expression, however, is possible for the Bessel functions of integral order.

The expression of u in terms of its boundary values for even n is considered briefly toward the end of this section, and in more detail in the following section.

After these preliminaries we indicate briefly the generalizations of (1.2), (1.3) for any number of dimensions n by a method similar to the Fourier integral method mentioned in Sec. 1. We resolve u, the solution of (3.1), into a Fourier integral in time:

$$u = \int_{-\infty}^{\infty} \exp(i\alpha t) U(x_1, x_2, \cdots, x_n)\, d\alpha, \tag{3.7}$$

where (it is assumed that) the integrand satisfies Eq. (3.1), and hence U is a solution of the n-dimensional equation

$$\nabla^2 U + k^2 U = 0, \qquad k = \alpha/c. \tag{3.8}$$

We apply Green's theorem to U and the Green's function G for the half space $x_n > 0$ and the condition $U = 0$ on $x_n = 0$:

$$G(P, P') = f_n(k, R) - f_n(k, R_1), \tag{3.9}$$

where R is the distance from P, R_1 the distance from its mirror image in $x_n = 0$, while $f_n(k, R)$ is a proper, spherically symmetric solution of (3.8), which behaves like

$$\text{Const.} \exp(ikR)/R^{(n-1)/2} \tag{3.10}$$

at infinity and like

$$\begin{cases} R^{2-n}/(n-2) & \text{for} \quad n > 2, \\ -\ln R & \text{for} \quad n = 2, \end{cases} \tag{3.11}$$

near $R = 0$. There results

$$U(P) = \frac{2}{\Omega_n} \int \cdots \int x_n U(x_1, x_2, \cdots, x_{n-1}, 0) \{\partial f_n(k, R)/R\, \partial R\}\, dx_1' \cdots dx_{n-1}', \tag{3.12}$$

and, upon utilization of Eq. (2.3),

$$U(P) = \frac{2}{\Omega_n} \int \cdots \int U(x_1, x_2, \cdots, x_{n-1}, 0) R^{n-1}\, \partial f_n(k, R)/\partial R\, d\omega_n . \tag{3.13}$$

In a similar way, by replacing G by the sum of the two right-hand terms in Eq. (3.9), one obtains

$$U(P) = 2 \int \cdots \int U_{x_n}(x_1, x_2, \cdots, x_{n-1}, 0) f_n(k, R)\, dx_1' \cdots dx_{n-1}' . \tag{3.14}$$

To obtain f_n we note that for spherically symmetric solutions Eq. (3.8) reduces to

$$\frac{\partial^2 U}{\partial R^2} + \frac{n-1}{R} \frac{\partial U}{\partial R} + k^2 U = 0. \tag{3.15}$$

Solutions of this equation are given by

$$U = R^{1-n/2} B_{n/2-1}(kR) \tag{3.16}$$

where B_ν is a Bessel function of order ν. We shall choose

$$f_n = \text{Const}\, R^{1-n/2} H^{(2)}_{n/2-1}(kR), \tag{3.17}$$

since the asymptotic form of this solution yields a behavior at infinity of the form (3.10) which is more in accord with a "retarded potential" solution corresponding to an expanding spherical wave, than would be the case with any other Bessel function.

The difference between even and odd n appears when we examine the expansions for $H_{n/2-1}$ (see [3], p. 198, Eq. (6)):

$$H_\nu^{(2)}(z) \cong \left(\frac{2}{\pi z}\right)^{1/2} \exp\,[-i(z - \nu\pi/2 - \pi/4)] \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2iz)^m}, \qquad (\nu, m) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2) \cdots [4\nu^2 - (2m-1)^2]}{2^{2m} m!}. \tag{3.18}$$

For general ν these expansions are asymptotic, and divergent. However, if $\nu = n/2 - 1$ where n is an odd integer, then $(\nu, m) = 0$ for $m \geqq (n-1)/2$, the series in (3.18) reduces to $(n-1)/2$ terms, and yields an exact representation for $H_\nu^{(2)}$.

Hence for odd n, one obtains from Eqs. (3.16), (3.17), upon adjusting the multiplicative constant in accordance with Eq. (3.11),

$$f_n(k, R) = \frac{R^{2-n} \exp(-ikR)}{(n-2)[(n/2) - 1, (n-3)/2]} \sum_{m=0}^{(n-3)/2} ((n/2) - 1, m)(2ikR)^{(n-3)/2-m} = \exp(ikR)\, R^{2-n} \sum_{m=0}^{(n-3)/2} D_{nm}(ikR)^m. \tag{3.19}$$

Substituting this expansion in Eq. (3.13) and recalling the factor exp $(i\alpha t)$ one obtains

$$U(P) \exp(i\alpha t) = -\frac{2}{\Omega_n} \int \cdots \int U_{x_n}(x_1', x_2', \cdots, x_{n-1}', 0) \sum_{m=0}^{(n-3)/2} \exp\,[i(\alpha t - kR)] D_{nm}(ikR)^m. \tag{3.20}$$

Now replace the left-hand member by $u(P, t)$; note that multiplication by the factor ik is equivalent to application of the operator $(1/c)(\partial/\partial t)$, and that t occurs only in the form $\exp\,[i\alpha(t - R/c)]$. Hence for functions u of the form $\exp(i\alpha t)\; U(x_1 \cdots x_n)$ Eq. (3.20) may be recast in the form (3.6).

For general wave functions, Eq. (3.6) now follows by means of Fourier integral superposition, as in Eq. (3.7).

A similar proof of Eq. (3.2) follows from Eq. (3.12). By differentiation of (3.19) one obtains

$$\frac{\partial f_n}{\partial R} = \exp(ik\,R)\, R^{1-n} \sum_{m=0}^{(n-1)/2} C_{nm}(ikR)^m, \tag{3.21}$$

where

$$C_{n0} = (2 - n)D_{n0}, \qquad C_{nm} = ikD_{n,m-1} + (m - n)D_{n,m}, \quad m > 0. \tag{3.22}$$

The case $n = 3$ has been considered in Sec. 1. For this case the series in Eq. (3.18) reduces to its first term, unity, and Eq. (3.19) yields

$$f_3 = \exp(-ikR)/R. \tag{3.23}$$

For $n = 5, 7$ one obtains

$$f_5(k, R) = \frac{\exp(ikR)}{3R^3}[1 + ikR], \tag{3.24}$$

$$f_7(k, r) = \frac{\exp(-ikR)}{5R^5}\left[1 + \frac{ikR}{2} + \frac{(ikR)^2}{6}\right]. \tag{3.25}$$

It is of interest to note that for *odd* n the functions (3.19), multiplied by the factor $\exp(i\alpha t)$, can be written in the form

$$u_n(R, t) = R^{2-n} \sum_{m=0}^{(n-3)/2} \frac{R^m D_{nm}}{c^m} \frac{\partial^m}{\partial t^m} F\left(t - \frac{R}{c}\right) \tag{3.26}$$

where

$$F(u) = \exp(i\alpha u). \tag{3.27}$$

Hence, by superposition over α, it follows that (3.26) is a solution of the wave equation (3.1) for an *arbitrary* function F. This is the general expanding spherical wave, and is analogous to $F(t - R/c)/R$ for $n = 3$, to which, indeed, it reduces for $n = 3$.

Returning to the case of even n, now one can no longer utilize the asymptotic Eq. (3.18) to advantage for the integral order Bessel functions. Equations (3.17), (3.11) yield (for both even and odd n)

$$f_n = \begin{cases} \pi i H_0(\alpha R/c) & \text{for } n = 2, \\ \dfrac{\pi i}{(n-2)\left(\dfrac{n}{2} - 2\right)!}\left(\dfrac{\alpha}{2c}\right)^{(n/2-1)} R^{1-n/2} H_{(n/2-1)}(\alpha R/c) & \text{for } n > 2. \end{cases} \tag{3.28}$$

Again, Eqs. (3.13), (3.14) can be used, combined with Fourier α-integration as in Eq. (3.7), to yield the desired expression of u in terms of its boundary values, but the result is rather complex. More attractive forms are considered in the following section.

4. Alternative methods in n dimensions. With the odd-dimensional case disposed of, one may treat the case of even n by immersing the space E_n of $(x_1, x_2, \cdots, x_n)$ in an odd-dimension space E_{n+1} obtained by adding one coordinate x_{n+1} to E_n, and regarding the solution u of Eq. (3.1) as a solution in E_{n+1} of the corresponding $(n + 1)$-dimensional wave equation. This is sometimes known as the "principle of descent" (see [5]).

As a first application of this "principle" we shall obtain for even n a general expression for $u_n(R, t)$, and expanding spherical wave. This will be done by immersing E_n in E_{n+1}, lining the x_{n+1}-axis with a uniform distribution per unit x_{n+1} of $(n + 1)$-dimensional point "sources", and adding the resulting wave functions given by Eq. (3.26) with n replaced by $(n + 1)$, for the elements of the wave components. Thus, for $n = 2$, we immerse the (x, y)-plane in an (x, y, z)-space, and obtain a wave of cylindrical symmetry by adding spherical wave elements originating from sources distributed uniformly over the z-axis.

For general even n the resulting wave function is independent of x_{n+1} and can be evaluated in the flat $x_{n+1} = 0$. There results

$$u_n(R, t) = 2\int_{-\infty}^{+\infty} \sum_{m=0}^{n/2-1} (R'')^{1+m-n} D_{n+1,m} c^{-m} F^{(m)}(t - R''/c)\, dx_{n+1}, \tag{4.1}$$

where

$$R'' = [R^2 + (x_{n+1})^2]^{1/2}. \tag{4.2}$$

Changing to $s = R''$ as a variable of integration, one obtains

$$u_n(R, t) = 2 \int_R^\infty \sum_m \frac{s^{2+m-n} D_{n+1,m}}{c^m (s^2 - R^2)^{1/2}} F^{(m)}(t - s/c)\, ds; \tag{4.3}$$

while introduction of

$$x_{n+1} = R \sinh v, \qquad s = R'' = R \cosh v, \tag{4.4}$$

yields

$$u_n(R, t) = \int_{-\infty}^{+\infty} \sum_m D_{n+1,m} (R \cosh v)^{2+m-n} c^{-m} F^{(m)}[t - (R/c) \cosh v]\, dv. \tag{4.5}$$

For $n = 2$, Fig. 4 shows in perspective the (x, y)-plane, its "immersion" in the 3-space E_3, and the various distance R, R'', $dx_3 = dz$ involved in Eqs. (4.1), (4.2).

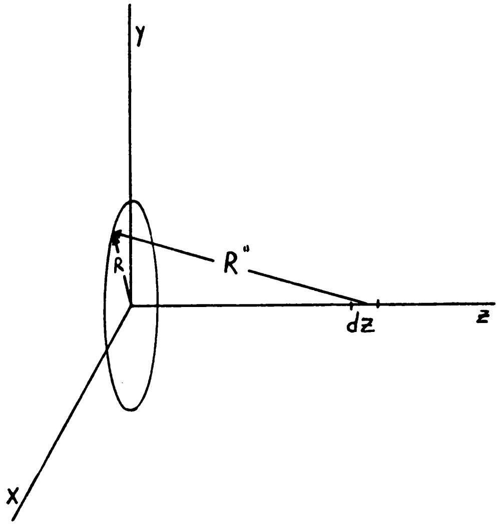

Fig. 4

If F satisfies the conditions

$$F(s) = 0 \quad \text{for} \quad s \leqq 0, \qquad f(0) = F'(0) = \cdots F^{(n/2-1)}(0), \tag{4.6}$$

the upper limits in the integrations in Eqs. (4.3), (4.5) are replaced by ct, $\cosh^{-1}(ct/R)$. For such a case, for $n = 2$, Fig. 5 shows for a fixed t, both in perspective and in the plane $z = 0$, the spherical wave fronts corresponding to the argument 0 for F.

For $n = 2, 4$, (with (4.6) holding), Eqs. (4.1)-(4.5) yield

$$u_2(R, t) = 2 \int_R^{ct} F(t - s/c)(s^2 - R^2)^{-1/2} ds = 2 \int_0^{\cosh^{-1}(ct/R)} F[t - (R/c) \cosh v]\, dv; \tag{4.7}$$

$$u_4(R, t) = \frac{2}{3}\int_R^{ct}\left[\frac{F(t - s/c)}{s^2} - \frac{F'(t - s/c)}{sc}\right]\frac{ds}{(s^2 - R^2)^{1/2}}$$
$$= \frac{2}{3}\int_0^{\cosh^{-1}(ct/R)}\left[\frac{F(t - (R/c)\cosh v)}{\cosh^2 v} - \frac{RF'(t - (R/c)\cosh v)}{\cosh v}\right]dv. \tag{4.8}$$

It is possible by integration by parts to eliminate the derivatives F', F'', from Eqs. (4.3), (4.5) leaving only F in the integrand.

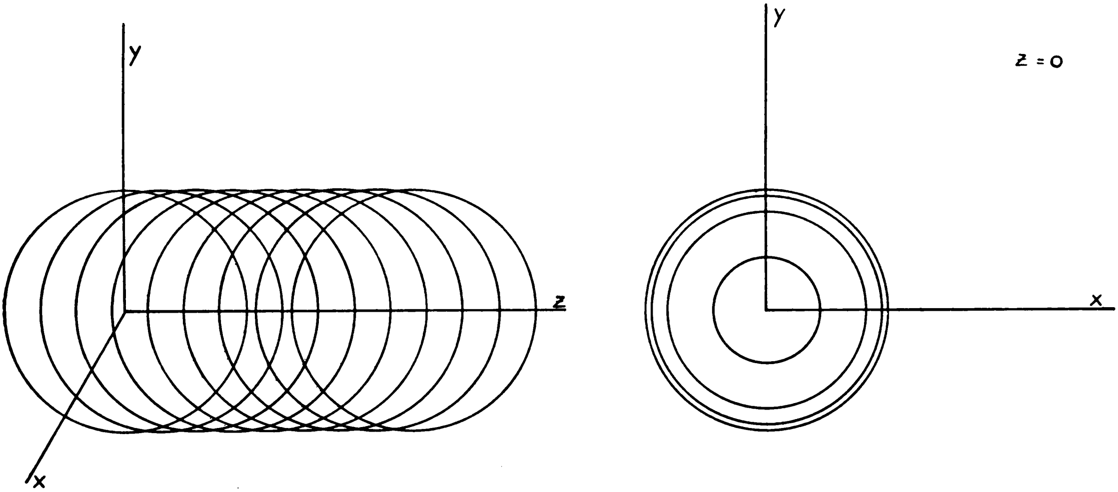

FIG. 5.

It will be noticed from Fig. 5 and Eqs. (4.1)-(4.8) that if F satisfies the conditions (3.2), the wave function $u_n(R, t)$ does not vanish outside the spherical shell

$$0 < R/c - t < \delta. \tag{4.9}$$

In this respect the case of even n differs from the odd one.

Of special interest, both for odd and even n, is the limiting wave function $V_n(R, t)$ approached by $u_n(R, t)$ by letting F above approach a proper constant multiple of the Dirac δ-function:

$$F(t) \to C\delta(t). \tag{4.10}$$

The function $V_n(R, t)$ may be regarded as the retarded potential solution due to pulse point source at the origin, at time $t = 0$, that is, it is the "retarded" solution of

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\Omega_n \delta(x_1, x_2, \cdots, x_n, t), \tag{4.11}$$

where

$$\delta(x_1 \cdots t) = \delta(x_1)\,\delta(x_2) \cdots \delta(t). \tag{4.12}$$

The functions V_n for n even or odd, are useful in deriving the following n-dimensional analogue of Kirchhoff's theorem. The solution of wave equation

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\Omega_n \rho(x_1, \cdots, x_n, t) \tag{4.13}$$

with a "source" P, is given by

$$u(P, t) = \int_{-\infty}^{t} dt' \int dP' \, \rho(P', t') V_n(R, t - t') \, dP' \tag{4.14}$$

where

$$R = PP'. \tag{4.15}$$

For odd n, recalling Eqs. (3.26), (4.10) and replacing $F(t)$ by $\delta(t)$, one may convert the right-hand member of Eq. (4.14) to a form resembling Eq. (1.7); for even n, however, the existence of the "tail" in the wave element originating from each source element prevents a true retarded potential solution from ever acquiring a form similar to Eq. (1.7).

Finally, the function $V_n(R, t)$ can be used to obtain analogues of Eqs. (1.2), (1.3) for n-dimensions. This will be illustrated for $n = 2$. Equations (4.7) now yield

$$V_2(R, t) = \begin{cases} [(ct)^2 - R^2]^{-1/2} & \text{for} \quad R < ct, \\ 0 & \text{for} \quad R < ct. \end{cases} \tag{4.16}$$

By utilizing this function, one obtains the following expression in $y > 0$ of the solution u of

$$\frac{\partial^2 u}{\partial x^2} + \frac{p^2 u}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \tag{4.17}$$

in terms of its values along $y = 0$, along the lines of Eqs. (1.2), (1.3):

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{t} dt' \int_{-\infty}^{+\infty} dx' [u(x', 0, t') y \; V_2(R, t - t')/R \; \partial R], \tag{4.18}$$

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{t} dt' \int_{-\infty}^{+\infty} dx' [u_y(x', 0, t') V_2(R, t - t')], \tag{4.19}$$

where

$$R^2 = (x - x')^2 + y^2.$$

An alternative way of arriving at the expression of the two-dimensional wave function $u(x, y)$ in $y > 0$ in terms of its values on $y = 0$, is again by applying the "method of descent", by considering u as a wave function in three dimensions, and applying Eqs. (1.1), (1.3) to the half space $y > 0$. First one obtains, since u is independent of z,

$$u(x, y, t) = (1/2\pi) \iint [u(x', 0, t - R''/c) \, y/(R'')^3 + u_t(x', 0, t - R''/c) \, y/cR''] \, dx' \, dz', \tag{4.20}$$

$$u(x, y, t) = -\frac{1}{2\pi} \iint [u_y(x', 0, t - R''/c)/R''] \, dx' \, dz'$$

where

$$(R'')^2 = (x - x')^2 + y^2 + (z')^2. \tag{4.21}$$

By a proper change of variables (and integration by parts) it is possible to convert the z'-integration of Eqs. (4.20), (4.21) into a t'-integration as in Eqs. (4.18), (4.19), and derive the latter equations from the former.

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