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**AN EXPRESSION FOR GREEN'S FUNCTION FOR A
PARTICULAR TRICOMI PROBLEM***

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1. Introduction. This paper is concerned with the simplest equation of mixed type, known as the Tricomi equation

$$T(u) = zu_{zz} + u_{zz} = 0 \quad (1)$$

where u is the dependent variable, x and z the independent variables. Equation (1) is elliptic when $z > 0$, hyperbolic when $z < 0$. In this latter half plane the characteristics of (1) are the lines defined by $3x \pm 2(-z)^{3/2} = \text{constant}$. The Tricomi problem consists of solving the equation $T(u) = 0$ in a domain Δ bounded by two concurrent characteristics AC and BC drawn in $z < 0$ and an arc AMB drawn in the half plane $z > 0$ when values of u are known along AMB and AC . Thus the mixed character of the equation is involved in the definition of the Tricomi problem and in fact the Tricomi problem is the typical problem for an equation of mixed type. The existence of a solution for such a problem has been proved by F. Tricomi [1]** himself in his fundamental paper. A quite different type of proof has been given by P. Germain and R. Bader [2], [3]. They have considered in particular the special case for which AMB is a "normal" curve, according to Tricomi's terminology—that is to say an arc defined by $(x - x_0)^2 + y^2 = R^2$, $3y = 2z^{3/2}$, with x_0 and R given constants and have shown that in such a case it is possible to give an explicit solution of the Tricomi problem. Such a Tricomi problem will be called a "normal" Tricomi problem. This result was quite interesting from a theoretical point of view and permitted a very simple proof of the existence of the solution of the Tricomi problem to be given. More recently [4], [5], and [6] it was shown that such a "normal" problem was of particular interest in the application of an approximate method to subsonic and transonic flows involving jets or wedges. However, the solution given previously was not found quite satisfactory from the computational point of view. It was emphasized, [2], that a transformation involving three parameters (transformation related to the Poincaré's geometry) can be associated with the Tricomi equation. As a result, the solution of a "normal" problem can be simply derived from the special case in which Δ is the region Δ_0 defined by $x > 0$ for $z > 0$, $3x > 2(-z)^{3/2}$ for $z < 0$. The values of u along oz ($x > 0$) are known; u is also given either along AC , or along the

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**Numbers in brackets refer to the bibliography at the end of the paper.

characteristic at infinity; in the former case, which will be called the direct problem, the solution must be regular at infinity [2]; in the latter case it will be called the conjugate problem. An inversion with center at the origin allows one to reduce one of these problems to the other. A few comments are needed on the definition of the Green's function of a Tricomi problem. This notion was introduced in [2], [3] for the case of a Tricomi equation and generalized in [7], [8] for a wider class of equations of mixed type. It arises when one looks for the possibility of writing the expression for the solution of a Tricomi problem as a linear functional of the data. It can be shown that to every point P inside Δ , one can make correspond a function $g_P(M)$, called the Green's function of the Tricomi problem for the given domain Δ . This function is continuous for M inside Δ , (if some singular lines or points are excluded), and has the following fundamental properties: Given one solution u of (1) in Δ , it is possible to find by application of the Green's formula the value of $u(P)$ in terms of some integrals involving only (besides the value of g_P and its derivatives), the value of u along AMB and AC . Moreover, g_P as function of the coordinate of M is a fundamental solution of (1) and g_P is zero on AMB and BC ; g_P as a function of the coordinates of P , M being kept fixed inside Δ , is a fundamental solution of (1) and takes the value zero when P is on AMB or on AC . In other words, as a function of M , $g_P(M)$ satisfies some boundary conditions for the conjugate problem; as a function of P , it satisfies some boundary conditions for the direct problem. The notion of fundamental solution* for an equation of mixed type is also discussed in [7], [8]. When the point P is in the elliptic half plane ($z > 0$), g_P as a function of M is regular everywhere in the open domain Δ , except in the neighborhood of P , near which it has the classical logarithmic singularity. When the point P is in the hyperbolic half plane

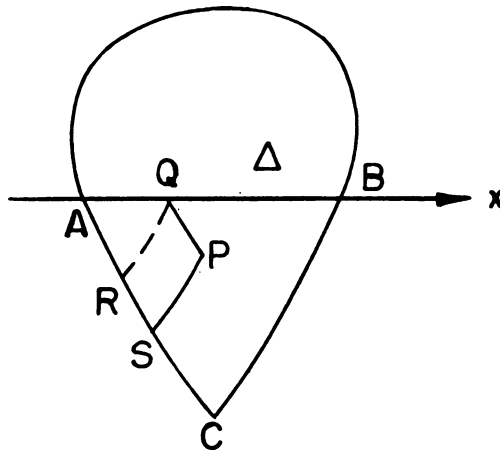


FIG. 1.

($z < 0$), the Green's function as function of M is singular on the characteristics shown in Fig. 1: it has some discontinuities along PQ and PS , proportional to the values of the Riemann function along these lines, and it becomes logarithmically infinite in the neighborhood of the reflected characteristic QR . A similar behavior is of course valid for g as a function of P when M is fixed inside Δ .

*In the usual French terminology such a solution is called "elementary".

The following developments are the result of two remarks. First it was shown in [7] and [8] how it is possible to build the Green's function of a strip for a class of differential equations, even if the strip lies in a mixed region. Second, for the Tricomi equation (1) new independent variables can be introduced in such a way that the Green's function of the Tricomi problem corresponding to Fig. 2 is transformed into the Green's function

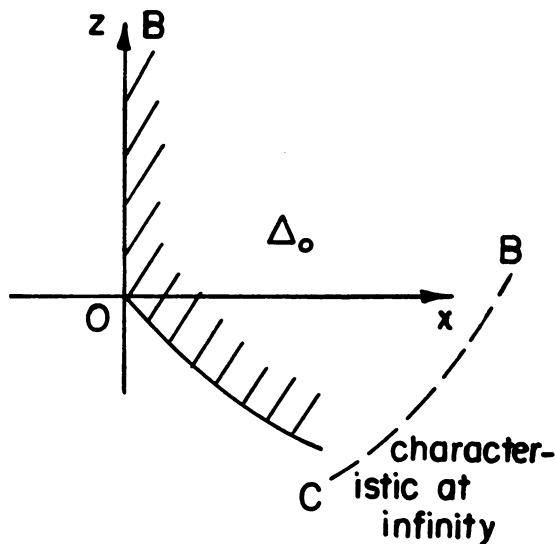


FIG. 2.

of a strip in a mixed domain. Thus it is possible to apply a previous technique with some minor differences in order to obtain the required result.

2. Transformations of the fundamental equation. The following definitions were introduced in [2]:

$$3y = 2z^{3/2}, \quad r^2 = x^2 + y^2, \quad x = rt. \tag{2}$$

In Δ_0 , r^2 as defined by (2) is positive (zero on the characteristic OC), and accordingly r will be assumed real and positive. With r and t as new independent variables, the operator $T(u)$ becomes:

$$T(u) = zL(u) = z\{u_{rr} + (1 - t^2)r^{-2}u_{tt} + (4/3)(r^{-1}u_r - r^{-2}tu_t)\}. \tag{3}$$

Now it is convenient to introduce the new variable ξ defined by $r = \exp \xi$, and with ξ and t as independent variables (3) becomes

$$T(u) = zL(u) = r^{-2}zM(u) = r^{-2}z\{(1 - t^2)u_{tt} + u_{\xi\xi} + 1/3(u_{\xi} - 4tu_t)\}. \tag{4}$$

The domain Δ_0 is mapped into the half plane $t > 0$ of the ξ, t plane. For $0 \leq t \leq 1$, the equation $M(u) = 0$ is of the elliptic type, and for $t > 1$, hyperbolic. Another form can be used with ξ and λ as independent variables where

$$\lambda = \int_t^1 (1 - v^2)^{-2/3} dv,$$

namely

$$\begin{aligned} T(u) &= r^{-2}zM(u) = r^{-2}z(1 - t^2)^{-1/3}N(u) \\ &= r^{-2}z(1 - t^2)^{-1/3}\{u_{\lambda\lambda} + (1 - t^2)^{1/3}(u_{\xi\xi} + (1/3)u_{\xi})\} \end{aligned} \tag{5}$$

The Green's function we are looking for is a fundamental solution of (1); precisely, such a fundamental solution e_P is a solution of

$$T(u) = \delta_{x_0, z_0} \tag{6}$$

where δ_{x_0, z_0} is the Dirac distribution [9] at the point $P(x_0, z_0)$. An equivalent definition is, that for every φ which is continuously twice differentiable and zero outside some compact subset of Δ_0 ,

$$\varphi(x_0, z_0) = \iint e_P T(\varphi) dx dz. \tag{7}$$

Thus, if we express e_P with the variables ξ and t (ξ_0 and t_0 are the values of these variables for the point P), (7) can be written

$$\varphi(\xi_0, t_0) = - \iint e_P z r^{-2} M(u) (3/2) r^3 z^{-2} d\xi dt$$

and e_P is a solution of

$$M(u) = -(2/3)r_0^{-1}z_0\delta_{\xi_0, t_0}; \tag{8}$$

similarly, with ξ and λ , e_P is found to satisfy

$$N(u) = (2/3)^{1/3}r_0^{-1/3}\delta_{\xi_0, \lambda_0}. \tag{9}$$

δ_{ξ_0, t_0} and $\delta_{\xi_0, \lambda_0}$ are the Dirac distributions for the two variables ξ, t , and ξ, λ respectively.

Although the equation $N(u) = 0$ does not belong, strictly speaking, to the class considered in [7], and [8], it can be studied by the same method. In order to show the extension of this method, a simple Dirichlet problem will be considered in the next section.

3. Singular Dirichlet problem for the region $x > 0, z > 0$.

A new expression of the Green's function will now be derived by the same method which will be used later for the Tricomi problem.

In the ξ, t plane this region is mapped into the strip $0 < t < 1$; in the ξ, λ plane into a similar strip $0 < \lambda < \lambda_1$, (λ_1 being the value of λ which corresponds to $t = 0$). We introduce the Fourier transform $U(\alpha, \lambda)$ of $u(\xi, \lambda)$, $U = \mathcal{F}u$, which for summable functions may be written

$$U = \mathcal{F}u = \int_{-\infty}^{+\infty} \exp(-2i\pi\alpha\xi) u d\xi, \quad u = \mathcal{F}^{-1}U = \int_{-\infty}^{+\infty} \exp(2i\pi\alpha\xi) U d\alpha \tag{10}$$

and use, as in [7], the extension of this transform to distributions [9]. It is easy to show [7] that the transform of (9) is

$$n(U) = U_{\lambda\lambda} + (1 - t^2)^{1/3} [(2/3)i\pi\alpha - 4\pi^2\alpha^2]U = (2/3)^{1/3}r_0^{-1/3} \exp(2i\pi\alpha\xi_0)\delta_\lambda. \tag{11}$$

where δ_λ is the Dirac distribution of one variable λ at the point $\lambda = \lambda_0$. In order to solve (11), the following notation is introduced: $S_1(\lambda, \alpha)$ and $S(\lambda, \alpha)$ are the solutions of $n(U) = 0$ which satisfy

$$S_1(\lambda_1, \alpha) = 0, \quad S(0, \alpha) = 0, \quad \frac{\partial}{\partial \lambda} S_1(\lambda_1, \alpha) = 1, \quad \frac{\partial}{\partial \lambda} S(0, \alpha) = 1.$$

The Fourier transform E_P of e_P is then defined by $E_P = (3/2)^{-1/3}r_0^{-1/3} \exp(-2i\pi\alpha\xi_0)E_P^*$

$$E_P^* = \begin{cases} hS_1(\lambda, \alpha)S(\lambda_0, \alpha), & \lambda_0 < \lambda < \lambda_1, \\ hS_1(\lambda_0, \alpha)S(\lambda, \alpha), & 0 < \lambda < \lambda_0, \end{cases} \tag{12}$$

h being such that the jump in the value of the first derivative of E_P^* with respect to λ at $\lambda = \lambda_0$ be equal to $+1$. Obviously $h^{-1} = S(\lambda_1, \alpha) = -S_1(0, \alpha)$.

On the other hand S_1 and S are easily found from the first Darboux solutions; set

$$s = 2i\pi\alpha + 1/6, \quad \tau = t^2, \tag{13}$$

one can write

$$\left. \begin{aligned} S_1(\lambda, \alpha) &= -tF(7/12 + s/2, 7/12 - s/2, 3/2, \tau), \\ S(\lambda, \alpha) &= 3/2(1 - \tau)^{1/3}F(5/12 + s/2, 5/12 - s/2, 4/3, 1 - t), \end{aligned} \right\} \tag{14}$$

where $F(a, b, c, \tau)$ denotes the usual hypergeometric function. Accordingly h can be expressed in terms of gamma functions

$$h = 2\Gamma(11/12 + s/2)\Gamma(11/12 - s/2)[\Gamma(1/2)\Gamma(1/3)]^{-1}. \tag{15}$$

Now we must investigate the most convenient form in which to write the inverse transform. The equation (11) will be identical to equation (20) of [7] (apart from some obvious changes in the notation) if we introduce the variable β defined by

$$4\pi^2\beta^2 = 4\pi^2\alpha^2 - (2/3)i\pi\alpha \tag{16}$$

and choose the branch which reduces for large values of $|\alpha|$ to $\beta \cong \alpha - i/12\pi$. Thus the asymptotic behavior* of the solutions of (11) for large values of $|\beta|$ are given by some formulae similar to formulae** (23) and (37) of [7]. The real axis of the α plane and the corresponding line $Re\{s\} = 1/6$ of the s plane are mapped into the contour C in the β plane (Fig. 3). Therefore that the right hand side of (12) is a distribution whose

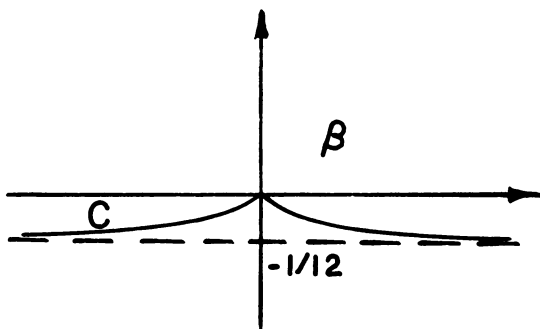


FIG. 3.

inverse transform can be written using an integral in the s plane and therefore the value of e_P is given by

$$e_P = (2/3)^{1/3}(2i\pi r_0^{1/3})^{-1} \int_{1/6-i\infty}^{1/6+i\infty} E_P^*(rr_0^{-1})^{s-1/6} ds \tag{17}$$

because $r = \exp \xi$. The reader will recognize in this formula the Mellin transform which in fact may be applied to the operator $L(u)$. From such a solution, the expression for

*The general study of the asymptotic behavior of these solutions is found in [10].

**Or the formulae (9) and (10) of [8].

the doublet at the point $M_1(0, z_1)$ may be derived easily. The Green's formula for equation (1) shows that a solution u defined in $x > 0, z > 0$ which is zero along Ox , is given by

$$u(P) = - \int_0^\infty zu(0, z)(e_P)_z dz.$$

This proves that the doublet $d_{M_1}(P)$ in $M_1(0, z_1)$ is

$$d_{M_1}(P) = -z_1 \frac{\partial}{\partial x} e_P(0, z_1, x_0, z_0). \tag{18}$$

It is possible to evaluate the right hand side of (17) and (18) by series expansions. For instance, if $\mathfrak{P}_p(\tau)$ is the polynomial of degree p

$$\mathfrak{P}_p(\tau) = F(-p, 11/6 + p, 3/2, \tau),$$

then

$$d_{M_1}(P) = (2/3)^{1/3} z_1 t_0 (1 - \tau_0)^{1/3} \sum_0^\infty 4\Gamma(3/2 + p)\Gamma(11/6 + p)[\pi\Gamma(4/3 + p)p]^{-1} \\ \times \mathfrak{P}_p(\tau_0) \begin{cases} r_1^{-4/3}(r_1 r_0^{-1})^{2p+2} & \text{if } r_1 < r_0, \\ r_0^{-4/3}(r_0 r_1^{-1})^{2p+3} & \text{if } r_1 > r_0. \end{cases}$$

In practical applications to boundary value problems the quantity $\partial/\partial x_0 d_{M_1}(0, z_0)$ is very important. The following result is easily obtained

$$\partial/\partial x_0 d_{M_1}(0, z_0) = \begin{cases} (3/2)^{1/3} 4\Gamma(11/6)\Gamma(3/2)[\pi\Gamma(4/3)]^{-1} r_1^{4/3} r_0^{-3} F(11/6, 3/2, 4/3, r_1^2 r_0^{-2}), & \text{if } r_1 < r_0, \\ (3/2)^{1/3} 4\Gamma(11/6)\Gamma(3/2)[\pi\Gamma(4/3)]^{-1} r_1^{-7/3} r_0^{2/3} F(11/6, 3/2, 4/3, r_0^2 r_1^{-2}), & \text{if } r_1 > r_0. \end{cases} \tag{19}$$

These results can be checked with those obtained by different methods (methods of discontinuous integrals or transforms [11], [12], and [13], or the direct method given in [2] and [3].

4. Green's function for the Tricomi problem. The same method can be applied in order to find an expression for the Tricomi problem of Fig. 2. The domain to be considered in the ξ, t plane is the half plane $t > 0$ which corresponds to the strip $\lambda_1 > \lambda > \lambda_2$ (λ_2 negative) in the ξ, λ plane. One must again form a solution of (11) with convenient boundary conditions. As previously, the solution $S_1(\lambda, \alpha)$ —see (14)—is needed. The second solution we need in order to define E_P as in (12) is defined by a property of asymptotic behavior. According to the general theory, [7, 8] we have two possibilities depending on the orientation of the solutions. It was shown that these two solutions, which correspond to the two Tricomi problems which can be defined in the domain Δ_0 , are 1) a function $H_1(\lambda, \alpha)$, a solution of $n(u) = 0$ which is real for $\beta = i\beta'$ (β' positive), see (16), and which tends uniformly toward zero for any $\lambda_2 < \lambda < \lambda_1$ when β' tends towards $+\infty$; 2) a function $H_2(\lambda, \alpha)$ which is simply defined by $H_2(\lambda, \beta) = H_1(\lambda, \beta e^{-i\pi})$. Let us note that (13), and (16) give $-4\pi^2\beta^2 = 4\pi^2\beta'^2 = s^2 - 1/36$ and $s \sim -2\pi\beta'$ for β' sufficiently large. Noting also that $H_1(\lambda, \alpha)$ for $\beta = i\beta'$ must tend towards zero

when $\lambda \rightarrow \lambda_2$, that is to say $t \rightarrow +\infty$, it is evident that H_1 and H_2 are given by the following expressions, solutions of $n(U) = 0$ (see [2] p. 11)

$$\begin{aligned} H_1 &= t(\tau - 1)^{-7/12+s/2} F(7/12 - s/2, 11/12 - s/2, 1 - s, [1 - \tau]^{-1}), \\ H_2 &= t(\tau - 1)^{-7/12-s/2} F(7/12 + s/2, 11/12 + s/2, 1 + s, [1 - \tau]^{-1}). \end{aligned} \tag{20}$$

Therefore, if $G_P^{(1)}$ or $G_P^{(2)}$ are the Fourier transforms of $g_P^{(1)}$ and $g_P^{(2)}$ —these are the two functions we are looking for—we may write as in (12),

$$G_P^{(j)} = (3/2)^{-1/3} r_0^{-1/3} \exp(-2i\pi\alpha\xi_0) G_i^* \quad (j = s, 2)$$

with

$$G_i^* = \begin{cases} h_i S_1(\lambda, \alpha) H_i(\alpha_0, \alpha), & \lambda_0 < \lambda < \lambda_1, \\ h_i S_1(\lambda_0, \alpha) H_i(\lambda_0, \alpha), & \lambda_2 < \lambda < \lambda_0, \end{cases} \tag{21}$$

h_i being such that the jump of the first derivative of G_i^* with respect to λ for $\lambda = \lambda_0$ is equal to $+1$. Thus h_i is the inverse of the Wronskian of S_1 and H_i , then

$$h_i = 3^{1/2} \Gamma(11/12 - js/2) \Gamma(7/12 - js/2) [2\Gamma(1/2) \Gamma(1 - js) \sin \pi(1/4 - js/2)]^{-1}. \tag{22}$$

Now, as in (17), $g_P^{(j)}$ is given by the Mellin transform

$$g_P^{(j)} = (2/3)^{1/3} (2i\pi r_0^{1/3})^{-1} \int_{1/6-i\infty}^{1/6+i\infty} G_i^*(r r_0^{-1})^{s-1/6} ds. \tag{23}$$

It is not difficult, of course, to verify for this result some properties of the Green's function which have been proved previously [2]. For instance, the "symmetry" property $g_P^{(1)}(M) = g_M^{(2)}(P)$, and the "inversion" property

$$g_P^{(1)}(M) = g_P^{(1)}(r, t; r_0, t_0) = (r_0 r^{-1})^{1/3} g_P^{(2)}(r_0^2 r^{-1}, t; r_0, t_0).$$

For practical purposes, the most important thing is to find the expression for the doublet $D_{M_1}^{(j)}(P)$ at the point $M_1(0, z_1)$ the doublet associated with the Tricomi problems in Δ_0 . The same argument used previously in order to derive (18) shows that

$$D_{M_1}^{(j)}(P) = -z_1 \frac{\partial}{\partial x} g_P^{(j)}(0, z_1, x_0, z_0) \tag{24}$$

and thus

$$D_{M_1}^{(j)}(P) = \left(\frac{2}{3}\right)^{1/3} \frac{z_1}{r_1 r_0^{1/3}} \frac{1}{2i\pi} \int_{1/6-i\infty}^{1/6+i\infty} h_i H_i(\lambda_0, \alpha) \left(\frac{r_1}{r_0}\right)^{s-1/6} ds. \tag{25}$$

In many applications, especially in the transonic problems we have in mind, the only thing which is important to know is the value of the normal derivative along Oz , a value which can be computed with an integral if we know the value of $\partial/\partial x_0 D_{M_1}^{(j)}(0, z_0) = K_j(z_1, z_0)$. This one can be expressed as some kind of generalized hypergeometric function. In fact

$$K_1(z_1, z_0) = -(3/2)^{1/3} 4r_1^{-1/2} r_0^{-7/6} G_{4.4}^{3.1} \left[\left(\frac{r_1}{r_0}\right)^2 \middle| \begin{matrix} 1/4, 1/12, 5/12, 3/4 \\ 1/4, 7/12, 11/12, 3/4 \end{matrix} \right], \tag{26}$$

$G_{p,q}^{m,n}$ being a Meijer's G -function, (see [14], p. 206). From a straightforward application of the formulae given in [14] p. 208, one obtains with the classical notation ${}_pF_q(a_1 \dots, a_p; b_1 \dots, b_q; z)$ for the generalized hypergeometric functions.

First case: $r_0 < r_1$,

$$K_1(z_1, z_0) = -(3/2)^{1/3} 16r_0^{1/3} [3^{3/2} \pi r_1^2]^{-1} {}_3F_2(4/3, 5/3, 1; 5/6, 7/6, r_0^2 r_1^{-2}). \tag{27}$$

Second case: $r_1 < r_0$,

$$\begin{aligned} K_1(z_1, z_0) = & -(3/2)^{1/3} 2 [3^{3/2} \pi r_0^{5/3}]^{-1} {}_3F_2(7/6, 5/6, 1; 1/3, 2/3; r_1^2 r_0^{-2}) \\ & - (2/3)^{2/3} r_1^{2/3} \Gamma(1/2) \Gamma(7/6) [\pi r_0^{7/3} \Gamma(2/3)]^{-1} {}_2F_1(3/2, 7/6; 2/3; r_1^2 r_0^{-2}) \\ & - (3/2)^{1/3} 5 r_1^{4/3} [3\pi r_0^3]^{-1} {}_2F_1(11/6, 3/2; 4/3; r_1^2 r_0^{-2}). \end{aligned} \tag{28}$$

These formulae seem to be convenient for numerical computations.

The values of $K_2(z_1, z_0)$ may be derived from (27) and (28) with

$$z_0 K_1(z_1, z_0) = z_1 K_2(z_0, z_1). \tag{29}$$

Let us conclude this section by recalling that $g_P^{(1)}$ is the Green's function for the direct Tricomi problem for P in the hyperbolic half plane (Fig. 4), this function has its

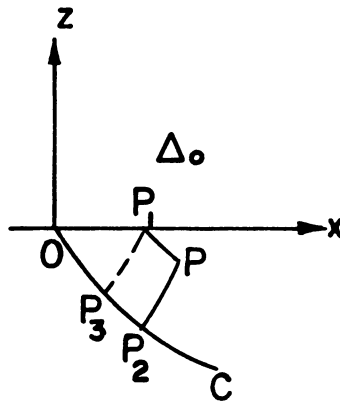


FIG. 4.

singularities along PP_1, PP_2 (discontinuities proportional to $(-z)^{-1/4}$) and its logarithmic singularities along P_1P_3 . Similarly, $g_P^{(2)}$ is the Green's function of the conjugate Tricomi problem in Δ_0 and has its singularities along PP_1, PP_2, PP_3 (Fig. 5). The doublet $D_M^{(1)}(P)$ is the function which allows one to solve the direct Tricomi problem where the value of the solution along OC is zero, $D_M^{(2)}(P)$ the doublet which allows one to solve the conjugate Tricomi problem, when the required solution is zero on the characteristic at infinity. When r is infinitely small the limiting value of $D_M^{(2)}(P)$ (r_0 finite) must be proportional (the factor may depend upon r_1) to the first Darboux solution which gives the asymptotic behavior at infinity of the flow around the profile of the Mach number 1. This principal value of $D_M^{(2)}(P)$ is obtained by considering the positive residue in (25). It is proportional to

$$\begin{aligned} r_0^{-5/3} t_0 (t_0^2 - 1)^{-4/3} F(4/3, 5/3, 5/2, [1 - t_0^2]^{-1}) \\ = Kr_0^{-3} \{ (r - x)^{1/3} (3x - r) - (r + x)^{1/3} (3x + r) \}, \end{aligned} \tag{30}$$

where K is a numerical constant.

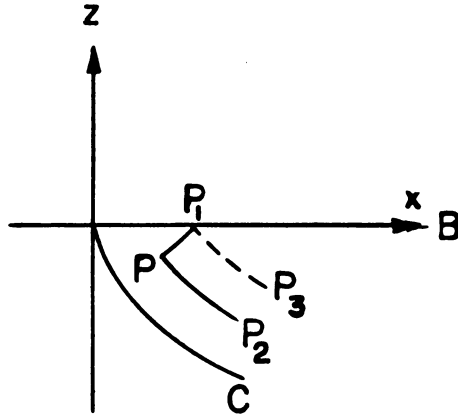


FIG. 5.

5. A special generalized Tricomi problem. It was often suggested, for instance in [15], that a generalized Tricomi problem could be considered, for which the solution is to be found in a domain Δ , bounded by an arc AB drawn in the elliptic half plane with end points A and B on the x axis, an arc AC in the hyperbolic half plane with a time-like direction if the direction AB is taken as the time direction, and an arc BC of a characteristic. The data for such a problem are the values of the solution u along the arc AB and the arc AC . By a transformation of the group of the Poincare geometry associated with the Tricomi equation, it is always possible to reduce this problem to a similar one in which BC is the characteristic at infinity. Now consider the special case for which the arcs OB and OC are such that t is constant along each of these arcs. The corresponding domain in the ξ, λ plane is mapped onto a strip $\lambda_4 < \lambda_3$ parallel to the ξ axis. Thus, it is clear that the previous method must allow one to give the solution of this generalized Tricomi problem.

In the ξ, λ plane we have to start with the fundamental solution $e_P^*(M)$ of $N(u) = 0$, which is "orientated" in the direction opposite to that of the ξ axis. $P(\xi_0, \lambda_0)$ is the singular point of $e_P^*(M)$ and the function has a singular behavior along the singular characteristics schematically represented on (Fig. 6); $e_P^*(M)$ has a discontinuity on each solid characteristic and a logarithmic singularity along each dotted characteristic. On the other hand $e_P^*(M)$ will be chosen in such a way that it vanishes when M lies on $\lambda = \lambda_3$ or $\lambda = \lambda_4$. The expression of such a solution was given in [7]. Let us call $S_3(\lambda, \alpha)$ and $S_4(\lambda, \alpha)$ the solutions of $n(U) = 0$ which satisfy the following conditions:

$$S_3(\lambda_3, \alpha) = 0, \quad S_4(\lambda_4, \alpha) = 0, \quad \partial/\partial\lambda S_3(\lambda_3, 0) = 1, \quad \partial/\partial\lambda S_4(\lambda_4, 0) = 1. \quad (31)$$

Using this notation, $e_P^*(M)$ is the inverse Fourier transform (ξ_0 is taken equal to 0 in (32)) of

$$E_P^* = \begin{cases} WS_3(\lambda, \alpha)S_4(\lambda_0, \alpha), & \lambda_0 < \lambda < \lambda_3, \\ WS_4(\lambda, \alpha)S_3(\lambda_0, \alpha), & \lambda_4 < \lambda < \lambda_0, \end{cases} \quad (32)$$

with*

$$W^{-1} = S_3(\lambda_4, \alpha) = -S_4(\lambda_3, \alpha). \quad (33)$$

*The functions $S_3(\lambda, \alpha)$, $S_4(\lambda, \alpha)$ may be easily expressed using the hypergeometric functions of the first Darboux solutions.

It was shown that expressions (32) are meromorphic functions with respect to the variable β defined by (16), which have poles for a sequence of real values of β and for a sequence of purely imaginary values of β and that, in order to obtain the fundamental solution with the orientation given by Fig. 6, the integral which gives the inverse Fourier trans-

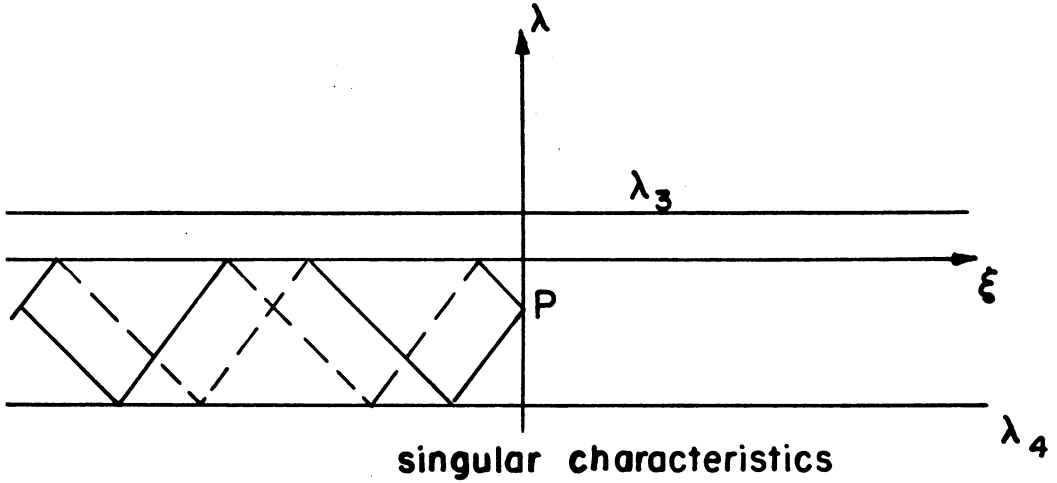


FIG. 6.

form of (32) must be taken in the β plane on a line which leaves the real poles below and the imaginary poles with positive ordinates above. Recalling the relations which join s and β : $s^2 = 1/6 - 4\pi^2\beta^2$, $s \sim 2i\pi\beta$ for $|\beta|$ large, it is clear that:

$$e_{\mathbb{P}}^*(M) = (2i\pi)^{-1} \int_{c_1} E_{\mathbb{P}}^*(rr_0^{-1})^{s-1/6} ds, \tag{34}$$

where c_1 is schematized in the figure 7. It was shown also that $e_{\mathbb{P}}^*(M)$ tends towards zero when ξ tends towards $+\infty$ for every $\lambda_4 < \lambda < \lambda_3$.

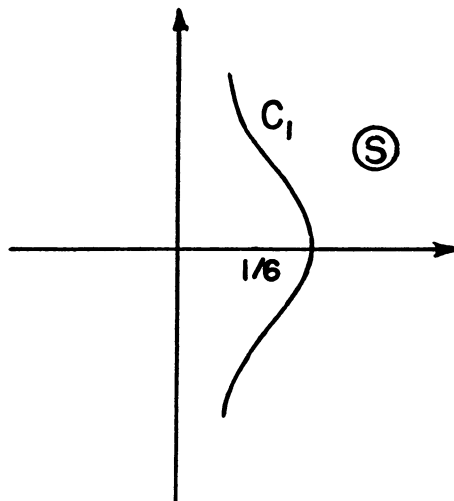


FIG. 7.

Now, according to (9) the Green's function $g_P(M)$ for the special problem we consider is

$$g_P(M) = (2/3)^{1/3} r_0^{-1/3} e_P^*(M). \quad (35)$$

This function is zero on the characteristic at infinity, on the arc OB and on the arc OC , but has a singular point in O . When P lies in the elliptic part of Δ_0 , $g_P(M)$ has the classical logarithmic singularity in the vicinity of P , when P lies in the hyperbolic part, $g_P(M)$ has a singular behavior along the characteristic lines schematized in Fig. 8. Although

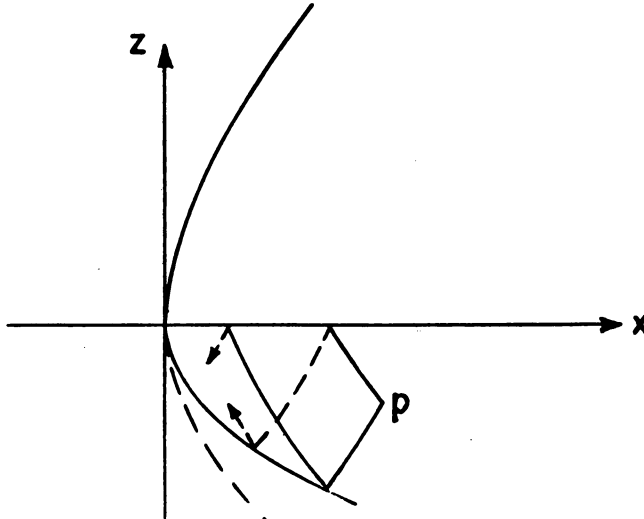


FIG. 8.

we do not want to enter into all the details, it is quite clear that the previous result allows us to state the uniqueness and existence theorems for the solution of this special generalized Tricomi problem when the value of the unknown function is prescribed along $OB(t = t_3)$ and $OC(t = t_4)$, by application of the Green's formula. A small difficulty arises in the vicinity of o because of the singularity of the Green's function near the origin, since the application of the Green's formula gives rise to a series of integrals. But this series can be shown to be convergent. Such a situation always arises in classical problems for hyperbolic equations in connection with what E. Picard [16] has called a fourth boundary value problem, (see for instance [17]).

6. Concluding remarks. The previous results allow one to simplify a great deal the proof of the existence of the solution of the Tricomi problem. To outline very briefly such a proof, it is first of all clear that the result of Section 4 gives the direct solution of the problem when the contour AMB (Fig. 1) is a normal contour. It is also possible to give a direct solution when the contour AMB is, in the x, y plane, any circular arc with end-points A and B , by using the result of Section 5. Now the general case when the contour AMB is arbitrary can be solved by using the Schwartz process as in [2] and the sequence of solutions obtained by this method converges towards the solution by application of the maximum principle [2].

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