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NOTE ON LINEAR PROGRAMMING*

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Statement of theorem. Consider a set of m linear equations in n unknowns x_i ;

$$a_{\alpha i}x_i = b_\alpha , \tag{1}$$

where Greek indices range from 1 to m and Latin from 1 to n. The usual summation convention on repeated indices is used; $a_{\alpha i}$ and b_α are constants. Linear programming is a method of obtaining a solution (if it exists) of Eq. (1) satisfying in addition the requirements

$$x_i \geq 0, \qquad \text{all } i, \tag{2}$$

$$c_i x_i = \text{minimum}, \tag{3}$$

where the c_i are constants. The fundamental theorem used in the simplex method of Dantzig (1) is that if one solution exists, then an equivalent solution can be found in which not more than m of the x_i are non-zero; further, those columns of the matrix $(a_{\alpha i})$ which correspond to such non-zero (x_i) will be linearly independent. The usual proof of this theorem (e.g. Ref. (2)) involves tedious geometrical considerations in n-dimensional space; it therefore seems worth-while to point out that a simple direct proof exists.

Proof of theorem. Suppose (x_i') satisfies conditions (1), (2), (3). Some—perhaps all—of these (x_i') will be non-zero; say for example that x_2' , x_3' , x_6' , are alone not zero. If firstly the corresponding columns $(a_{\alpha 2}$, $a_{\alpha 3}$, $a_{\alpha 6})$ were linearly dependent, then a set of three constants K_2 , K_3 , K_6 (not all zero) would exist such that

$$A(K_2 a_{\alpha 2} + K_3 a_{\alpha 3} + K_6 a_{\alpha 6}) = 0, \qquad \text{all } \alpha \tag{4}$$

for any arbitrary constant A. Then because of Eq. (4), the new set (x_i'') defined by

$$\begin{aligned} x_i'' &= x_i' - AK_i \qquad \text{for } i = 2, 3, 6, \\ x_i'' &= 0 \qquad\qquad\quad \text{for other } i \end{aligned} \tag{5}$$

satisfies Eq. (1) and, for sufficiently small A, also Eq. (2). Clearly however $c_i x_i'' < c_i x_i'$ for appropriate A, unless

$$c_2 K_2 + c_3 K_3 + c_6 K_6 = 0. \tag{6}$$

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Consequently, either the hypothesis of linear dependency has led to a contradiction so that the three columns are linearly independent, or alternatively Eq. (6) must hold. But if Eq. (6) holds, then

$$c_i x_i'' = c_i x_i' \qquad \text{for all } A,$$

and A can be chosen so that at least one of x_2'', x_3'', x_6'' vanishes, so that an equivalent solution involving fewer columns has been obtained; the same analysis may now be applied to the new solution.

It follows that, eventually, an equally good solution utilizing only linearly independent columns will be obtained. Since not more than m columns can be linearly independent, the theorem is proved.

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HEAT FLOW IN A HALF SPACE*

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This heat flow problem has a more difficult boundary condition than usual. A formal solution is obtained by the combined use of Laplace and Fourier sine transforms.

Let the region $x > 0$ have an initial temperature which is a function of x only, $f(x)$. Assume that there is radiation of heat from the surface $x = 0$ to a finite slab, and from the other surface of the slab to surroundings whose temperature is a prescribed function of the time, $g(t)$. The slab is assumed to be thin and of high thermal conductivity so that its temperature may be considered to be uniform throughout at any time. The heat exchange by radiation between adjacent surfaces will be taken to be proportional to the difference in temperatures of the two surfaces. Mathematically, then, the problem may be stated as follows:

$$\frac{\partial u(x, t)}{\partial t} = \kappa \frac{\partial^2 u(x, t)}{\partial x^2} \qquad \text{for} \qquad x > 0,\ t > 0, \tag{1}$$

$$u(x, 0) = f(x), \tag{2}$$

$$\frac{\partial u(0, t)}{\partial x} = a[u(0, t) - v(t)], \qquad t > 0 \tag{3}$$

$$h \frac{\partial v(t)}{\partial t} = a[u(0, t) - v(t)] - b[v(t) - g(t)], \qquad t > 0 \tag{4}$$

$$v(0) = V. \tag{5}$$

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