strain be derivable from displacement (an irrotational vector is derivable from a potential).[2]

The proof of the theorem rests on the fact that σ''_{ij} may be chosen as the solution to the following mixed boundary value problem for σ''_{ij}, u''_i, and ϵ''_{ij}:

$$\begin{aligned} \sigma''_{ij,i} &= \sigma_{ij,i} + f_i^0 \quad \text{in } V, \\ \epsilon''_{ij} &= L(\sigma''_{ij}) = \tfrac{1}{2}(u''_{i,j} + u''_{j,i}) \quad \text{in } V, \\ \sigma''_{ij}n_i &= \sigma_{ij}n_i - T_j^0 \quad \text{on } S_B, \\ u''_i &= u_i^0 \quad \text{on } S_A. \end{aligned}$$

Then the choice $\sigma'_{ij} = \sigma_{ij} - \sigma''_{ij}$ satisfies

$$\begin{aligned} \sigma'_{ij,i} &= -f_j^0 \quad \text{in } V, \\ \sigma'_{ij}n_i &= T_j^0 \quad \text{on } S_B, \end{aligned}$$

and so σ'_{ij} and σ''_{ij} constitute the elements of the required decomposition.

Theorem 2. A symmetrical (strain) tensor ϵ_{ij} defined in V may be written

$$\epsilon_{ij} = \epsilon'_{ij} + \epsilon''_{ij},$$

where ϵ'_{ij} and ϵ''_{ij} have the properties

(c) $\epsilon''_{ij} = \frac{1}{2}(u''_{i,j} + u''_{j,i})$, where u''_i is a (displacement) vector that takes on the values u_i^0 on S_A,

(d) the (stress) tensor $\sigma'_{ij} = H(\epsilon'_{ij})$, where H is the inverse of L, satisfies

$$\begin{aligned} \sigma'_{ij,i} &= -f_j^0 \quad \text{in } V, \\ \sigma'_{ij}n_i &= T_j^0 \quad \text{on } S_B. \end{aligned}$$

This theorem follows immediately from the application of Theorem 1 to the stress tensor $\sigma_{ij} = H(\epsilon_{ij})$ with the result that the decomposition $\sigma_{ij} = \sigma'_{ij} + \sigma''_{ij}$ obeys conditions (a) and (b). Then the strain tensors $\epsilon'_{ij} = L(\sigma'_{ij})$ and $\epsilon''_{ij} = L(\sigma''_{ij})$ provide the elements of the required decomposition of ϵ_{ij}.

[2]The reader is cautioned not to infer that σ''_{ij} satisfies the conventional Beltrami-Michell compatibility equations, since these involve equilibrium conditions that σ''_{ij} need not satisfy.

ON VARIATIONAL PRINCIPLES AND GALERKIN'S PROCEDURE FOR NON-LINEAR ELASTICITY*

By BERNARD BUDIANSKY and CARL E. PEARSON (*Harvard University*)

Variational principles are discussed by Greenberg [1] for the following problem of non-linear elasticity** in the region V bounded by the surface $S = S_A + S_B$:

$$\begin{aligned} \sigma_{ij,i} &= 0 \quad \text{in } V, \\ \epsilon_{ij} &= \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } V, \\ u_i &= u_i^0 \quad \text{on } S_A, \\ \sigma_{ij}n_i &= T_j^0 \quad \text{on } S_B, \end{aligned}$$

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**The non-linearity is incorporated only in the stress-strain relations; displacements are still assumed small.

where

$$\epsilon_{ij} = \epsilon_{ij}(\sigma_{11}, \sigma_{12}, \cdots),$$

$$\sigma_{ij} = \sigma_{ij}(\epsilon_{11}, \epsilon_{12}, \cdots).$$

The unambiguous formulation of the variational principles requires that the quantities $\sigma_{ij} d\epsilon_{ij}$ and $\epsilon_{ij} d\sigma_{ij}$ be exact differentials; it is accordingly assumed that $\partial\sigma_{ij}/\partial\epsilon_{pq} = \partial\sigma_{pq}/\partial\epsilon_{ij}$ and that $\partial\epsilon_{ij}/\partial\sigma_{pq} = \partial\epsilon_{pq}/\partial\sigma_{ij}$. Then it is shown in (1) that

I. If u_i is a solution, with $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, then

$$\delta\left[\int_V \int_0^{\epsilon_{ij}} \sigma_{ij}(\epsilon_{11}, \epsilon_{12}, \cdots)\, d\epsilon_{ij}\, dV - \int_{S_B} T_j^0 u_j\, dS_B\right] = 0 \tag{1}$$

for all δu_i satisfying $\delta u_i = 0$ on S_A . Here the upper limit in the inner integral of the first term is the ϵ_{ij} state corresponding to the u_i solution.

II. If σ_{ij} is a solution, then

$$\delta\left[\int_V \int_0^{\sigma_{ij}} \epsilon_{ij}(\sigma_{11}, \sigma_{12}, \cdots)\, d\sigma_{ij}\, dV - \int_{S_A} u_j^0 \sigma_{ij} n_i\, dS_A\right] = 0 \tag{2}$$

for all $\delta\sigma_{ij}$ satisfying $\delta\sigma_{ij,i} = 0$ in V and $\delta\sigma_{ij} n_i = 0$ on S_B . The upper limit of the inner integral in the first term is the σ_{ij} state of the solution.

In the practical application of these principles the procedure is to enforce (at least approximately) the satisfaction of Eqs. (1) and (2). The question therefore arises whether the following inverse theorems are valid:

III. If (1) holds for a particular u_i satisfying $u_i = u_i^0$ on S_A , and for all δu_i satisfying $\delta u_i = 0$ on S_A , then this u_i is a solution.

IV. If (2) holds for a particular σ_{ij} satisfying $\sigma_{ij,i} = 0$ in V, $\sigma_{ij} n_i = T_j^0$ on S_B , and for all $\delta\sigma_{ij}$ satisfying $\delta\sigma_{ij,i} = 0$ in V, $\delta\sigma_{ij} n_i = 0$ on S_B , then this σ_{ij} is a solution.

As will be shown, the validity of III can be readily established by elementary means, but the proof of IV is less obvious. To prove III, we note that the volume integral in (1) may be transformed as follows:

$$\delta \int_V \int_0^{\epsilon_{ij}} \sigma_{ij}\, d\epsilon_{ij}\, dV = \int_V \sigma_{ij} \delta\epsilon_{ij}\, dV = \int_V \sigma_{ij} \delta u_{j,i}\, dV$$

$$= \int_V (\sigma_{ij} \delta u_j)_{,i}\, dV - \int_V \sigma_{ij,i} \delta u_j\, dV.$$

Then use of Green's theorem and substitution back into (1) gives

$$-\int_V \sigma_{ij,i} \delta u_j\, dV + \int_{S_B} (\sigma_{ij} n_i - T_j^0) \delta u_j\, dS_B = 0.$$

It then follows from the freedom of choice available for δu_i that $\sigma_{ij,i} = 0$ in V and $\sigma_{ij} n_i = T_j^0$ on S_B , and so, since $u_i = u_i^0$ on S_A , it must be true that u_i is indeed a solution.

The verification of IV does not seem to be possible in so direct a fashion and so recourse will be made to the introduction of the following auxiliary theorem:

V. If ϵ_{ij} is a (strain) field in V that satisfies the requirement

$$\int_V \epsilon_{ij} \sigma_{ij}\, dV - \int_{S_A} u_j^0 \sigma_{ij} n_i\, dS_A = 0 \tag{3}$$

for all σ_{ij} that obey the admissibility conditions $\sigma_{ij,i} = 0$ in V, $\sigma_{ij}n_i = 0$ on S_B, then ϵ_{ij} is derivable from a displacement field u_i that equals u_i^0 on S_A.

The proof of V depends on the proposition proved in [2] that ϵ_{ij} can be decomposed into $\epsilon'_{ij} + \epsilon''_{ij}$, where ϵ''_{ij} is derivable from a u''_i that equals u_i^0 on S_A, and ϵ'_{ij} provides, through Hooke's law, a stress $\sigma'_{ij} = H(\epsilon'_{ij})$ that satisfies $\sigma'_{ij,i} = 0$ in V and $\sigma'_{ij}n_i = 0$ on S_B. Then (3) may be written as

$$\int_V \epsilon'_{ij}\sigma_{ij}\,dV + \int_V \epsilon''_{ij}\sigma_{ij}\,dV - \int_{S_A} u_j^0\sigma_{ij}n_i\,dS_A = 0.$$

By the principle of virtual work the last two integrals cancel for all admissible σ_{ij}, and the admissible choice of $\sigma_{ij} = H(\epsilon'_{ij})$ then gives

$$\int_V \epsilon'_{ij}H(\epsilon'_{ij})\,dV = 0.$$

But since the integrand is necessarily positive for $\epsilon'_{ij} \neq 0$ (since it is essentially the strain-energy-density of linear elasticity) it follows that $\epsilon'_{ij} = 0$. Hence $\epsilon_{ij} = \epsilon''_{ij}$, and so has the properties stated in V.

The proof of IV is now immediate; from (2)

$$\int_V \epsilon_{ij}\delta\sigma_{ij}\,dV - \int_{S_A} u_j^0\delta\sigma_{ij}n_i\,dS_A = 0$$

for all admissible $\delta\sigma_{ij}$. Hence, by V, ϵ_{ij} is derivable from a u_i that equals u_i^0 on S_A, and therefore σ_{ij} is a solution.

The auxiliary theorem V also provides a theoretical basis for the use of a Galerkin-type procedure in the approximate solution of non-linear elasticity problems. Such a procedure could consist of the following steps:

(a) Assume an approximate solution of the form

$$\sigma^*_{ij} = \sigma^{(0)}_{ij} + \sum_{k=1}^{N} a_k\sigma^{(k)}_{ij},$$

where

$$\sigma^{(k)}_{ij,i} = 0, \qquad k = 0, 1, 2, \cdots, N$$
$$\sigma^{(0)}_{ij}n_i = T_j^0, \qquad \text{on } S_B$$
$$\sigma^{(k)}_{ij}n_i = 0, \qquad \text{on } S_B \qquad \text{for } k = 1, 2, 3, \cdots, N$$

and where the constants a_k are to be chosen so that

(b) $$\int_V \epsilon_{ij}(\sigma^*_{11}, \sigma^*_{12}, \cdots)\lambda^{(k)}_{ij}\,dV - \int_{S_A} u_j^0\lambda^{(k)}_{ij}n_i\,dS_A = 0, \qquad k = 1, 2, 3, \cdots, N$$

where the $\lambda^{(k)}_{ij}$ are arbitrary symmetrical tensor fields that satisfy $\lambda^{(k)}_{ij,i} = 0$ in V and $\lambda^{(k)}_{ij}n_i = 0$ on S_B.

The $\lambda^{(k)}_{ij}$ need not coincide with the $\sigma^{(k)}_{ij}$; if however they do coincide, this Galerkin procedure becomes equivalent to a Rayleigh-Ritz procedure based on (2) and the same expansion σ^*_{ij}.

All of the above results may be considered applicable to theories of plasticity of the deformation type if no "unloading" takes place (see [1]); in addition the various theorems can easily be generalized to the case of non-zero body force.

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NOTE ON THE EQUATIONS OF SHALLOW ELASTIC SHELLS*

BY P. M. NAGHDI (*University of Michigan*)

In the following, a system of differential equations is deduced for thin shallow elastic shells (of uniform thickness) with small displacements, which include the effect of transverse shear deformation. The corresponding equations of the classical theory, where the effect of transverse shear deformation is neglected, are contained in the works of Marguerre [1] and Green and Zerna [2], and have also been employed recently by E. Reissner [3, 4]. Although the equations sought may be obtained (through appropriate approximations) from the results given in [5], considerable space will be conserved if the basic equations of the theory are referred to Cartesian coordinates.

Let the equation of the middle surface of the shell be written in the form $z = z(x_1, x_2)$ and let:** w denote the displacement in the z-direction and u_i and β_i be respectively the displacements of the middle surface and the change of the slope of the normal to the middle surface, along the Cartesian coordinate axes x_1 and x_2. With the notation N_{ij}, M_{ij}, and V_i, for the stress resultants, stress couples, and transverse stress resultants, respectively, the stress differential equations of equilibrium are

$$N_{ij,j} + p_i = 0, \tag{1a}$$

$$V_{i,i} + [z_{,j}N_{ij}]_{,i} + q = 0, \tag{1b}$$

$$V_i = M_{ij,j}, \tag{1c}$$

where comma denotes partial differentiation and p_1, p_2, and q are the components of the load intensity in x_1, x_2, and z directions, respectively.

The stress strain relations, which include the effect of transverse shear deformation, may be written as

$$\begin{aligned}\epsilon_{ij} &= \tfrac{1}{2}[(u_{i,j} + u_{j,i}) + (z_{,j}w_{,i} + z_{,i}w_{,j})] \\ &= \frac{1}{C}[-\nu N_{kk}\delta_{ij} + (1 + \nu)N_{ij}],\end{aligned} \tag{2a}$$

$$M_{ij} = \tfrac{1}{2}D[2\nu\beta_{k,k}\delta_{ij} + (1 - \nu)(\beta_{i,j} + \beta_{j,i})], \tag{2b}$$

$$\beta_i = -w_{,i} + \frac{6}{5Gh}V_i, \tag{2c}$$

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**Throughout this note, the Latin indices i, j, and k have the range 1 and 2 only. Repeated indices imply summation over all the values the indices may take. Whenever one of the repeated indices is placed in parentheses [as in Eq. (3)], the summation convention is suspended for that index.