A METHOD FOR THE CONSTRUCTION OF GREEN'S FUNCTIONS*

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Summary. A method is outlined for the determination of the Green's function associated with any partial differential equation for arbitrary domains. The Green's function is obtained as the solution of an integral equation. A method of solution of this equation is discussed which yields the Green's function as the limit of an infinite sequence of functions. Convergence of this sequence is proved for the case of Helmholtz' equation. An example from the theory of heat conduction is solved in detail.

1. Introduction. The use of Green's and related functions provides one of the most powerful tools for the solution of many boundary value problems of theoretical physics and applied mechanics [1, 2]. Its chief drawback lies in the difficulty of determining the Green's function, for the particular partial differential equation concerned, in the case of domains with complicated geometrical configurations. For certain differential equations general methods are available; for instance, the Green's function of the Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = q(x, y)u(x, y); \qquad q \geq 0 \tag{1}$$

can be derived for an arbitrary domain as the solution of an integral equation [1]. For boundary value problems not governed by Eq. (1), general methods are usually not available, and it would therefore be desirable to devise a procedure for the construction of Green's and similar functions whose basic principles are independent of any specific partial differential equation. A procedure of this type is described in this paper.

In the method proposed, the Green's function in question is also obtained as the solution of an integral equation, the derivation of which is presented in Sec. 2. For simplicity's sake, this derivation is restricted to a general two-dimensional problem with a single dependent variable, and deals only with the boundary value problem associated with Green's function. It possesses, however, a simple physical interpretation, and can be readily extended to other problems. The proposed method is thus also applicable to three-dimensional domains, to systems of equations in several dependent variables, or to other boundary conditions. It can be directly applied to the determination of Neumann's function, and to the solution of boundary value problems in the theory of elasticity [3] or in the theory of heat conduction [4] (see Sec. 4).

Two methods of solution of the basic integral equation are discussed in the paper; they are referred to as the series and sequence methods, respectively, according to the form in which the solution is most readily obtained. The series method is in effect a Liouville-Neumann expansion, so that the Green's function appears in the form of an infinite series whose properties are well known [5]. The sequence method, on the other hand, has not been previously studied; it yields the Green's function as the limit of an infinite sequence of functions. The convergence of this sequence is examined in this

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paper for the case of the example of Sec. 4 (which deals with a solution of the heat-conduction equation), and then in Sec. 5 for the general case of Eq. (1).

2. The basic integral equation. Consider the problem of the determination of the solution $u(x, y)$ of a given linear partial differential equation for a domain D bounded by a closed curve C, under the boundary condition that the function u assume prescribed values on C. It is assumed that an integral representation of u in terms of its boundary values is available; for the present purposes it will be taken to be of the form:

$$u(Q) = \int_C u(P) \frac{\partial G(Q, P)}{\partial n_P} ds_P \tag{2}$$

where P and Q are points in $D + C$, s is the length parameter of the boundary curve C, and n denotes the direction of the interior normal on C. The Green's function $G(P, Q)$, considered as a function of P with fixed Q, satisfies the conditions that

(i) $G(P, Q) = 0$ for P on C

(ii) $G(P, Q)$ exhibits a singularity of prescribed character (depending, that is, on the specific differential equation in question) as P approaches Q, and is regular elsewhere in $D + C$

(iii) $G(P, Q)$ is a solution of the given differential equation, if considered as a function of P with Q fixed (and a solution of the corresponding adjoint differential equation if considered as a function of Q with P fixed).

Suppose now that a function $G_0(P, Q)$ is available, which satisfies conditions (ii) and (iii) above, but not (i); then one may write

$$G(P, Q) = G_0(P, Q) + g_0(P, Q), \tag{3}$$

where the function $g_0(P, Q)$, considered as a function of P with Q fixed, is a regular solution of the given differential equation, and assumes on C known boundary values equal to the negative of those assumed there by $G_0(P, Q)$. Clearly then

$$g_0(P, Q) = -\int_C G_0(R, Q) \frac{\partial G(P, R)}{\partial n_R} ds_R . \tag{3a}$$

Substitution into Eq. (3) then yields the following basic relation:

$$G(P, Q) = G_0(P, Q) - \int_C G_0(R, Q) \frac{\partial G(P, R)}{\partial n_R} ds_R \tag{4}$$

which may be regarded as a linear integral equation with a singular kernel G_0, for the desired Green's function G.

The above derivation shows that the extensions mentioned in the Introduction can be readily carried out; as an example, consider the boundary value problem obtained by prescribing the normal derivatives of u on C, rather than u itself. It is then assumed that the desired solution can be written as

$$u(Q) = -\int_C N(Q, P) \frac{\partial u(P)}{\partial n_P} ds_P , \tag{5}$$

where the Neumann function $N(P, Q)$ satisfies conditions (ii) and (iii) above, and the relation

$$\frac{\partial N(P, Q)}{\partial n_P} = 0 \quad \text{for} \quad P \text{ on } C \tag{5a}$$

in place of (i). The integral equation analogous to (4) then becomes in this case

$$N(P, Q) = G_0(P, Q) + \int_C N(P, R) \frac{\partial G_0(R, Q)}{\partial n_R} ds_R . \tag{6}$$

One method of solution of the above integral equations is provided by the Liouville-Neumann expansion. With Eq. (4) as an example, substitution of the expression for G given by the right-hand side of Eq. (4) into the integrand contained in that equation, and repetition of this operation, gives the Green's function explicitly in the form of the following series:

$$\begin{aligned} G(P, Q) = G_0(P, Q) &- \int_C G_0(P_1, Q) \frac{\partial G_0(P, P_1)}{\partial n_{P_1}} ds_{P_1} \\ &+ \int_C \int_C G_0(P_1, Q) \frac{\partial G_0(P_2, P_1)}{\partial n_{P_1}} \frac{\partial G_0(P, P_2)}{\partial n_{P_2}} ds_{P_2} ds_{P_1} \\ &- \int_C \int_C \int_C G_0(P_1, Q) \frac{\partial G_0(P_2, P_1)}{\partial n_{P_1}} \frac{\partial G_0(P_3, P_2)}{\partial n_{P_2}} \frac{\partial G_0(P, P_3)}{\partial n_{P_3}} ds_{P_1} ds_{P_2} ds_{P_3} + \cdots . \end{aligned} \tag{7}$$

Each term of this series depends only on the known function G_0 and on the geometry of the given domain, and can therefore be evaluated. The conditions for the convergence of expansion of this type have been extensively investigated in the past [5], and will therefore not be entered into here. This method of solution will be called in this paper the series method.

For the integral equations arising in the present problem, it is possible to devise an alternative method of solution, often more advantageous, as outlined in the following section.

3. Sequence method of solution. The function G_0 may be regarded as an initial guess (say the zero-th approximation) for the desired function G; other approximations may be obtained by considering the partial sums associated with the series of Eq. (7). The first approximation G_1, for example, may be taken as

$$G_1(P, Q) = G_0(P, Q) - \int_C G_0(P_1, Q) \frac{\partial G_0(P, P_1)}{\partial n_{P_1}} ds_{P_1} . \tag{8}$$

This function satisfies all the conditions that were required of G_0, and may therefore be used in its stead in Eq. (4) to obtain a new integral equation for G in terms of G_1. The whole process may then be repeated, the next step yielding the second approximation G_2 in terms of G_1, and still a new integral equation for G in terms of G_2. With the notation

$$I_{ii}(P, Q) = \int_C G_i(R, Q) \frac{\partial G_i(P, R)}{\partial n_R} ds_R \tag{8a}$$

the recurrence formula linking the $(i + 1)$th and ith approximations is

$$G_{i+1}(P, Q) = G_i(P, Q) - I_{ii}(P, Q). \tag{8b}$$

The Green's function G and the ith approximation are connected by the integral equation

$$G(P, Q) = G_i(P, Q) - I_i(P, Q), \tag{8c}$$

where

$$I_i(P, Q) = \int_C G_i(R, Q) \frac{\partial G(P, R)}{\partial n_R} ds_R . \tag{8d}$$

It may be noted that the relation

$$I_{i+1}(P, Q) = I_i(P, Q) - I_{ii}(P, Q) \tag{8e}$$

follows directly from Eqs. (8b) and (8c).

Equation (8b) is the basic equation of the sequence method of solution; there still remains to determine, however, whether the sequence $G_0 , G_1 , G_2 , \cdots$ converges. In the next section a particular example from the theory of heat conduction is solved both by the series and sequence methods; the methods are compared and convergence is proved in both cases. In Sec. 5 the case of Eq. (1) is studied, and it is found that convergence is assured provided the initial function G_0 is chosen in a prescribed manner.

The solution procedures discussed above have been called series and sequence methods, depending on the form the solution most readily assumes. Clearly, however, each solution could be expressed in either form, and thus the two methods could be compared. If, for example, the third term of the sequence is referred to the original function G_0 , the following result is obtained:

$$\begin{aligned} G_2(P, Q) = G_0(P, Q) &- 2 \int_C G_0(P_1 , Q) \frac{\partial G_0(P, P_1)}{\partial n_{P_1}} ds_{P_1} \\ &+ 2 \int_C \int_C G_0(P_1 , Q) \frac{\partial G_0(P_2 , P_1)}{\partial n_{P_1}} \frac{\partial G_0(P, P_2)}{\partial n_{P_2}} ds_{P_2} ds_{P_1} \\ &- \int_C \int_C \int_C G_0(P_1 , Q) \frac{\partial G_0(P_2 , P_1)}{\partial n_{P_1}} \frac{\partial G_0(P_3 , P_2)}{\partial n_{P_2}} \frac{\partial G_0(P, P_3)}{\partial n_{P_3}} ds_{P_3} ds_{P_2} ds_{P_1} \end{aligned} \tag{9}$$

and similarly for all other terms of the sequence. Comparison with the partial sums of the series of Eq. (7) shows that there is no direct relationship between the two sequences, and thus the two methods represent distinct procedures of solution. A choice of methods is thus available, and the advantages of one over the other may depend on the particular problem under consideration. It is however reasonable to expect that the sequence method will exhibit faster convergence since it makes use at all times of the best approximation available rather than reverting at all times to the original guess G_0 . This expectation was fulfilled in the example of Sec. 4. where the sequence method is clearly preferable. The sequence method does not require the evaluation of multiple integrals; it furthermore provides an automatic check of the accuracy of any one approximation, since evaluation of the integral I_{ii} of Eq. (8a) requires calculation of the boundary values of G_i .

4. Example. As an illustration of the methods of the previous sections, a simple problem from the one-dimensional theory of heat conduction is solved here. It is required to find the Green's function for the semi-infinite solid $x > 0$, i.e. the temperature in that body at x at a time t, due to a unit source released at x_1 at a time t_1 . This function will be written as $G(x, x_1 , t - t_1)$; it is well known that its value is [4, p. 295]:

$$G(x, x_1 , t - t_1) = \frac{1}{2[\pi\kappa(t - t_1)]^{1/2}} \left\{ \exp\left[-\frac{(x - x_1)^2}{4\kappa(t - t_1)} \right] - \exp\left[-\frac{(x + x_1)^2}{4\kappa(t - t_1)} \right] \right\}, \tag{10}$$

where κ is the thermal diffusivity; this expression will be used as a check on the present procedures. The temperature $u(x, t)$ of a body bounded by a curve C, on which its value $u(x_c, t)$ is prescribed, is then [4, p. 294]:

$$u(x, t) = \kappa \int_0^t \left[u(x_1, t_1) \frac{\partial G(x_1, x, t - t_1)}{\partial x_1} \right]_{x_1 = x_c} dt_1, \tag{10a}$$

provided that the initial temperature $u(x, 0)$ is zero. This relation takes, in the present problem, the place of Eq. (2); the corresponding basic integral equation analogous to (4) is:

$$G(x, x_1, t) = G_0(x, x_1, t) - \kappa \int_0^t G_0(x_c, x_1, t_1) \frac{\partial G(x_c, x, t - t_1)}{\partial x_c} dt_1. \tag{11}$$

The initial guess G_0 will here be taken as

$$G_0(x, x_1, t - t_1) = \frac{1}{2[\pi\kappa(t - t_1)]^{1/2}} \exp\{-(x - x_1)^2/[4\kappa(t - t_1)]\}; \tag{11a}$$

that is, as the temperature in an infinite body at x at a time t due to a unit source released at x_1 at time t_1. Comparison of Eqs. (10) and (11a) shows that the difference between G and G_0 is the effect of a unit sink released at $(-x_1)$.

(*a*) *Solution by sequence method.* Proceeding as in Sec. 3, the first approximation G_1 is

$$\begin{aligned} G_1(x, x_1, t) &= G_0(x, x_1, t) - \frac{x}{8\kappa\pi} \int_0^t \frac{1}{[t_1(t - t_1)]^{3/2}} \exp\left\{-\frac{x_1^2}{4\kappa t_1} - \frac{x^2}{4\kappa(t - t_1)}\right\} dt_1 \\ &= \frac{1}{2[\pi\kappa t]^{1/2}} \left\{\exp\left[-\frac{(x - x_1)^2}{4\kappa t}\right] - \frac{1}{2}\exp\left[-\frac{(x + x_1)^2}{4\kappa t}\right]\right\}. \end{aligned} \tag{12}$$

The integral in Eq. (12) may be reduced to the known form

$$\frac{4}{\pi^{1/2}} \int_0^x \exp\left(-\tau^2 - \frac{a^2}{\tau^2}\right) d\tau = e^{-2a} \operatorname{erfc}\left(\frac{a}{x} - x\right) - e^{2a} \operatorname{erfc}\left(\frac{a}{x} + x\right) \tag{12a}$$

with the aid of the transformation

$$t_1 = \frac{4\kappa t^2 \tau^2}{x^2 + 4\kappa t\, \tau^2}. \tag{12b}$$

Inspection of G_1 shows that the first approximation differs from the correct result by the magnitude of the sink appearing at $(-x_1)$. Further approximations may now be calculated; it is found that the magnitude m_i of this sink in the ith approximation varies as follows:

$$m_0 = 0; \quad m_1 = (1/2); \quad m_2 = (7/8); \quad m_3 = (127/128); \cdots. \tag{12c}$$

The general term of this sequence is

$$m_i = 1 - (1/2)^{(2^i - 1)} \tag{12d}$$

and thus

$$\lim_{i \to \infty} m_i = 1 \tag{12e}$$

and the sequence converges to the correct value. It may be noted that convergence is extremely rapid, and furthermore that the calculation of any one approximation requires only the evaluation of an integral of the same form as that of Eq. (12a).

(*b*) *Solution by series method.* The first two terms of the series solution analogous to Eq. (7) are identical with those of Eq. (12); the third term has the form

$$\kappa^2 \int_0^t \int_0^{(t-t_1)} G_0(x_c\,, x_1\,, t_1)G_0(x_c\,, x, t_2)\left[\frac{\partial^2 G_0(x_3\,, x_2\,, t - t_1 - t_2)}{\partial x_3\,\partial x_2}\right]_{\left[\substack{x_2 = x_c \\ x_3 = x_c}\right]} dt_2\, dt_1\,. \quad (13)$$

The integrals resulting from substitution of G_0 into Eq. (13) reduce, with the aid of transformation (12b) either to the form of Eq. (12a) or to the form

$$\frac{4}{\pi^{1/2}} \int_0^\infty \tau^2 \exp\left(-\tau^2 - \frac{a^2}{\tau^2}\right) d\tau = (2a + 1)e^{-2a}. \quad (13a)$$

The final result is

$$G(x, x_1\,, t) = \frac{1}{2[\pi\kappa t]^{1/2}}\left\{\exp\left[-\frac{(x - x_1)^2}{4\kappa t}\right] - \left(0 + \frac{1}{2} + \frac{1}{4} + \cdots\right)\exp\left[-\frac{(x + x_1)^2}{4\kappa t}\right]\right\} \quad (13b)$$

which shows that the strength of the sink at $(-\,x_1)$ increases geometrically with each term. Convergence is thus assured, though it is not as rapid as that of Eq. (12c).

5. Convergence of the sequence method in the special case of Eq. (1). Equation (1) was chosen for this detailed investigation of the sequence method because of its importance in theoretical physics, and because its properties have been widely studied (see, for example, the comprehensive treatise of Bergman and Schiffer [1], which will be taken as a general reference for this section).

The singularity required of the Green's function $G(P, Q)$ of Eq. (1) as P approaches Q is a logarithmic infinity; it will further be specified that

$$\lim_{r\to 0} \int_{c_r} \frac{\partial G(P, Q)}{\partial n_P}\, ds_P = -1, \quad (14)$$

where c_r is a circle of radius r around Q. Functions which satisfy Eq. (14) and conditions (ii) and (iii) of Sec. 2 will be called fundamental singularities; between any two such functions, say $S_1(P, Q)$ and $S_2(P, Q)$ the relation

$$S_1(P, Q) - S_2(Q, P) = \int_C \left[S_1(R, Q)\,\frac{\partial S_2(R, P)}{\partial n_R} - S_2(R, P)\,\frac{\partial S_1(R, Q)}{\partial n_R}\right] ds_R \quad (14a)$$

must hold [1, p. 270]. Green's function is of course an example of a fundamental singularity, and it follows from Eq. (14a) by setting $S_1(P, Q) = G(P, Q)$ and $S_2(Q, P) = G(Q, P)$ that it is symmetric, in the case of the present (self-adjoint) differential equation, i.e.

$$G(P, Q) = G(Q, P). \quad (14b)$$

It is interesting to note that in the case of Eq. (1) the basic integral equation of the present method, Eq. (4), follows directly from Eqs. (14a) and (14b) by setting $S_1(P, Q) =$

$G_0(P, Q)$ and $S_2(Q, P) = G(Q, P)$ and transposing terms. Equation (6) can be similarly derived.

In order to examine the convergence of the sequence $G_0, G_1, G_2, \cdots$ defined by Eq. (8b), the following inductive reasoning will be employed. It will be first assumed that a particular member of the sequence, say G_k, has the following properties (in addition to those previously prescribed for all members of the sequence):

(i) G_k is continuous and twice continuously differentiable in a domain D^*, bounded by a curve C^*, which contains the given domain D; except possibly on the bounding curve C of D,

(ii) $G_k(P, Q) = 0$ for P on C^*, (15)

(iii) $G_k(P, Q) = G_k(Q, P)$. (15a)

It will then be shown (a) that the function G_{k+1} also enjoys properties (i), (ii) and (iii) above, and (b) that under these conditions the sequence $G_k, G_{k+1}, G_{k+2}, \cdots$ converges. If then the initial function G_0 is chosen so as to satisfy the above conditions, convergence of the sequence $G_0, G_1, G_2, \cdots$ will clearly be assured.

To prove statement (a) first, consider the identity

$$\begin{aligned} I_{kk}(P, Q) &= \int_C G_k(R, Q) \frac{\partial G_k(R, P)}{\partial n_R} \, ds_R \\ &= E_{D^*-D}\{G_k(R, P); G_k(R, Q)\} + \int_{C^*} G_k(R, Q) \frac{\partial G_k(R, P)}{\partial n_R} \, ds_R \end{aligned} \tag{16}$$

which follows from Green's theorem [1, p. 260]. The last integral of Eq. (16) vanishes because of Eq. (15), and the energy integral E, extended over the domain indicated in the subscript, is defined as follows:

$$\begin{aligned} E\{u(R, B); v(R, Q)\} = \iint \Bigg\{ &\frac{\partial u(R, P)}{\partial x_R} \frac{\partial v(R, Q)}{\partial x_R} \\ &+ \frac{\partial u(R, P)}{\partial y_R} \frac{\partial v(R, Q)}{\partial y_R} + q(R) u(R, P) v(R, Q) \Bigg\} \, dx_R \, dy_R \, . \end{aligned} \tag{16a}$$

It is thus clear that I_{kk} is symmetrical, i.e.

$$I_{kk}(P, Q) = I_{kk}(Q, P) \tag{17a}$$

and that therefore

$$I_{kk}(P, Q) = \int_C G_k(R, P) \frac{\partial G_k(R, Q)}{\partial n_R} \, ds_R = \int_C G_k(P, R) \frac{\partial G_k(R, Q)}{\partial n_R} \, ds_R \tag{17b}$$

and finally that

$$I_{kk}(P, Q) = 0 \quad \text{for} \quad P \text{ on } C^*. \tag{17c}$$

Inspection of Eqs. (17a, b, c) proves statement (a) above.

Before proceeding with the proof of statement (b), it is convenient to establish the following inequality

$$\sum_{i=1}^{N} \sum_{j=1}^{N} I_{kk}(P_i, P_j) x_i x_j \geq 0 \tag{18}$$

for any choice of points P_i and P_j in D and of real numbers x_i and x_j. The quadratic form in question is in fact identical with the expression

$$\sum_{i=1}^{N} \sum_{j=1}^{N} E_{D^*-D}\{G_k(R, P_i); G_k(R, P_j)\} x_i x_j = E_{D^*-D}\left\{\sum_{i=1}^{N} G_k(R, P_i)x_i \ ; \ \sum_{j=1}^{N} G_k(R, P_j)x_j\right\} \geq 0. \tag{18a}$$

It is furthermore shown in [1, p. 279] that if the function $q(x, y)$ of Eq. (1) is analytic in x and y throughout $D + C$, then the equality of Eq. (18) can hold (the trivial case $x_1 \equiv 0$ excepted) only if

$$I_{kk} = 0. \tag{18b}$$

Consider now the quantity I_{i+1}; it may be written [see Eqs. (8c-e)] as:

$$I_{i+1}(P, Q) = I_i(P, Q) - I_{ii}(P, Q) = \int_C G_i(R, P) \frac{\partial}{\partial n_R} [G(R, Q) - G_i(R, Q)]\, ds_R = -\int_C I_i(R, P) \frac{\partial I_i(R, Q)}{\partial n_R}\, ds_R = E_D\{I_i(R, P); I_i(R, Q)\}. \tag{19}$$

From the latter energy integral, by means of a reasoning analogous to that which led from Eq. (18a) to inequality (18), it follows that the quadratic form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} I_k(P_i , P_j)x_i x_j \ ; \qquad k \neq 0$$

is positive definite. It follows from Eqs. (8e) and (18) that the quadratic form associated with $(I_k - I_{k+1})$ is also positive definite, and that therefore the relation

$$\sum_{i=1}^{N} \sum_{j=1}^{N} I_k(P_i , P_j)x_i x_j > \sum_{i=1}^{N} \sum_{j=1}^{N} I_{k+1}(P_i , P_j)x_i x_j > 0; \qquad k \neq 0 \tag{19a}$$

holds. This implies that the sequence of the quadratic forms associated with I_k is a bounded decreasing sequence and therefore converges (and does so, in fact, monotonically). According to Cauchy's convergence theorem, then, corresponding to any given positive number ϵ, however small, there exists a number $m(m \geq k)$ such that the inequality

$$\left[\sum_{i=1}^{N} \sum_{j=1}^{N} I_m(P_i , P_j)x_i x_j\right] - \left[\sum_{i=1}^{N} \sum_{j=1}^{N} I_{m+p}(P_i , P_j)x_i x_j\right] < \epsilon \tag{20}$$

holds for all positive integral values of p. With the aid of Eq. (8c) this may be written as

$$\sum_{i=1}^{N} \sum_{j=1}^{N} [I_m(P_i , P_j) - I_{m+p}(P_i , P_j)]x_i x_j = \sum_{i=1}^{N} \sum_{j=1}^{N} [G_m(P_i , P_j) - G_{m+p}(P_i , P_j)]x_i x_j < \epsilon. \tag{20a}$$

This inequality must hold for arbitrarily selected x_i's; if they are all chosen zero except $x_1 = 1$, say, it follows that

$$G_m(P_1 , P_1) - G_{m+p}(P_1 , P_1) < \epsilon, \tag{21a}$$

where P_1 may designate any one point in the domain considered. Similarly, one may write

$$G_m(P_2, P_2) - G_{m+p}(P_2, P_2) < \epsilon \tag{21b}$$

for another point P_2. The quadratic forms in (20a) are however positive definite, so that their principal minors of order 2 must be positive; in other words:

$$\begin{aligned} &| G_m(P_1, P_2) - G_{m+p}(P_1, P_2) | \\ &\quad < [G_m(P_1, P_1) - G_{m+p}(P_1, P_1)]^{1/2}[G_m(P_2, P_2) - G_{m+p}(P_2, P_2)]^{1/2} < \epsilon. \end{aligned} \tag{21c}$$

Inequalities (21a) and (21c) then show that the conditions of Cauchy's theorem are everywhere satisfied by the sequence $G_k, G_{k+1}, \cdots$, and that therefore this sequence converges.

There now remains to select the initial function G_0 so as to satisfy the conditions listed earlier in this section. It is clear that this will be the case if G_0 is chosen as the Green's function of a domain D^* which includes D. This is probably the simplest choice, though by no means the only one; for example, the Neumann function of D^* will also be a suitable selection, though probably less efficient in view of the relation [1, p. 383]:

$$N(P, Q) \geq G(P, Q) \tag{22}$$

for any domain. The proof in this case is analogous to that just presented.

The above proof shows that convergence is assured, but it indicates that the limit reached need not necessarily be, from a general standpoint, the desired Green's function; for example, Neumann's function also satisfies Eq. (18b). This situation arises from the fact that in the set up of the sequence method, only the first two terms of the series expansion of Eq. (7) were employed; on the other hand, it appears impractical to apply the method on the basis of additional terms. This drawback of the sequence method is not serious, however, from the viewpoint of a calculation procedure. It is usually unlikely that any function but Green's will result, since the portion of the expansion (7) used pertains to Green's and no other function; and in the example of the previous section no such difficulty was encountered. It was mentioned in Sec. 3, moreover, that at each step of the sequence method a running check is automatically kept on the improvement obtained. Thus, if it should happen that at any one stage of the calculations convergence appears unsatisfactory, the work already performed is not wasted; the integral equation may be set up with the last (and still best) approximation available, and the series method applied to that.

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