be $v_{i,i} = 0$ and $\epsilon_{ijl}v_jv_l(v_{i,j} + v_{j,i}) = 0$ with the transformation

$$\sigma_{ij} = \left(\frac{K}{2} \ln v_k^2 + C\right)\delta_{ij} + \frac{K}{v_k^2} v_i v_j .$$

(v) *Plane stress of a Mises solid.* The yield criterion is $3I_2 - I_1^2 = 2K^2$. Substitution from (5) and (6) shows that $M$ and $N$ satisfy

$$M^2 + MNv_k^2 + N^2(v_k^2)^2 = K^2.$$


## References

1. N. Inoue, 1952, *J. Phys. Soc. Japan*, **7**, 119 and 518 (1952); *J. Aero. Sci.*, 19, 783 (1952); *Doshisha Eng. Review*, Special Paper No. 1 (1952).
2. R. Hill, *J. Mech. Phys. Solids* 2, 110 (1954).
3. Ernst Preiswerk, Nat. Advisory Comm. Aeronaut. Tech. Mem. No. 934, (1940).


# PHYSICAL INTERPRETATION OF PHYSICAL COMPONENTS OF STRESS AND STRAIN*

By DANIEL FREDERICK (*Virginia Polytechnic Institute*)

**1. Introduction.** The purpose of this paper is to give a physical interpretation to the several different sets of physical components of stress and strain which have been introduced in the literature by different authors and to present and interpret some new sets. The question of the physical interpretation of the physical components of stress and strain arises when one expresses the equations of mechanics involving these in a general coordinate system through the use of tensor calculus. Since the tensor form of the equations can be obtained readily, it is of practical interest to have a direct physical interpretation of the physical components which are related to the tensor quantities.

The reader is assumed to be familiar with the elements of tensor calculus as presented in the book by Synge and Schild [1] and their basic notation will be used here. All quantities are referred to the ordinary physical space which is flat and has a positive definite metric form.

**2. The physical components of the stress tensor.** These will be introduced through the equations of equilibrium using the four sets of components of the stress vectors which act on the faces of an infinitesimal curvilinear tetrahedron associated with the covariant (contravariant) base vectors $\bar{g}_i$ ($\bar{g}^i$). When these equilibrium equations are compared with corresponding tensor equations, four different sets of physical components arise for each triad.

(*a*) *Physical components of a vector.* The four sets of physical components associated with a vector $\bar{Z}$ in general coordinates $x^r$ will be divided into two groups. The first are called orthogonal components, since they are obtained by projecting the vector orthogonally onto the base vectors, and will be denoted by the capital letter $O$ to the left of the base letter. Those components along the covariant or contravariant base triads which add by the vector or parallelogram law to give the original vector are called parallelogram components. They are marked with the capital letter $P$ to the left of the base letter. Used as subscripts (superscripts) these letters refer to components along

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the covariant (contravariant) base triad. The relations between the physical components $X_{(i)}$ and the tensor components $X_i$ , $X^i$ are shown in Table I. These were given by Ricci and Levi-Civita [2] in 1901. For orthogonal coordinates where the metric tensor $a_{ij} = 0$, for $i \neq j$, the four sets are identically the same.

TABLE I*

| Covariant base triad | Contravariant base triad |
|---|---|
| ${}_PX_{(i)} = X^i(a_{ii})^{1/2}$ | ${}^PX_{(i)} = X_i(a^{ii})^{1/2}$ |
| ${}_0X_{(i)} = X_i(a_{ii})^{-1/2}$ | ${}^0X_{(i)} = X^i(a^{ii})^{-1/2}$ |

Also, the $X_{(i)}$ components are shown in Figure 1 for plane $x^r$ coordinates. Note: Unless otherwise signified the range convention is assumed to operate for all indices.

(*b*) *Notation for the physical components of stress.* Since the stress vectors associated with either triad can be resolved into the four components described above, it is possible

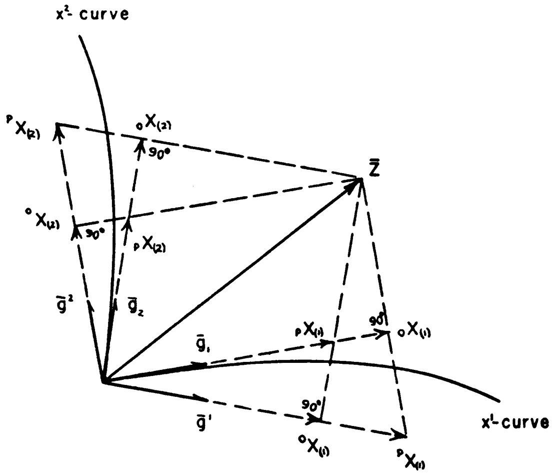


FIG. 1. Physical components of a vector in plane $x^r$ coordinates.

to introduce eight distinct sets of physical components of stress. The four sets associated with the covariant (contravariant) base triad $T_B(T^B)$ will be denoted by placing $(ij)$ as subscripts (superscripts); while the letters $O$ and $P$ to the left of the base letter $T$ have the meanings described above.

(*c*) *The* ${}_PT_{(i)}$ *components.* This set of physical components arises upon resolving the stress vectors associated with the triad $T_B$ into parallelogram components along the covariant base directions. In order to relate these to a stress tensor, the equation of force equilibrium along the $x^k$ curve will be written and compared with a corresponding tensor equation.

To carry this out, consider the infinitesimal tetrahedron associated with the triad $T_B$ shown in Figure 2, where the face perpendicular to the unit normal vector $\bar{n}$ will

*The summation convention is not used here.

be $dS$ and the face normal to $\bar{g}^i$ will be $dS_{(i)}$. The stress vector on the face $dS[dS_{(i)}]$ is $\bar{t}[\bar{t}_{(i)}]$. These elemental areas are related by the formula

$$dS_{(i)} = n_i(a^{ii})^{1/2}dS. \tag{1}$$

Also, consider the components of force ${}_PT_{(ij)}\, dS_{(i)}$ on the face $dS_{(i)}$. These three forces, for $j = 1, 2, 3$, are projected onto the $x^k$ curve by multiplying by the cosine of

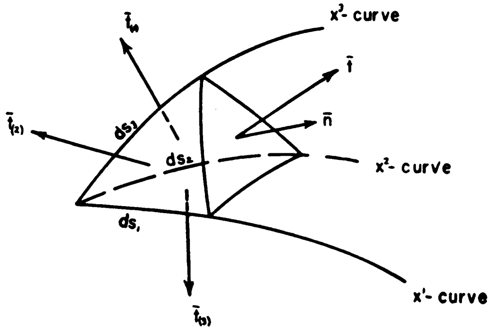


FIG. 2. The curvilinear tetrahedron with faces normal to the contravariant base vectors.

the angle between the $x^i$ and $x^k$ directions which is $a_{kj}\ (a_{kk}a_{jj})^{-1/2}$ and summing over $j$. The result is

$$\sum_{j=1}^{3} {}_PT_{(ij)}dS_{(i)}a_{kj}(a_{kk}a_{jj})^{-1/2}. \tag{2}$$

Summing over $i$ in Eq. (2) gives the sum of the force components on all three faces $dS_{(i)}$ in the direction of the $x^k$ curve and this must be equal to the component of force on the face $dS$ in the $x^k$ direction, which is ${}_0t_{(k)}\ dS$. Hence

$$\sum_{i,j=1}^{3} {}_PT_{(ij)}dS_{(i)}a_{kj}(a_{kk}a_{jj})^{-1/2} = {}_0t_{(k)}dS. \tag{3}$$

Using Table I and Eq. (1), this becomes

$$t_k = \sum_{i,j=1}^{3} {}_PT_{(ij)}a_{kj}(a^{ii}/a_{jj})^{1/2}n_i\ . \tag{4}$$

Comparing this with the tensor equation

$$t_k = T^i_{.k}n_i \tag{5}$$

yields a relation between the mixed tensor components and the ${}_PT_{(ij)}$ physical components

$$T^i_{.k} = \sum_{j=1}^{3} {}_PT_{(ij)}a_{kj}(a^{ii}/a_{jj})^{1/2}. \tag{6}$$

By raising the index $k$, there results

$$T^{ij} = {}_PT_{(ij)}(a^{ii}/a_{jj})^{1/2}. \tag{7}$$

This is tabulated in Table II with the seven other relations which can be found by a similar procedure. The numbers refer to the reference where the relation was given.

TABLE II

| Covariant base triad | Contravariant base triad |
|---|---|
| $T^{ij} = {}_PT_{(ij)}(a^{ii}/a_{jj})^{1/2}$ [3][4] | $T^{i}_{\cdot j} = {}_PT^{(ij)}(a_{jj}/a_{ii})^{1/2}$ [5] |
| $T^{i}_{\cdot j} = {}_0T_{(ij)}(a^{ii}a_{jj})^{1/2}$ | $T_{ij} = {}_0T^{(ij)}(a_{ii}a_{jj})^{1/2}$ |
| $T^{i}_{\cdot j} = {}^PT_{(ij)}(a^{ii}/a_{jj})^{1/2}$ | $T_{ij} = {}^PT^{(ij)}(a_{ii}/a_{jj})^{1/2}$ |
| $T^{ij} = {}^0T_{(ij)}(a^{ii}a^{jj})^{1/2}$ [1] | $T^{i}_{\cdot j} = {}^0T^{(ij)}(a^{ii}a_{jj})^{1/2}$ |

For the special case of orthogonal coordinates these eight sets of physical components become identical.

**3. Physical components of infinitesimal strain.** The material of this paragraph is presented in a form which presumes that the reader is familiar with the notation and presentation of strain as given by Green and Zerna [4]. Assuming infinitesimal strains, these authors present the physical components

$$\gamma_{(ii)} = \frac{\gamma_{ii}}{a_{ii}} \tag{8}$$

and, for $i \neq j$,

$$\gamma_{(ij)} = [2\gamma_{ij} - a_{ij}(\gamma_{(ii)} + \gamma_{(jj)})][a_{ii}a_{jj} - (a_{ij})^2]^{-1/2}. \tag{9}$$

In these equations $\gamma_{ij}$ is the covariant strain tensor and $a_{ij}$ is the metric for the undeformed state. The physical components $\gamma_{(ii)}$ are changes of length per unit of length for line elements along the coordinate curves, while $\gamma_{(ij)}$ is the change of angle between the covariant base vectors $\bar{g}_i$ and $\bar{g}_j$ .

Another useful set of physical components of infinitesimal strain can be obtained by considering the line elements along and the angles between the contravariant base vectors. Following the procedure used by Green and Zerna for $\gamma_{(ii)}$ and $\gamma_{(ij)}$ , the final expressions for the corresponding strains $\gamma^{(ii)}$ and $\gamma^{(ij)}$ are the following:

$$\gamma^{(ii)} = \frac{\gamma^{ii}}{a^{ii}} \tag{10}$$

$$\gamma^{(ij)} = [2\gamma^{ij} - a^{ij}(\gamma^{(ii)} + \gamma^{(jj)})][a^{ii}a^{jj} - (a^{ij})^2]^{-1/2}. \tag{11}$$

For orthogonal coordinates $\gamma^{(ii)} = \gamma_{(ii)}$ and $\gamma^{(ij)} = \gamma_{(ij)}$ .

**4. Conclusions.** For stress (strain), there are eight (two) convenient sets of physical components which might be used for solving boundary value problems in elasticity involving general coordinates.

Although the results are presented for a triad tied to the $x^r$ coordinates, the same ideas can be extended to a general point using the theory of non-holonomic coordinates as explained by Truesdell [6].

BIBLIOGRAPHY

1. J. L. Synge and A. Schild, *Tensor calculus*, 2nd ed., University of Toronto Press, Toronto, 1952, pp. 142–146
2. G. Ricci and T. Levi-Civita, *Methodes de calcul differential absolu et leurs applications*, Math. Ann. **54**, 125–201 (1901)
3. H. Neuber, *Die Grundgleichungen der elastischen Stabilität in allgemeinen Koordinaten und ihre Integration*, Z. angew. Math. u. Mech. (6) **23**, 321–330 (1943)
4. A. E. Green and W. Zerna, *Theoretical elasticity*, Oxford at the Clarendon Press, 1954, pp. 18–22, 57–61
5. C. Truesdell, *The physical components of vectors and tensors*, Z. angew. Math. u. Mech. (10/11) **33**, 345–356 (1953)
6. C. Truesdell, *Remarks on the paper, "The physical components of vectors and tensors"*, Z. angew. Math. u. Mech. (1/2) **34**, 69–70 (1954)


# A NOTE ON THE DECOMPOSITION OF STRESS AND STRAIN TENSORS*

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A well-known theorem of vector analysis states that an arbitrary vector field can be decomposed into the sum of two fields, one of which is irrotational and the other solenoidal. Analogous theorems for stress and strain tensor fields are presented in this note.

Theorem 1. A symmetrical (stress) tensor $\sigma_{ij}$ defined in the region $V$ bounded by the surface $S$ may be written

$$\sigma_{ij} = \sigma'_{ij} + \sigma''_{ij} ,$$

where $\sigma'_{ij}$ and $\sigma''_{ij}$ have the following properties:

(a) $\sigma'_{ij,i} = -f_i^0$ in $V$,
$\sigma'_{ij}n_i = T_i^0$ on a portion $S_B$ of $S$,
where $f_i^0$ and $T_i^0$ are prescribed[1], and $n_i$ is the unit normal to $S$.

(b) the (strain) tensor $\epsilon''_{ij}$ derived from $\sigma''_{ij}$ by the Hookean relation

$$\epsilon''_{ij} = L(\sigma''_{ij}) = \frac{1}{E}\left[(1+\nu)\sigma''_{ij} - \nu\sigma''_{kk}\delta_{ij}\right]$$

is related to some (displacement) vector field $u''_i$ by

$$\epsilon''_{ij} = \tfrac{1}{2}(u''_{i,j} + u''_{j,i}),$$

where $u''_i$ takes on the prescribed value $u_i^0$ on $S_A = S - S_B$ .

In other words, the theorem states that any stress field can be decomposed into the sum of two fields, one of which obeys prescribed conditions on internal and surface equilibrium and the other of which provides (through Hooke's law) strains that satisfy internal and surface compatibility requirements. The analogy with the corresponding result for vector fields is evident: the equilibrium condition involves the divergence of the stress tensor (a solenoidal vector is divergence-free) and compatibility requires that

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[1]Here $f_i^0$ and $T_i^0$ have the character of body force and surface traction, so that if $S_B$ is the entire surface, they must be so chosen as to satisfy overall static equilibrium.