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NOTE ON THE EQUATIONS OF SHALLOW ELASTIC SHELLS*

BY P. M. NAGHDI (*University of Michigan*)

In the following, a system of differential equations is deduced for thin shallow elastic shells (of uniform thickness) with small displacements, which include the effect of transverse shear deformation. The corresponding equations of the classical theory, where the effect of transverse shear deformation is neglected, are contained in the works of Marguerre [1] and Green and Zerna [2], and have also been employed recently by E. Reissner [3, 4]. Although the equations sought may be obtained (through appropriate approximations) from the results given in [5], considerable space will be conserved if the basic equations of the theory are referred to Cartesian coordinates.

Let the equation of the middle surface of the shell be written in the form $z = z(x_1, x_2)$ and let:** w denote the displacement in the z-direction and u_i and β_i be respectively the displacements of the middle surface and the change of the slope of the normal to the middle surface, along the Cartesian coordinate axes x_1 and x_2. With the notation N_{ij}, M_{ij}, and V_i, for the stress resultants, stress couples, and transverse stress resultants, respectively, the stress differential equations of equilibrium are

$$N_{ij,j} + p_i = 0, \tag{1a}$$

$$V_{i,i} + [z_{,j}N_{ij}]_{,i} + q = 0, \tag{1b}$$

$$V_i = M_{ij,j}, \tag{1c}$$

where comma denotes partial differentiation and p_1, p_2, and q are the components of the load intensity in x_1, x_2, and z directions, respectively.

The stress strain relations, which include the effect of transverse shear deformation, may be written as

$$\begin{aligned} \epsilon_{ij} &= \tfrac{1}{2}[(u_{i,j} + u_{j,i}) + (z_{,j}w_{,i} + z_{,i}w_{,j})] \\ &= \frac{1}{C}[-\nu N_{kk}\delta_{ij} + (1+\nu)N_{ij}], \end{aligned} \tag{2a}$$

$$M_{ij} = \tfrac{1}{2}D[2\nu\beta_{k,k}\delta_{ij} + (1-\nu)(\beta_{i,j} + \beta_{j,i})], \tag{2b}$$

$$\beta_i = -w_{,i} + \frac{6}{5Gh}V_i, \tag{2c}$$

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**Throughout this note, the Latin indices i, j, and k have the range 1 and 2 only. Repeated indices imply summation over all the values the indices may take. Whenever one of the repeated indices is placed in parentheses [as in Eq. (3)], the summation convention is suspended for that index.

where ϵ_{ij} are the components of strain at the middle surface; δ_{ij} is the Kronecker delta; h denotes the shell thickness and is assumed to be uniform; E, G, and ν are Young's modulus, shear modulus, and Poisson's ratio, respectively; $C = Eh$ and $D = Eh^3/[12(1 - \nu^2)]$.

Introduction of Airy's stress function F into Eqs. (1a) by the relations

$$N_{ij} = \left(\nabla^2 F - \int p_{(i)}\, dx_{(i)}\right)\delta_{ij} - F_{,ij}\,, \tag{3}$$

where $\nabla^2 \equiv \partial^2/\partial x_i \partial x_i$, and their substitution into the relevant compatibility equation yields

$$\nabla^2\nabla^2 F = C\{2z_{,12}w_{,12} - z_{,11}w_{,22} - z_{,22}w_{,11}\} + \left\{\left[\int p_1\, dx_1\right]_{,22} + \left[\int p_2\, dx_2\right]_{,11} - \nu p_{i,i}\right\}, \tag{4}$$

which in various forms may be found in [1], [2], and [3].

From (2b, c) and (1c), there follows

$$\tfrac{1}{2}D\{(1 - \nu)\nabla^2\beta_i + (1 + \nu)\beta_{j,ij}\} - \tfrac{5}{6}Gh(\beta_i + w_{,i}) = 0, \tag{5}$$

which can also be expressed in terms of V_i and w. Next, substitution of (2b) into (1b, c) with the aid of (2c) and (3) results in

$$D\nabla^4 w - (1 - \lambda\nabla^2)[(\nabla^2 z)\nabla^2 F - z_{,ij}F_{,ij}] = (1 - \lambda\nabla^2)q - \left\{\left[z_{,1}\int p_1\, dx_1\right]_{,1} + \left[z_{,2}\int p_2\, dx_2\right]_{,2}\right\}, \tag{6}$$

where $\lambda = h^2/5(1 - \nu)$ and $\nabla^4 = \nabla^2\nabla^2$. It may be noted that elimination of the function F between (4) and (6) yields an eight-order partial differential equation in w alone, namely

$$D\nabla^4\nabla^4 w - (1 - \lambda\nabla^2)\left[(\nabla^2 z)\nabla^2 - z_{,ij}\frac{\partial^2}{\partial x_i\, \partial x_j}\right][\text{Right side of Eq. (4)}] = \nabla^4[\text{Right side of Eq. (6)}], \tag{7}$$

which for cylindrical shells, upon the neglect of transverse shear deformation ($\lambda = 0$), reduces to the corresponding equation given by Donnell [6].

Equations (4), (5), and (6) constitute the required differential equations for shallow shells. An alternative formulation in terms of displacements may sometimes be preferable; this is achieved by reducing the system of Eqs. (1) and (2) to five partial differential equations in u_i, β_i, and w. Thus, in addition to (5) we have

$$\frac{C}{2(1 + \nu)}\left\{\left(\nabla^2 u_i + \frac{1 + \nu}{1 - \nu}u_{j,ij}\right) + \left[(w_{,i}\nabla^2 z + z_{,i}\nabla^2 w) + \frac{1 + \nu}{1 - \nu}(z_{,ij}w_{,j} + z_{,j}w_{,ij})\right]\right\} + p_i = 0 \tag{8}$$

and

$$
\begin{aligned}
D\nabla^4 w - (1 - \lambda\nabla^2)\left(\frac{C}{1-\nu^2}\right)\Big\{\nu(\nabla^2 z)u_{k,k} + \frac{1-\nu}{2}\,[z_{,i}(\nabla^2 u_j \\
+ z_{,j}\nabla^2 w) + z_{,ij}(u_{i,j} + u_{j,i})] + \frac{1+\nu}{2} z_{,i}[(\nabla^2 z)w_{,i} + u_{i,jj} \\
+ z_{,ij}w_{,i} + z_{,i}w_{,ij}] + (1-\nu)z_{,i}z_{,ij}w_{,j}\Big\} = [\text{Right side of Eq. (6)}].
\end{aligned}
\tag{9}
$$

In conclusion, it may be mentioned that as in [5] for vibration problems of shallow shells, the effect of rotatory inertia can be easily added to the above equations.

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ON THE ERRORS IN ANALOGUE SOLUTIONS OF HEAT CONDUCTION PROBLEMS*

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1. Introduction. Many practical heat conduction questions lead to problems not amendable to the methods of classical mathematical physics. Consequently, numerous methods for approximating the solution have been developed; many of these procedures result from the replacement of the differential equation by either finite difference equations, which are usually solved by means of a digital computer, or by systems of ordinary differential equations, which are often treated using an analogue device of some kind. Our interest will be confined to the latter of these types.

Useful analogue machines for the solution of heat conduction problems include differential analyzers [3, 4] and resistance-capacitance circuits [1, 6]. In either case, the derivatives with respect to one variable are retained and all others are replaced by finite differences, and the resulting system of ordinary differential equations is solved. For the heat flow equation in one space variable, it is possible to retain either the space derivative or the time derivative; however, keeping the time derivative is much more desirable from a least work point of view [4]. With more than one space variable, this is the only approach that can treat the space variables in a symmetric manner.

The replacement of the space derivatives by finite differences results in an error in

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