# CHARACTERISTIC VALUES OF ARBITRARY MATRICES*

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**1. Introduction.** The determination of the characteristic values of matrices is of paramount importance in many branches of the applied sciences, and consequently, much attention is being given to the improvement of existing relevant methods as well as to the development of new ones.

The method described below for the determination of the characteristic values of arbitrary matrices possessing complex elements, aside from being new, is believed to be of promise. It is based on a theorem of I. Schur which asserts that any matrix of complex elements can be reduced to triangular form by means of sequence of unitary transformations which generate the characteristic values of the matrix along the diagonal of the triangularized form; see, for example, MacDuffee [1]. In applying this theorem, as described in detail below, the triangular part of the matrix initially possessing the lower norm is chosen for annihilation. Then an arbitrary element in that triangle, e.g. an element of above-average absolute value, is selected as the pivot of a certain unitary transformation which is determined in such a manner as to reduce the total norm of the triangle. The construction of the unitary transforms having this norm reducing property is founded on the real roots of associated cubic polynomials with properly determined coefficients.

The entire process is iterated until the norm has been decreased to a desired tolerance.

In the following sections, then, the effect of the transformation upon the matrix is investigated, and the transformed elements are exhibited individually. In Sec. 3 the behavior of hermitian and skew-hermitian matrices is considered briefly. The construction of the norm reducing cubic is discussed in Sec. 4. The remaining two sections are devoted to the applicability of the proposed method to high speed computing machinery, and to a simple example illustrating the procedure.

**2. The unitary transformations.** Let us consider the $n$-dimensional matrix of complex elements

$$A = \begin{bmatrix} a_{11} & & \cdots & & a_{1n} \\ & \ddots & & & \\ \cdot & a & \cdots & b & \cdot \\ \cdot & \vdots & \ddots & \vdots & \cdot \\ \cdot & c & \cdots & d & \cdot \\ & & & \ddots & \\ a_{n1} & & \cdots & & a_{nn} \end{bmatrix}. \tag{1}$$

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Here the elements $a$, $b$, $c$, $d$, are, respectively, in the (ii), (ij), (ji), and (jj) position of $A$. It is assumed here that the "upper" triangle of $A$ is to be annihilated. The element $b$ will be called the "pivot" of the transformation. As shown below the matrix $A$ can be reduced to triangular form by an infinite sequence of unitary transformations $T_p$, $p = 0, 1, 2, \cdots$, so that

$$A_{p+1} = T_p^* A_p T_p', \qquad A_0 \equiv A,$$

where $T_p^* = \overline{T}_p'$, denotes the conjugate transpose of $T_p$. As $p$ grows beyond bound the diagonal elements of $A_p$ then should approach the characteristic values of $A$. We choose the transforms to be of the type

$$T = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \ddots & & & \\ \cdot & t & \cdot \quad \cdot & -(1-r^2)^{1/2} & \cdot \\ & \cdot & \ddots & \cdot & \\ \cdot & \cdot & 1 & \cdot & \cdot \\ & \cdot & & \ddots \quad \cdot & \cdot \\ \cdot & (1-r^2)^{1/2} & \cdot & \cdot \quad \bar{t} & \cdot \\ & & & \ddots & \\ 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix} \qquad (2)$$

with

$$t = r \exp(i\theta), \qquad r \text{ real.}$$

Thus $T$ contains $t$, $-(1 - r^2)^{1/2}$, $(1 - r^2)^{1/2}$, and the complex conjugate $\bar{t}$ of $t$ in the positions corresponding to $a$, $b$, $c$, $d$, and agrees with the unit matrix everywhere else.[1] The number $t$ will be appropriately determined later. Clearly $T$ is unitary: $T^{-1} = T^*$, and

$$T^* = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \ddots & & & \cdot \\ \cdot & \bar{t} & \cdot \quad \cdot & (1-r^2)^{1/2} & \cdot \\ \cdot & \cdot & \ddots & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & & \ddots \quad \cdot & \cdot \\ \cdot & -(1-r^2)^{1/2} & \cdot & \cdot \quad t & \cdot \\ & & & \ddots & \\ 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}. \qquad (3)$$

[1]This transformation was suggested by S. Pines, Republic Aviation Corporation.

Let us now form

$$A_1 = T^{-1}AT \equiv (a_{ij}^{(1)}).$$

For the sake of simplicity the subscripts "0" have been deleted here. In calculating $A_1$ it is convenient to partition $T$ as

$$T = \begin{bmatrix} I_{i-1} & O_{i-1,j-i+1} & O_{i-1,n-j} \\ O_{j-i+1,i-1} & S_{j-i+1} & O_{j-i+1,n-j} \\ O_{n-j,i-1} & O_{n-j,j-i+1} & I_{n-j} \end{bmatrix} \tag{4}$$

with $I_k$ denoting a unit matrix of $k$ dimensions, $0_{r,s}$ denoting a rectangular matrix of zeros having $r$ rows and $s$ columns, and, finally, $S$ denoting a square matrix of $j - i + 1$ dimensions. Similarly, let us denote the corresponding parts of $A$ by

$$A = \begin{bmatrix} B & C & D \\ E & F & G \\ H & J & K \end{bmatrix}, \tag{5}$$

having, respectively, the dimensions

$B$: $(i - 1) \times (i - 1)$ $C$: $(i - 1) \times (j - i + 1)$ $D$: $(i - 1) \times (n - j)$

$E$: $(j - i + 1) \times (i - 1)$ $F$: $(j - i + 1) \times (j - i + 1)$ $G$: $(j - i + 1) \times (n - j)$

$H$: $(n - j) \times (i - 1)$ $J$: $(n - j) \times (j - i + 1)$ $K$: $(n - j) \times (n - j)$.

It is then observed that

$$A_1 = \begin{bmatrix} B & CS & D \\ S^{-1}E & S^{-1}FS & S^{-1}G \\ H & JS & K \end{bmatrix}. \tag{6}$$

The elements of the central part $S^{-1} FS$ of $A_1$ are thus seen to be constituted as follows: Let

$$R = (1 - r^2)^{1/2} \cdot r^{-1}, \qquad z = R \exp (i\theta). \tag{7}$$

Then

$$a_1 = r^2[a + R^2d + zc + \bar{z}b], \tag{8}$$

$$b_1 = t^2[b - z^2c + z(d - a)], \tag{9}$$

$$c_1 = t^2[c - \bar{z}^2b + \bar{z}(d - a)], \tag{10}$$

$$d_1 = r^2[d + R^2a - zc - \bar{z}b], \tag{11}$$

for the corner elements of $S^{-1}FS$.

The other transformed elements occur only in the $i$th and $j$th rows and columns of the affected components of $A_1$, and are obtained as follows:

(i) $i$th row: $$a_{ik}^{(1)} = \bar{t}(\ {}_{ik} + za_{jk}), \tag{12}$$

(ii) $j$th row: $$a_{jk}^{(1)} = t(a_{jk} - \bar{z}a_{ik}), \tag{13}$$

(iii) $i$th column: $$a_{ki}^{(1)} = t(a_{ki} + \bar{z}a_{kj}), \tag{14}$$

(iv) $j$th column: $$a_{kj}^{(1)} = \bar{t}(a_{kj} - za_{ki}). \tag{15}$$

The range of $k$ is given by:

For $CS$: Formulas (14) and (15) for $I_1 : 1 \leq k \leq i - 1$,
For $JS$: Formulas (14) and (15) for $I_2 : j + 1 \leq k \leq n$,
For $S^{-1}E$: Formulas (12) and (13) for $I_1$,
For $S^{-1}G$ Formulas (12) and (13) for $I_2$,
For $S^{-1}FS$: Formulas (12) through (15) for $I_3 : i + 1 \leq k \leq j - 1$.

In addition to the relationships derived above it must also be remembered that unitary transformations leave trace and norm invariant:

$$\sum_{i=1}^{n} a_{ii}^{(p)} = \text{constant}, \qquad \sum_{i,j=1}^{n} | a_{ij}^{(p)} |^2 = \text{constant}. \tag{16}$$

The determination of suitable choices of real-valued $R$, $\theta$ is discussed in Sec. 4. Once $R$, $\theta$ are known the quantities $r$, $t$ are obtained as

$$r = \pm(1 + R^2)^{-1/2}, \qquad t = r \exp(i\theta). \tag{17}$$

The sign of $r$ does not affect the new corner elements $a_1$, $b_1$, $c_1$, $d_1$. However, the signs of the other elements of $A_1$ are changed if the sign of $r$ is changed. To fix matters it is advisable to assign to $r$ the same sign as is possessed by $R$. It is seen that $| r | < 1$ for $R \neq 0$, while for $R = 0$ also $z = 0$, and $r = 1$. In the latter case the transformation leaves the diagonal elements unchanged, rendering zero values of $R$ of no interest.

**3. Hermitian and skew hermitian matrices.** Let us pause to consider the effect of the transformation $T$ upon hermitian and skew hermitian matrices.

*A. The matrix A is hermitian.* Here $a_{ki} = \bar{a}_{ik}$ for all $i$ and $k$. Thus the diagonal elements are real, and $c = \bar{b}$. Equation (8) becomes

$$a_1 = r^2[a + R^2d + z\bar{b} + \bar{z}b],$$

so that $a_1$ is real if $R$ is real, which will be assumed in the following. Equation (11), or the fact that $a_1 + d_1 = a + d$, shows the diagonal element $d_1$ also to be real. Equations (9) and (10) reveal that $c_1 = \bar{b}_1$, and the other transformations (12) through (15) indicate that $a_{ki}^{(1)} = \bar{a}_{ik}^{(1)}$, in general. Thus $A_1$, and, similarly, $A_p$ is also hermitian.

*B. The matrix A is skew hermitian.* In this case $a_{ki} = -\bar{a}_{ik}$ for $i, k = 1, 2, \cdots, n$, so that $a_{ii} = 0$ for all $i$, $c = -\bar{b}$. Equation (8) leads to

$$a_1 = r^2[zc - \overline{zc}],$$

making $a_1$ a pure imaginary number, or zero. Since $a_1 + d_1 = 0$, $d_1 = -a_1$ is also purely imaginary. Further, it turns out that $c_1 = -\bar{b}_1$, and $a_{ki}^{(1)} = -\bar{a}_{ik}^{(1)}$, $i \neq k$, in general.

Continuing with the second iteration we find that

$$a_2 = r^2[(1 - R^2)a_1 + \bar{z}b_1 - z\bar{b}_1],$$
$$d_2 = -a_2 .$$

Thus $a_2$, $d_2$ are also pure imaginaries. Next, $c_2 = -\bar{b}_2$, and $a_{ki}^{(2)} = -\bar{a}_{ik}^{(2)}$. The successive transformed matrices $A_p$ all show the same behavior, namely, that the skew hermitian character is preserved except for the elements of the diagonal which become pure imaginaries.

The preservation of the hermitian and quasi skew hermitian nature of the transformed matrices is clearly of importance in that it essentially halves the amount of computational labor involved in the entire calculation.

As has been shown by the author elsewhere, hermitian and skew hermitian matrices can be diagonalized by unitary transformations of the type (2) by the annihilation of the pivot.

**4. Construction of the norm reducing cubic.** It is desired to construct the variable $t$ in the transformation matrix $T$ in such a manner that the sum of the squares of the absolute values of all the super-diagonal elements in the transformed matrix $A$ is decreased. Now this sum, for the affected elements of the original matrix $A$, is

$$\begin{aligned} M &= \sum_1 (|a_{ki}|^2 + |a_{kj}|^2) + \sum_2 (|a_{ik}|^2 + |a_{jk}|^2) \\ &\qquad + \sum_3 (|a_{ik}|^2 + |a_{kj}|^2) + |b|^2 \qquad (18) \\ &\equiv \sum_1 + \sum_2 + \sum_3 + |b|^2, \end{aligned}$$

and the summations are to be extended over $I_i$ for $\sum_i$, $i = 1, 2, 3$, with the intervals $I_i$ as defined in Sec. 2. For the transformed matrix $A_1$ the corresponding sum is

$$M_1 = \sum_1^{(1)} + \sum_2^{(1)} + \sum_3^{(1)} + |b_1|^2, \qquad (19)$$

where the superscripts "1" indicate that the $a_{st}$ are to be replaced by the corresponding $a_{st}^{(1)}$.

Now the transformation formulas, together with the fact that $r^2(1 + R^2) = 1$, lead to

$$\sum_1^{(1)} = \sum_1, \qquad \sum_2^{(1)} = \sum_2,$$

$$\begin{aligned} \sum_3^{(1)} &= r^2 \sum_3 + r^2R^2 \sum_3 (|a_{jk}|^2 + |a_{ki}|^2) \\ &\qquad + r^2 \sum_3 [(za_{jk}\bar{a}_{ik} + \bar{z}\bar{a}_{jk}a_{ik}) - (za_{ki}\bar{a}_{kj} + \bar{z}\bar{a}_{ki}a_{kj})]. \end{aligned}$$

Consequently,

$$\begin{aligned} M_1 - M &= (r^2 - 1) \sum_3 + r^2R^2 \sum_3 (|a_{jk}|^2 + |a_{ki}|^2) \\ &\qquad + 2r^2R \sum_3 [|a_{jk}| \, |a_{ik}| \cos(\theta + \alpha_{jk} - \alpha_{ik}) \\ &\qquad - |a_{ki}| \, |a_{kj}| \cos(\theta + \alpha_{ki} - \alpha_{kj})] + |b_1|^2 - |b|^2, \end{aligned}$$

where in general $a_{ij} = |a_{ij}| \exp(i\alpha_{ij})$.

Next it is found that

$$\begin{aligned} |b_1|^2 = r^4\{|b|^2 + R^4|c|^2 + R^2|d-a|^2 - 2R^2|b||c|\cos(2\theta-\beta+\gamma) \\ + 2R|b|[|d|\cos(\theta+\delta-\beta) - |a|\cos(\theta+\alpha-\beta)] \\ - 2R^3|c|[|d|\cos(\theta+\gamma-\delta) - |a|\cos(\theta+\gamma-\alpha)]\}, \end{aligned} \tag{20}$$

where again $a = |a|\exp(i\alpha)$, etc.

For $|r| < 1$, then,

$$M_1 - M < RF(R, \theta),$$

where

$$F(R, \theta) = C_3R^3 + C_2R^2 + C_1R + C_0, \tag{21}$$

$$C_3 = |c|^2 \tag{22}$$

$$C_2 = -2|c|[|d|\cos(\theta+\gamma-\delta) - |a|\cos(\theta+\gamma-\alpha)] \tag{23}$$

$$C_1 = |d-a|^2 - 2|b||c|\cos(2\theta-\beta+\gamma) + \textstyle\sum_3(|a_{ki}|^2 + |a_{jk}|^2) \tag{24}$$

$$\begin{aligned} C_0 = 2|b|[|d|\cos(\theta+\delta-\beta) - |a|\cos(\theta+\alpha-\beta)] \\ + 2\textstyle\sum_3(|a_{ik}||a_{jk}|\cos(\theta+\alpha_{jk}-\alpha_{ik}) - |a_{ki}||a_{kj}|\cos(\theta+\alpha_{ki}-\alpha_{kj})). \end{aligned} \tag{25}$$

For any value of $\theta$ the cubic polynomial $F(R, \theta)$ has at least one real root; the real root of largest absolute value leads to an

$$r = (R^2+1)^{-1/2}$$

of least value such that $M_1 < M$. Thus the system

$$F(R, \theta) = 0, \qquad \partial F(R, \theta)/\partial\theta = 0 \tag{26}$$

will produce the $z = R\exp(i\theta)$ required in the transformation.

It is sufficient to restrict the values of $\theta$ to the interval $\langle 0, \pi\rangle$. If, namely, $R$ is a real root of $F(R, \theta) = 0$ associated with a value of $\theta \in \langle 0, 2\pi\rangle$, then, if we put $\theta_1 = \theta - \pi$, $R_1 = -R$, because of

$$C_2(\theta_1) = -C_2(\theta), \qquad C_0(\theta_1) = -C_5(\theta),$$

it follows that $F(R_1, \theta_1) = 0$. However,

$$z_1 \equiv R_1\exp(i\theta_1) = R\exp(i\theta) = z,$$

and, if sgn $r$ = sgn $R$, also

$$t_1 \equiv -r\exp(i\theta_1) = r\exp(i\theta) = t.$$

The value of $z$ as determined by the system (26) will produce the greatest possible reduction per iteration in the superdiagonal norm. Any other number $z$ obtained by solving $F(R, \theta) = 0$ alone for arbitrary $\theta \in \langle 0, \pi\rangle$ will bring about a smaller decrease in the norm. However, such alternate values of $z$ may still be of interest.

The choice of

$$\theta = 2^{-1}(\beta - \gamma) \tag{27}$$

recommends itself for this purpose, for the following reasons:

a. If $A$ is hermitian, then $\gamma = -\beta$, and $\theta = \beta$. In particular, if $A$ is symmetric, then $\beta = 0$ or $\pi$, and, consequently, also $\theta = 0$ or $\pi$. Thus $z = R$ or $-R$, and $T$ remains free of imaginaries.

b. $A$ is skew hermitian. Now $\gamma = \pi - \beta$, $\theta = \beta - \pi/2$. For skew symmetric matrices, then, depending on whether $\beta = 0$ or $\pi$, $\theta = -\pi/2$ or $\pi/2$, so that $z = -Ri$ or $Ri$, as desired.

c. Above choice of $\theta$ simplifies the calculation of the cubic somewhat. As may be inferred from Eqs. (23), (24), (25), the arguments $\theta + \gamma - \delta$, $\theta + \gamma - \alpha$, $\theta - \beta + \delta$, $\theta - \beta + \alpha$, $2\theta - \beta + \gamma$, become, respectively, $2^{-1}(\beta + \gamma) - \delta \equiv \varphi$, $2^{-1}(\beta + \gamma) - \alpha \equiv \psi$, $-\varphi$, $-\psi$, and 0.

As the number of iterations increases, the super-diagonal elements in the $p$th transformed matrix $A_p$ tend toward zero. Consequently also $b \to 0$, $a_{ik} \to 0$, $a_{kj} \to 0$ for $i < k < j$. Simultaneously, $F(R, \theta) \to RG(R, \theta)$, where

$$G(R, \theta) = |c|^2 R^2 - 2|c|[|d|\cos(\theta + \gamma - \delta) - |a|\cos(\theta + \gamma - \alpha)]R + |d - a|^2 + \sum_3 (+). \tag{28}$$

But the discriminant of $G(R, \theta) = 0$ equals $(|d|\cos(\theta + \gamma - \delta) - |a|\cos(\theta + \gamma - \alpha))^2 - |d - a|^2 - \sum_3$, and is thus negative. Therefore, as the iterations proceed the largest real root $R$ tends to zero, so that the matrix tends to its triangularized form.

**5. Computational procedure.** The foregoing discussion indicates how the procedure may be applied in practice.

A method of selecting the pivot $b$ must be decided upon. Once the triangle of lower norm has been determined one may take as pivot the element of largest absolute value, any above-average element, proceed cyclically to the "next" element in that triangle, or take $b$ in some other fashion. Pivots lying in the super-diagonal ($j = i + 1$) simplify the computation somewhat in that they cause the sums $\sum_3$ in Eqs. (24) and (25) to vanish. With more experience better choices of pivots may become available.

Next, the selection of angles $\theta_i$ of rotation must be made for insertion into the coefficients of the cubic, and ultimately, for the transformation itself. Here, again, more experience will provide better clues.

Thus, one starts with the calculation of $N' = \sum_{i<j} |a_{ij}|^2$, $N'' = \sum_{i>j} |a_{ij}|^2$, and the determination of the "upper" triangle by means of $N''' = \min(N', N'')$. For checking purposes,

$$N^2(A) = N' + N'' + D, \quad \text{with} \quad D = \sum_i |a_{ii}|^2,$$

as well as $tr(A) = \sum_i a_{ii}$ should be computed.

1. Once the pivot $b$ in $A_p$ has been selected, $|a|$, $|b|$, $|c|$, $|d|$, $|a_{ik}|$, $|a_{jk}|$, $|a_{ki}|$, $|a_{kj}|$ for $i < k < j$ are calculated. From $a = a' + ia''$, $\cos\alpha = a'|a|^{-1}$, $\sin\alpha = a''|a|^{-1}$, $\cdots$, and similarly, $\cos\alpha_{kj}$, $\sin\alpha_{kj}$ are obtained.

2. Knowing $\theta_i$, $i = 1, 2, \cdots, m$, there are computed the cos functions occuring in the $C_k$. Thus, for example,

$$\cos(\theta_i + \gamma - \delta) = (|c||d|)^{-1}[\cos\theta_i(c'd' + c''d'') - \sin\theta_i(c''d' - c'd'')]. \tag{29}$$

3. The coefficients $C_k$ of the cubic (21) are now determined, and the largest real root $R_i$ of $F(R_i, \theta_i) = 0$ is found. This step is repeated for the other $\theta_i$ chosen. Then the $R$ of largest absolute value among the $R_i$ is ascertained, as well as its associated $\theta$.

4. Now $z, r, t$ are computed by Eqs. (7), (17), and the elements of $A_{p+1}$ are determined by formulas (8)–(15). They are checked by Eqs. (16).

1a. A pivot is selected in $A_{p+1}$, and the next iteration is begun, as described in step 1.

As may be inferred from this computational outline, the procedure lends itself readily to easy programming on modern high speed computing machinery.

A count of arithmetical operations reveals that approximately $8n + 8(j - i) + 8$ multiplications of complex numbers are involved per iteration, aside from the solution of the cubic. For an average value of $(n + 1)/3$ for $j - i$ this amounts to about $(32/3)(n + 1)$ multiplications per iteration. For each eligible element to act once as a pivot—which is the least one might expect—$n(n - 1)/2$ iterations would be required. This would result in $n - 1$ transformations of each diagonal element, at a cost of about $(16/3)n^3$ multiplications. This estimate seems reasonable in view of counts of $(4/3)n^3$ multiplications for the reduction of real symmetric matrices [2].

**6. Example.** The following simple example illustrates some of the salient features of the method. Let us take

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}. \tag{30}$$

The characteristic values of $A$ are

$$\lambda_1 = -2, \qquad \lambda_2 = 1 + 2i, \qquad \lambda_3 = 1 - 2i.$$

We note that $tr(A) = 0$, $N^2(A) = 19$. Further,

$$D_0 = \sum_{i=1}^{3} | a_{ii} |^2 = 2, \qquad D = \sum_{i=1}^{3} | \lambda_i |^2 = 14.$$

In carrying out the reduction of $A$ to triangular form the element of largest absolute value was taken in general as the pivot, and the angle of rotation $\theta$ was determined according to Eq. (27).

Since $N' = 8$, $N'' = 9$, we have $N''' = N'$; the matrix $A$, as it appears in Eq. (30), has already its "upper" half in the proper place.

1. The element $b = -2$ is the pivot, whence $a = 1, c = 2, d = 0$. Further, $\alpha = \gamma = \delta = 0, \beta = \pi$.

2. Next $\theta = 2^{-1}(\beta - \gamma) = \pi/2$, whence $\varphi = 2^{-1}(\beta + \gamma) - \delta = \pi/2$, $\psi = 2^{-1}(\beta + \gamma) - \alpha = \pi/2$.

3. Therefore, $C_3 = 4, C_2 = 0, C_1 = -2, C_0 = 0$, and $F(R) = 4R^3 - 2R$. Thus $R = 2^{-1/2}, \theta = \pi/2$.

4. It follows that $z = 2^{-1/2} i$, $r = (2/3)^{1/2}$, $t = (2/3)^{1/2}i$, and

$$A_1 = \begin{bmatrix} .66667 + 1.88561i & .57735 & .66667 + .47140i \\ 1.15470 + 1.63299i & -1 & -1.15470 - 1.63299i \\ -.66667 - .47140i & .81650i & .33333 - 1.88561i \end{bmatrix}. \tag{31}$$

1a. Now the element $a_{23}^{(1)}$ becomes the pivot. This leads to $\theta = 72^\circ 22'$, and $F_1(R) = .66667R^3 + 3.10395\, R^2 + 2.06733\, R - 7.36853$. It turns out that $R = 1.13916$, and consequently,

$$A_2 = \begin{bmatrix} .66667 + 1.88561i & .61638 + .71724i & -.00428 - .32494i \\ .75643 - .75390i & -1.57773 - .64200i & -.69069 - .50386i \\ -.70462 - 1.74055i & 1.53811 + .97889i & .91106 - 1.24361i \end{bmatrix}. \quad (32)$$

After eight iterations there is obtained

$$A_8 = \begin{bmatrix} .95014 + 2.11992i & -.22712 + .15730i & .13567 + .69079i \\ -1.31007 - .28045i & -1.97139 - .07342i & .24333 - .52467i \\ -.76294 - .14967i & -.91102 + .58694i & 1.02125 - 2.04650i \end{bmatrix}. \quad (33)$$

The real parts of the characteristic values are thus already determined to better than 5 per cent, while the imaginary parts are off by less than 6 per cent.

**7. Conclusion.** A new method has been presented for the determination of the characteristic values of arbitrary matrices having complex elements. The example shown above, as well as additional computational evidence, justify the belief that the approach is a promising one.

## References

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2. W. Givens, *A method of computing eigenvalues and eigenvectors* . . . , Nat. Bur. Standards, Appl. Math. Ser., No. 29, 1953