On comparing (10) and (13) one finds

$$Y(t) = \exp(At), \qquad Z(t) = \int_0^t \exp[A(t-u)]\,du; \tag{15}$$

thus (11) follows from (14) and (15).

Remark added in proof. In order that (6) and (9) be equivalent, it is not necessary that (8) hold always; rather, it is $f[\Sigma(t)] = f[\Sigma_1(t - t_d)]$ that is required. But this is possible even when f is not constant for intervals of length at least $4t_d$. In some cases (e.g. (5)) a minimum of t_d is sufficient.

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NOTES ON MATRIX THEORY—X
A PROBLEM IN CONTROL*

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Summary. In the theory of control processes, it is important to be able to calculate $\int_0^\infty (x, Bx)dt$ without having to solve explicitly the differential equation $dx/dt = Ax$, $x(0) = c$. A method for doing this is presented in this paper, generalizing one due to Anke for nth order linear differential equations.

1. Introduction. In a recent paper [1], Anke showed that the expression

$$J = \int_0^\infty x^2\,dt \tag{1}$$

could be computed as a rational function of the coefficients, $a_1, a_2, \cdots, a_n$ in the differential equation for x,

$$\frac{d^{(n)}x}{dt^n} = a_1 \frac{d^{(n-1)}x}{dt^{n-1}} + \cdots + a_n x, \tag{2}$$

and the initial values $x(0) = c_1$, $x'(0) = c_2, \cdots, x^{(n-1)}(0) = c_{n-1}$, without solving the equation explicitly, provided that all the solutions of (2) approached zero as $t \to \infty$. This is equivalent to the condition that all the roots of the equation

$$r^n + a_1 r^{n-1} + \cdots + a_n = 0 \tag{3}$$

have negative real parts. Determinantal criteria for this were first given by Hurwitz.

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In this paper we wish to consider the more general problem of determining the value of

$$J = \int_0^\infty (x, Bx)\, dt \tag{4}$$

under the assumption that x is the solution of

$$\frac{dx}{dt} = Ax, \qquad x(0) = c, \tag{5}$$

where the characteristic roots of A have negative real parts; i.e., a stability matrix.

2. The analytic procedure. Let x be the solution of (1.5) and compute, for F a constant symmetric matrix,

$$\begin{aligned} \frac{d}{dt}(x, Fx) &= \left(\frac{dx}{dt}, Fx\right) + \left(x, F\frac{dx}{dt}\right) \\ &= (Ax, Fx) + (x, FAx) \\ &= (x, (A'F + FA)x). \end{aligned} \tag{1}$$

From this it follows that

$$\int_0^\infty \frac{d}{dt}(x, Fx)\, dt = \int_0^\infty (x, (A'F + FA)x)\, dt \tag{2}$$

or

$$-(c, Fc) = \int_0^\infty (x, (A'F + FA)x)\, dt,$$

since, by assumption $x(t) \to 0$ as $t \to \infty$.

Hence if F is determined by the relation

$$A'F + FA = B, \tag{3}$$

we have a solution to the problem posed above.

3. The matrix problem. The question arises as to the existence of a solution of the equation in (2.3). Fortunately the problem can be resolved very simply by means of a transcendental procedure. It is well known that one solution of this equation is

$$F = -\int_0^\infty e^{A't} B e^{At}\, dt, \tag{1}$$

well defined for A a stability matrix.

Since this solution exists for arbitrary B, it follows that it is unique. Hence (2.3) can always be solved by the standard determinantal method.

It is interesting to note that in the (2×2) case, the determinant of the three unknown elements in the symmetric matrix F is the product of tr A by det A. The conditions that A be a stability matrix are that tr $A < 0$, det $A > 0$.

It is tempting to conjecture that the factors of the determinant of the $N(N + 1)/2$ unknown elements in the symmetric matrix F in the general case constitute a set of Hurwitz criteria for the matrix. In the case $N = 3$, the determinant is of degree 6. This leaves room for a quadratic factor in addition to the linear factor, tr A, and the cubic factor det A.

If a systematic method of obtaining these factors existed, the problem of determining stability criteria directly in terms of the elements of A, rather than in terms of the coefficients of the characteristic polynomial of A, would be resolved.

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NOTES ON CONTROL PROCESSES—I. ON THE MINIMUM OF MAXIMUM DEVIATION*

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Summary. The functional equation technique of the theory of dynamic programming is applied to the problem of determining the minimum of the maximum deviation of a system from a preassigned state.

Mathematically this reduces, in certain cases, to choosing a vector y, subject to certain constraints, so as to minimize

$$J(y) = \operatorname*{Max}_{0 \le t \le T} \| x - z \|,$$

where

$$\frac{dx}{dt} = \phi(x, y), \qquad x(0) = c.$$

1. Introduction. Suppose that we have a physical system S whose state at any time t is specified by an n-dimensional vector $x(t)$. Suppose further that $x(t)$ is determined for $t \geq 0$ as the solution of the vector differential equation

$$\frac{dx}{dt} = \phi(x, y), \qquad x(0) = c, \tag{1.1}$$

where y is a vector to be chosen so as to control the behavior of $x(t)$. Many versions of this problem occur in variational analysis and in applied fields such as mathematical economics and servomechanism engineering.

The problem we wish to discuss here is that of choosing y, a vector function of t, so as to minimize the maximum deviation of $x(t)$ from a given vector $z(t)$, over a fixed interval $0 \leq t \leq T$. Although this criterion frequently corresponds most closely to the physical criterion, it is seldom used as a mathematical criterion because of the analytic intractability of the maximum functional. It is interesting to note, however, that Chevychev's research on the minimum of the maximum deviation of polynomials over an interval arose from similar problems arising in the theory of linkages, a study arising from the Watt steam engine.

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