Bibliography

1. R. Bellman, *The theory of dynamic programming*, Bull. Am. Math. Soc. **60**, 503-516 (1954)
2. R. Bellman, *On a class of variational problems*, Quart. Appl. Math. (to appear)
3. R. Bellman, I. Glicksberg, and O. Gross, *On the 'bang-bang' control problem*, Quart. Appl. Math. (to appear)
4. R. Bellman, I. Glicksberg, and O. Gross, *Some non-classical problems in the calculus of variations*, Proc. Am. Math. Soc. (to appear)

A NETWORK PROOF OF A THEOREM ON HURWITZ POLYNOMIALS AND ITS GENERALIZATION*

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Bückner has recently given a canonical form for Hurwitz polynomials. Since network functions and Hurwitz polynomials are intimately related, we shall give an alternate proof of his theorem by network theoretic methods and a statement of a similar theorem. Finally, a general method for obtaining such theorems will be indicated.

Bückner[1] has shown that: "If the polynomial $f(p)$, normalized so that $f(0) = 1$, can be written as

$$f(p) = \begin{vmatrix} 1 + a_1p & -1 & 0 & \cdot & \cdot & 0 \\ 1 & a_2p & -1 & & & \\ 0 & 1 & a_3p & -1 & & \\ 0 & 0 & 1 & a_4p & -1 & \\ \cdot & & & & & \\ \cdot & & & & & -1 \\ 0 & \cdot & & 0 & 1 & a_np \end{vmatrix}$$

where $a_i > 0$ $(i = 1, 2, \cdots n)$, then $f(p)$ is a Hurwitz polynomial and conversely."

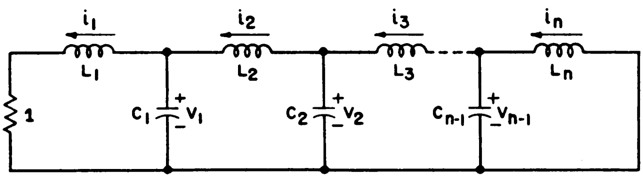

Fig. 1.

Consider the network shown on Fig. 1. If the network variables shown on the figure are used, the network equations can be written as:

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[1]H. Bückner, *A formula for an integral occurring in the theory of linear servomechanisms and control systems*, Quart. Appl. Math. (3) **10**, 205-213 (1952).

$$
\begin{aligned}
-L_1 p i_1 &= i_1 - v_1 \\
-C_1 p v_1 &= i_1 \qquad + i_2 \\
-L_2 p i_2 &= \quad + v_1 \qquad - v_2 \\
-C_2 p v_2 &= \qquad\quad + i_2 \qquad - i_3 \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}
$$

This set of linear equations in the variables i_1, v_1, i_2, v_2, i_3, v_3, $\cdots$ has the following characteristic determinant:

$$
\Delta = \begin{vmatrix}
1 + L_1 p & -1 & 0 & \cdot & \cdot & \cdot & 0 \\
+1 & C_1 p & -1 & & & & \\
0 & 1 & L_2 p & -1 & & & \\
\cdot & & 1 & C_2 p & -1 & & \\
\cdot & & & & & & \\
 & & & & & & -1 \\
0 & \cdot & \cdot & & 0 & +1 & L_n p
\end{vmatrix} \tag{1}
$$

It is physically obvious[2] that as long as $L_i > 0$, $C_i > 0$ $(i = 1, 2, \cdots n)$ the network shown on Fig. 1 is stable, hence the roots of $\Delta(p) = 0$ lie in the left half plane, i.e., the polynomial in p defined by the right hand side of (Eq. 1) is Hurwitz.

The converse is proved as follows. Consider any Hurwitz polynomial $f(p)$ normalized so that $f(0) = 1$. Let

$$f(p) = m(p) + n(p),$$

where $m(p)$ is an even polynomial and $n(p)$ is an odd polynomial. It is well known that $n(p)/m(p)$ is a reactance function and that if $n(p)/m(p)$ is considered to be impedance it is the driving point impedance of a reactive ladder network[3] such as the one shown[4] on Fig. 2. Since $f(p)$ is Hurwitz it follows that the L_i's and C_i's are positive. If a one-ohm

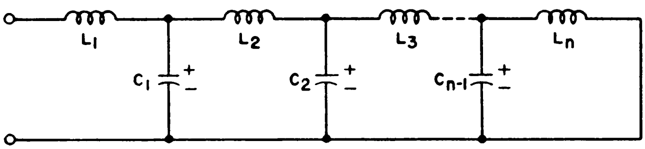

FIG. 2.

[2]See for example, H. W. Bode, *Network analysis and feedback amplifier design*, D. Van Nostrand Co., New York, 1945, Sec. 7.12 in particular.

[3]W. Cauer, *Theorie der linearen Wechselstromschaltungen*, J. W. Edwards, Ann Arbor, 1948, Chap. 5.

[4]The particular network shown on Fig. 2 assumes that the degree of $n(p)$ is higher than the degree of $m(p)$. If it were not the case, one needs only to consider $m(p)/n(p)$ as the impedance of the network to be synthesized. Again the network of Fig. 2 would be obtained except that the last inductance L_n would become an open circuit.

resistor is connected to the terminals of the network of Fig. 2, the network of Fig. 1 is obtained. Considering the way the network has been obtained, its natural frequencies are given by

$$1 + \frac{n(p)}{m(p)} = 0,$$

i.e. by the algebraic equation

$$m(p) + n(p) = 0.$$

However, we have shown previously that the characteristic polynomial of this network can be expressed in the form $\Delta(p) = 0$, where $\Delta(p)$ is given by (Eq. 1). As a result the polynomials $\Delta(p)$ and $f(p)$ have the same roots. Since both reduce to unity when $p = 0$ they are identical.

A similar theorem is immediately suggested by this proof.

Theorem. If the polynomial $f(p)$, normalized so that $f(0) = 1$, may be written as

$$f(p) = \begin{vmatrix} 1 + a_0 p & 0 & -1 & 0 & -1 & 0 & -1 & \cdot & \cdot & \cdot & -1 \\ 0 & a_1 p & -1 & 0 & 0 & 0 & 0 & & & & 0 \\ 1 & 1 & a_2 p & 0 & 0 & 0 & 0 & & & & 0 \\ 0 & 0 & 0 & a_3 p & -1 & 0 & 0 & & & & \\ 1 & 0 & 0 & 1 & a_4 p & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & a_5 p & -1 & & & & \\ \cdot & \cdot & & & & & & & & & \\ \cdot & \cdot & & & & & & & & & \\ \cdot & \cdot & & & & & & & & & \\ 0 & & & & & & & \cdot & & & -1 \\ 1 & & & & & & 0 & & & 1 & a_{2n} p \end{vmatrix}$$

with the $a_i > 0$ $(i = 0, 1, 2 \cdots)$ then $f(p)$ is a Hurwitz polynomial and conversely.

The proof of this theorem is analogous to that of the previous theorem except that

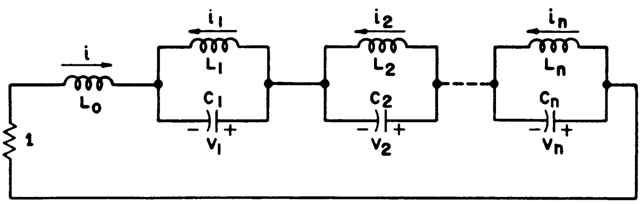

Fig. 3.

the Foster canonical form is used instead of the Cauer form. Refer to Fig. 3, where the notation is defined. The network equations are written as

$$\begin{aligned}
-L_0 pi &= i && - v_1 && - v_2 && - v_3 && - v_4 \\
-L_1 pi_1 &= && - v_1 \\
-C_1 pv_1 &= i + i_1 \\
-L_2 pi_2 &= && && - v_2 \\
-C_2 pv_2 &= i && + i_2 \\
-L_3 pi_3 &= && && && - v_3 \\
-C_3 pv_3 &= i && && + i_3 \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}$$

This set of network equations in the variables $i, i_1, v_1, i_2, v_2, \cdots$ has the given characteristic determinant. The proof of the converse follows that of the previous theorem if the continued fraction expansion is replaced by a partial fraction expansion.

From the point of view presented here it follows that any general synthesis procedure will lead to a canonical representation of Hurwitz polynomials, e.g., for any Hurwitz polynomial $f(p)$ there is a constant K such that the rational function $K/f(p)$ can be synthesized as the transfer impedance of a lossless network operating between two one-ohm resistors[5]. The resulting determinant has the same form as Bückner's except that the last diagonal element has the form $1 + a_n p$.

NOTE ON A PROBLEM CONSIDERED BY TIFFEN*

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In a paper published in the *Quarterly Journal of Mechanics and Applied Mathematics* [2], Tiffen considered the two dimensional elastic problem of determining the stress distribution in a semi-infinite plane with a parabolic boundary. Muskhelishvili [1] had previously given a general solution to this problem which yields the results much more easily than the method used by Tiffen. Muskhelishvili's method of solution avoids the rather cumbersome expressions obtained by Tiffen and effects a considerable saving in the work involved in obtaining the results. Since it may be that not everyone working in the field is familiar with Muskhelishvili's work, the solutions using his general result are given below.

Consider the region exterior to the parabola in the z-plane,

$$x^2 = a^2(a^2 - 2y).$$

This region is mapped conformally on to the upper half of the ζ-plane. $\zeta = \xi + i\eta$, by the mapping function

$$z(\zeta) = -\frac{i}{2}(\zeta + ia)^2.$$

[5]S. Darlington, *Synthesis of reactance 4-poles*, J. Math. and Phys. **18**, 257-353 (September 1939).

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