BIHARMONIC EIGENVALUE PROBLEM OF THE SEMI-INFINITE STRIP*

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Abstract. A basic problem in the evaluation of residual stresses in simple elastic structures concerns the determination of the stress and deformation state produced by self-equilibrating, but otherwise arbitrary, normal and shear tractions acting on the edge $x = 0$ of a semi-infinite elastic strip ($0 \leq x \leq \infty$, $-1 \leq y \leq 1$) which is free along the edges $y = \pm 1$. This strip is known to experience, in accordance with St. Venant's principle, inappreciable stresses at distances $x \gtrsim 2$ from the loaded edge, in spite of the very large stresses it may experience in the vicinity of the edge. An earlier paper [*The end problem of rectangular strips*, J. Appl. Mech. (1953)] based on the variational principle, established approximate eigenfunctions (modes of response) and eigenvalues (laws of oscillation and decay) for the various possible self-equilibrating end tractions. In this paper we give a rigorous solution of the end problem. This solution is obtained in two steps. First we solve the two "mixed" end problems: the parallel edges $y = \pm 1$ of the strip are free, and along the vertical edge $x = 0$ (a) the shear displacement is given, the normal stress is zero, (b) the normal displacement is given, the shear stress is zero. These two problems are solved by extending the strip to the left, to $-\infty$, and finding the tractions that must be applied at $y = \pm 1$ ($x < 0$) and at $x = -\infty$, so that one have $\sigma_x = 0$, $\tau = 0$, respectively, at $x = 0$, while the edge values of the displacements (more specifically, of dv/dy and u) are orthogonal polynomials in y (Horvay-Spiess polynomials and Legendre polynomials, respectively). The corresponding stress functions $K_n(x, y)$, $J_n(x, y)$ are found in the form of Fourier integrals plus polynomial terms. For $x \geq 0$ they may be rewritten as real parts of $\sum C_{Knk} \Phi_k$, $\sum C_{Jnk} \Phi_k$, where $\Phi_k = z_k^{-2} e^{-z_k x} (\cos z_k y - y \cot z_k \sin z_k y)$ or $z_k^{-2} e^{-z_k x} (\sin z_k y - y \tan z_k \cos z_k y)$, and $\sin 2z_k \pm 2z_k = 0$. An alternate procedure for determining the coefficients C_{Knk}, C_{Jnk}, based on a formula of R. C. T. Smith, which bypasses the extension of the strip to $x = -\infty$, is also furnished. The second phase of the solution of the "pure" end problem—along the short edge (a) the shear stress is given, normal stress is zero, (b) the normal stress is given, shear stress is zero—consists in recombining the biharmonic eigenfunctions K_n, J_n within each class into functions $H_n(x, y)$, $G_n(x, y)$, so that the $x = 0$ values of $H_{n,xy}$, $G_{n,yy}$ constitute two complete orthonormal sets of (transcendental) functions in y into which the given boundary stresses may be expanded.

1. Introduction. We propose to solve the biharmonic eigenvalue problem of the semi-infinite strip. More specifically, we shall establish functions $H_n(x, y)$, $G_n(x, y)$ (even in y for even n, odd in y for odd n) such that

(a) H_n, G_n are biharmonic functions

$$\nabla^4 H_n = \nabla^4 G_n = 0. \tag{1}$$

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(b) H_n, G_n satisfy homogeneous boundary conditions of zero normal stress, zero shear stress along the long edges (star denotes boundary value at $y = +1$)[1]:

$$H^*_{n,xx} = G^*_{n,xx} = 0, \qquad H^*_{n,xy} = G^*_{n,xy} = 0. \tag{2}$$

(c) H_n gives zero normal stress, G_n gives zero shear stress along the short edge (the "degree" sign denotes boundary value at $x = 0$)

$$H^\circ_{n,yy} = 0, \qquad G^\circ_{n,xy} = 0. \tag{3}$$

(d) The edge values

$$t_n(y) \equiv -H^\circ_{n,xy}, \qquad s_n(y) \equiv G^\circ_{n,yy} \tag{4}$$

constitute two complete orthonormal sets of functions into which prescribed self-equilibrating edge tractions τ°, σ°_x, i.e., tractions satisfying the conditions

$$\int_{-1}^{+1} \tau^\circ \, dy = \int_{-1}^{+1} \sigma^\circ_x \, dy = \int_{-1}^{+1} \sigma^\circ_x y \, dy = 0 \tag{5}$$

may be expanded:

$$\tau^\circ(y) = \sum_n \langle \tau^\circ, t_n \rangle t_n(y), \qquad \sigma^\circ_x(y) = \sum_n \langle \sigma^\circ_x, s_n \rangle s_n(y). \tag{6}$$

It follows that

$$\begin{aligned} H(x, y) &= \sum_n \langle \tau^\circ, t_n \rangle H_n(x, y) \\ G(x, y) &= \sum_n \langle \sigma^\circ_x, s_n \rangle G_n(x, y) \end{aligned} \tag{7a,b}$$

are the stress functions of the two problems. We used the notation

$$\langle f, g \rangle \equiv \int_{-1}^{+1} f(y) g(y) \, dy, \qquad | f | \equiv \langle f, f \rangle^{1/2}. \tag{8}$$

We arrive at the solutions H_n, G_n of the "pure" end problems by first solving the easier, "mixed" end problems pertaining to determination of stress functions K_n and J_n, which comply with conditions (a), (b), and with the modified conditions (c′) and (d′)[2]:

(c′) $$K^\circ_{n,yy} = 0, \qquad I^\circ_{n,xx} = 0, \qquad (I_{n,x} \equiv J_n), \tag{9}$$

(d′) $$K^\circ_{n,xx} = 1 - y^2, y - y^3, 1 - 8y^2 + 7y^4, \cdots \qquad (n = 2, 3, 4, \cdots) \tag{10}$$

$$I^\circ_{n,yy} = (-1 + 3y^2)/2, (-3y + 5y^3)/2, \cdots . \tag{11}$$

The functions $K^\circ_{n,xx}$ will be recognized as the (unnormalized) Horvay-Spiess polynomials Q_2, Q_3, Q_4, $\cdots$ (denoted formerly, except for normalization factors, by f_0,

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[1]Clearly, because of the evenness or oddness of the functions H_n, G_n, the data at $y = +1$ also specify the data at $y = -1$. We use as distance unit the semiwidth of the strip, as stress unit the modulus of elasticity.

[2]The advantage of regarding I_n rather than $J_n \equiv I_{n,x}$ as the basic function will become apparent in the sequel.

f_1, f_2, $\cdots$; see [1], Eq. (12)), with leading coefficient chosen as 1, which constitute a complete orthogonal set with respect to the boundary condition

$$Q_n(+1) = 0 \tag{12}$$

while the functions $I^\circ_{n,yy}$ are the well-known Legendre polynomials P_2, P_3, P_4, $\cdots$ forming a complete orthogonal set with respect to the boundary condition

$$P_n(+1) = 1. \tag{13}$$

We disregard the first two Legendre polynomials, $P_0 = 1$, $P_1 = y$, which represent rigid body motions, and do not give rise to stresses. In contrast, the singular functions

$$Q_0 = 1, \qquad Q_1 = y \tag{14}$$

which violate condition (12) (the condition of $\sigma_x(0, \pm 1) = \sigma_y(0, \pm 1) = 0$) are of considerable interest. It should be remembered that the corresponding stress functions, K_0, K_1 may be written as linear combinations $\sum_2^\infty c_i K_i$ of the complete set of functions K_n, $n \geq 2$, and so their separate consideration is somewhat redundant. Nevertheless, a direct determination of K_0, K_1 is of great practical value; we shall therefore list their direct formulas along with those of K_2, K_3, $\cdots$. (The symbol K_0 was previously denoted by 2Φ in [4] and by $2\mathcal{K}_1$ in [7], the symbol J_2 was previously denoted by $6\mathcal{G}$ in [7].)

It is clear that solution of the (a), (b), (c′), (d′) problem resolves the end problem of given shear displacement, zero normal stress, and given normal displacement, zero shear stress. For, if K is a stress function, then

$$K_{,yy} = \sigma_x, \qquad K_{,xx} = \nu K_{,yy} + dv/dy \tag{15}$$

and, because of condition (c′), the edge values

$$\begin{bmatrix} K^\circ_{,xx} \\ K^\circ_{,yy} \end{bmatrix} = \begin{bmatrix} dv^\circ/dy \\ 0 \end{bmatrix} \tag{16}$$

are properly specified. Similarly for a stress function $I_{,x}$ we have

$$I_{,xx} = -\int_{-1}^{y} \tau\, dy, \qquad I_{,yy} = \nu I_{,xx} + u, \tag{17}$$

hence, because of condition (c′), the edge values

$$\begin{bmatrix} I^\circ_{,xx} \\ I^\circ_{,yy} \end{bmatrix} = \begin{bmatrix} 0 \\ u^\circ \end{bmatrix} \tag{18}$$

are again properly specified. Thus, the two mixed end problems have the stress functions

$$\begin{aligned} K(x, y) &= \sum_2^\infty \frac{\langle K^\circ_{n,xx}, dv^\circ/dy\rangle}{|K^\circ_{n,xx}|^2} K_n(x, y) \\ J(x, y) \equiv I_{,x}(x, y) &= \sum_2^\infty \frac{\langle I^\circ_{n,yy}, u^\circ\rangle}{|I^\circ_{n,yy}|^2} I_{n,x}(x, y) \end{aligned} \tag{19}$$

as solutions[3].

[3]It is obvious that the edge value problem—I and $\nabla^2 I$ are specified along $x = 0$— is also solved in terms of Eqs. (19).

Once we have determined the functions $K_n(x, y)$, $J_n(x, y)$ appropriate to the mixed end problems, it is easy to calculate the shear stress boundary values $-K^\circ_{n,xy}$ and the normal stress boundary values $J^\circ_{n,yy}$. We shall obtain these in integral form

$$-\tau^\circ_n \equiv K^\circ_{n,xy} = \frac{2}{\pi}\int_0^\infty \left[\mathcal{A}(\lambda)\begin{Bmatrix} ch\ \lambda y \\ sh\ \lambda y \end{Bmatrix} + \mathcal{B}(\lambda) y \begin{Bmatrix} sh\ \lambda y \\ ch\ \lambda y \end{Bmatrix} + \mathcal{P}(\lambda, y) \right] \frac{d\lambda}{\lambda^n}, \tag{20}$$

$\sigma^\circ_n \equiv J^\circ_{n,yy} =$ similar,

where $\mathcal{P}(\lambda, y)$ represents polynomial terms in λ and y, progressing to such power in λ as to compensate for the singularities introduced by the terms $\mathcal{A}/\lambda^n$, $\mathcal{B}/\lambda^n$ (see Table II). Knowing the integral representations of $\tau^\circ_n(y)$ it is easy to orthonormalize these expressions, writing

$$\begin{aligned} t_2(y) &= b_{22}\tau^\circ_2(y), \\ t_4(y) &= b_{44}\tau^\circ_4(y) + b_{42}\tau^\circ_2(y), \\ t_6(y) &= b_{66}\tau^\circ_6(y) + b_{64}\tau^\circ_4(y) + b_{62}\tau^\circ_2(y) \\ &\cdot \qquad \cdot \qquad \cdot \end{aligned} \tag{21}$$

and choosing the b_{nk} so that (δ_{nm} = Kronecker delta)

$$\langle t_n , t_m \rangle = \delta_{nm} . \tag{22}$$

It follows that

$$\begin{aligned} H_2(x, y) &= b_{22}K_2(x, y), \\ H_4(x, y) &= b_{44}K_4(x, y) + b_{42}K_2(x, y) \\ &\cdot \qquad \cdot \qquad \cdot \end{aligned} \tag{23}$$

are the stress functions which resolve the edge value problem τ° = given, $\sigma^\circ_x = 0$, in the form of (7a), and

$$\begin{aligned} G_2(x, y) &= a_{22}J_2(x, y), \\ G_4(x, y) &= a_{44}J_4(x, y) + a_{42}J_2(x, y), \\ &\cdot \qquad \cdot \qquad \cdot \end{aligned} \tag{24}$$

where the a_{nk} are determined from the requirement that

$$\begin{aligned} s_2(y) &= a_{22}\sigma^\circ_2(y), \\ s_4(y) &= a_{44}\sigma^\circ_4(y) + a_{42}\sigma^\circ_2(y), \\ &\cdot \qquad \cdot \qquad \cdot \\ \langle s_n , s_m \rangle &= \delta_{nm} \end{aligned} \tag{25}$$

are the stress functions which resolve the edge value problem σ°_x = given, $\tau^\circ = 0$, in the form of (7b).

2. The functions K_n, I_n, H_n, G_n. Let z_k denote the first quadrant roots $z_2, z_4, \cdots$, and $z_3, z_5, \cdots$ of

$$\sin 2z_k + 2z_k = 0, \qquad \sin 2z_k - 2z_k = 0 \tag{26}$$

respectively. We recall that the Fadle-Papkovitch solutions [5], [6], [2], of $\nabla^4\Phi = 0$ $(x \geq 0)$,

$$\begin{aligned} \Phi_k(x, y) &= z_k^{-2}e^{-z_k x}(\cos z_k y - y \cot z_k \sin z_k y) \qquad (k = 2, 4, 6, \cdots) \\ \Phi_k(x, y) &= z_k^{-2}e^{-z_k x}(\sin z_k y - y \tan z_k \cos z_k y) \qquad (k = 3, 5, 7, \cdots) \end{aligned} \tag{27}$$

satisfy the homogeneous stress conditions (2) along the long edges of the strip, but produce *both* σ_x° and τ° values, and *both* u° and v° displacements along the short edge. What is desired are, however, such combinations of Φ_k which give *either* σ_x° stress *or* τ° stress, making the other stress zero[4]. Such are the functions K_n, J_n and H_n, G_n.

In the remainder of this section we give the results of the analysis. The analysis itself will be carried out in Secs. 3 and 4.

In Table I below we list, up to $n = 6$, the expansion coefficients in terms of the Fadle-Papkovitch functions Φ_k, of the eigenfunctions

$$K_n(x, y) = \Re \sum_k C_{nk}\Phi_k(x, y) \tag{28}$$

TABLE I. Formulas for the coefficients C_{nk} in the expansions of K_n and I_n, Eqs. (28), (29). For n = even, $C'_{nk} \equiv C_{nk} \cos^2 z_k/\sin^2 z_k$, for n = odd, $C'_{nk} \equiv C_{nk}/\cos z_k$ is shown. The subscript k of z_k is omitted for the sake of simpler notation.

K_n	dv°/dy	C'_{nk}
K_0	1	2
K_1	y	$-2z^{-1}$
K_2	$1 - y^2$	$4z^{-2}(2 + \cos^2 z)$
K_3	$y - y^3$	$-4z^{-3}(6 - \cos^2 z)$
K_4	$1 - 8y^2 + 7y^4$	$8z^{-4}[42(3 + 2\cos^2 z) - z^2(34 + 3\cos^2 z)]$
K_5	$y - 4y^3 + 3y^5$	$-8z^{-5}[30(9 - 2\cos^2 z) - z^2(18 - \cos^2 z)]$
K_6	$1 - 19y^2 + 51y^4 - 33y^6$	$32z^{-6}[1485(4 + 3\cos^2 z) - 18z^2(111 + 19\cos^2 z) + 2z^4(13 + \cos^2 z)]$

I_n	u°	C'_{nk}
I_2	$(-1 + 3y^2)/2$	$-6z^{-2}$
I_3	$(-3y + 5y^3)/2$	$30z^{-3}$
I_4	$(3 - 30y^2 + 35y^4)/8$	$30z^{-4}[7(2 + \cos^2 z) - 3z^2]$
I_5	$(15y - 70y^3 + 63y^5)/8$	$-210z^{-5}[3(6 - \cos^2 z) - z^2]$
I_6	$(-5 + 105y^2 - 315y^4 + 231y^6)/16$	$-210z^{-6}[99(3 + 2\cos^2 z) - 6z^2(15 + 2\cos^2 z) + 2z^4]$

[4]A similar problem may be formulated also with respect to the end displacements u°, v°. See, however, the footnote[11].

appropriate to dv°/dy = given by (10), $\sigma_x^\circ = 0$, and of the eigenfunctions

$$I_n(x, y) = \Re \sum_k C_{nk}\Phi_k(x, y) \tag{29}$$

appropriate to u° = given by (11), $\tau^\circ = 0$, where

$$\begin{gathered} \Re \equiv \text{"real part of"} \\ \sum \equiv \sum_{z_2, z_4, \cdots} \quad \text{for} \quad n = \text{even}, \qquad \sum \equiv \sum_{z_3, z_5, \cdots} \quad \text{for} \quad n = \text{odd} \end{gathered} \tag{30}$$

TABLE II. Formulas for the edge values of the stresses.

K_n	τ_n°
K_0	$-\frac{4}{\pi}\int_0^\infty k_{\sigma e}\, d\lambda$
K_1	$-\frac{4}{\pi}\int_0^\infty [k_{\sigma 0} + k_{\tau 0}]\, d\lambda/\lambda^2$
K_2	$\frac{8}{\pi}\int_0^\infty [2k_{\sigma e} - k_{\tau e}]\, d\lambda/\lambda^2$
K_3	$\frac{8}{\pi}\int_0^\infty [6k_{\sigma 0} + (6 + \lambda^2)k_{\tau 0}]\, d\lambda/\lambda^4$
K_4	$-\frac{16}{\pi}\int_0^\infty \left[(126 + 34\lambda^2)k_{\sigma e} - (84 + 3\lambda^2)k_{\tau e} + \frac{7}{2}\lambda^2 y(1 - y^2)\right] d\lambda/\lambda^4$
K_5	$-\frac{16}{\pi}\int_0^\infty \left[(270 + 18\lambda^2)k_{\sigma 0} + (270 + 78\lambda^2 + \lambda^4)k_{\tau 0} - \frac{3}{8}\lambda^4(1 - 6y^2 + 5y^4)\right] d\lambda/\lambda^6$
K_6	$\frac{64}{\pi}\int_0^\infty \left[(5940 + 1998\lambda^2 + 26\lambda^4)k_{\sigma e} - (4455 + 342\lambda^2 + 2\lambda^4)k_{\tau e} + \frac{\lambda^2}{2} y(1 - y^2)\left\{495 - \frac{3}{8}\lambda^2(1 - 33y^2)\right\}\right] d\lambda/\lambda^6$

J_n	σ_n°
J_2	$\frac{12}{\pi}\int_0^\infty j_{\sigma e}\, d\lambda/\lambda^2$
J_3	$\frac{60}{\pi}\int_0^\infty [j_{\sigma 0} + j_{\tau 0}]\, d\lambda/\lambda^4$
J_4	$\frac{60}{\pi}\int_0^\infty [(14 + 3\lambda^2)j_{\sigma e} - 7j_{\tau e}]\, d\lambda/\lambda^4$
J_5	$\frac{420}{\pi}\int_0^\infty [(18 + \lambda^2)j_{\sigma 0} + (18 + 4\lambda^2)j_{\tau 0}]\, d\lambda/\lambda^6$
J_6	$\frac{420}{\pi}\int_0^\infty \left[(297 + 90\lambda^2 + 2\lambda^4)j_{\sigma e} - (198 + 12\lambda^2)j_{\tau e} + \frac{33}{4}\lambda^2(1 - 3y^2)\right] d\lambda/\lambda^6$

and summation extends over the first quadrant roots of (26). The two procedures, one due to the author, the other based on a method of R. C. T. Smith, by which the coefficients C_{nk} may be determined, are described in Secs. 3 and 4, respectively.

Anticipating the results of Sec. 3, and utilizing the functions $k(\lambda, y)$, $j(\lambda, y)$ listed in Appendix I, the edge stresses τ_n° appropriate to K_n and the edge stresses σ_n° appropriate to J_n are found to have the integral representations listed in Table II. We give these expressions up to $n = 6$. Numerical integration, performed as in [4], then leads to the edge values of the stresses $\tau_0^\circ, \tau_1^\circ, \cdots, \sigma_4^\circ$ displayed in Table III. The scalar products $\langle \tau_n^\circ \tau_k^\circ \rangle$, $\langle \sigma_n^\circ \sigma_k^\circ \rangle$, obtained by numerical integration over the tabulated values, are listed in Table IVa[5].

TABLE III. Edge values of the stresses.

$y =$	0	.2	.4	.6	.8	.9	.95	1.0⁻
τ_0°	0	−.10038	−.20638	−.32458	−.46371	−.54490	−.58929	−.63662
τ_1°	.18917	.17263	.11981	.01867	−.16549	−.32227	−.43786	−.63662
τ_2°	0	−.14339	−.27328	−.36970	−.38606	−.31681	−.22945	0
τ_3°	.15800	.13177	.05739	−.05029	−.15417	−.16970	−.14317	0
τ_4°	0	−.3644	−.5143	−.29678	.27376	.54880	.56353	0
σ_2°	.58820	.53678	.37249	.05747	−.51596	−1.0022	−1.3588	−1.9665
σ_3°	0	.47538	.81663	.83568	.1329	−.86634	−1.7833	−3.7827
σ_4°	−.87136	−.57794	.19518	1.0672	1.0413	−.25672	−1.8781	−6.3727

TABLE IV. (a) *Scalar products of* τ_n°, σ_n°. (b) *The expansion coefficients* b_{nk}, a_{nk} *of* t_n, s_n *in terms of* τ_k°, σ_k°, *Eqs.* (21), (25)*.

nk	$\langle \tau_n^\circ \tau_k^\circ \rangle$	$\langle \sigma_n^\circ \sigma_k^\circ \rangle$	b_{nk}	a_{nk}
22	.1598	.7912	2.501	−1.124
33	.02726	1.590	6.056	−.7931
44	.2848	2.722	−1.939	−.6279
42	.05470	.3836	.6634	.3044

The orthogonalization processes (22), (25), finally lead, in accordance with (23), (24), to the expansion coefficients b_{nk}, a_{nk} of H_n, G_n in terms of K_k, J_k, as shown in Table IVb. The boundary stresses t_2, t_3, t_4 produced by H_n and the boundary stresses s_2, s_3, s_4 produced by G_n are plotted in Figs. 1 and 2. For comparison, the corresponding distributions based on orthonormalized self-equilibrating polynomials are also shown. (It will be recalled that the $f_k(y)$ polynomials are orthogonal, the $f_k'(y)$, $f_k''(y)$ polynomials are not. They may be readily orthonormalized into functions

$$T_n = \sum_2^n B_{nk} f_k', \qquad S_n = \sum_2^n A_{nk} f_k'' \tag{31}$$

as shown in [2].) Note that the disagreement of the two sets of curves is not a measure of the inaccuracy of the variational approach, but reflects merely a rotation in function space from one set of orthogonal axes to another.

*The last two digits of the entries of Tables IVa,b are uncertain. See footnote[5].

[5]Because this numerical integration was carried out over the sparsely determined values, at $y = 0$, .2, .4, .6, .8, .9, .95, 1.0, the entries of the present Tables IVa,b must be regarded as preliminary estimates. However, a more accurate determination would require a tremendous investment in time and personnel; these are not available to the author.

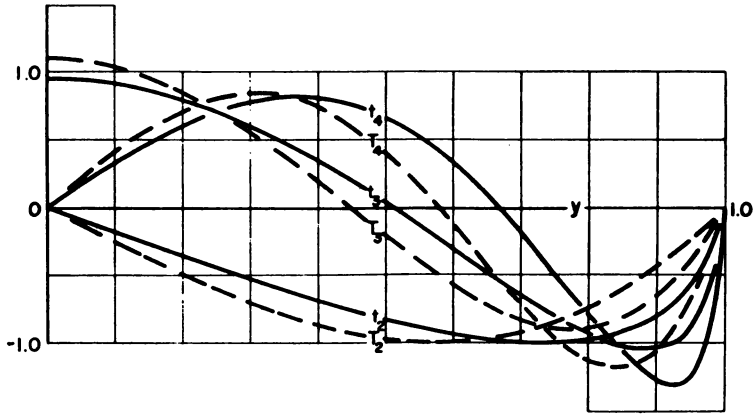

FIG. 1. The first three orthonormal boundary shear tractions. Full lines:

$$t_n(y) \equiv -H^{\circ}_{n,xy} \equiv -\sum_{i=2}^{n} b_{ni} K^{\circ}_{i,xy} \; ;$$

dashed lines:

$$T_n(y) \equiv \sum_{i=2}^{n} B_{ni} f_i' \; .$$

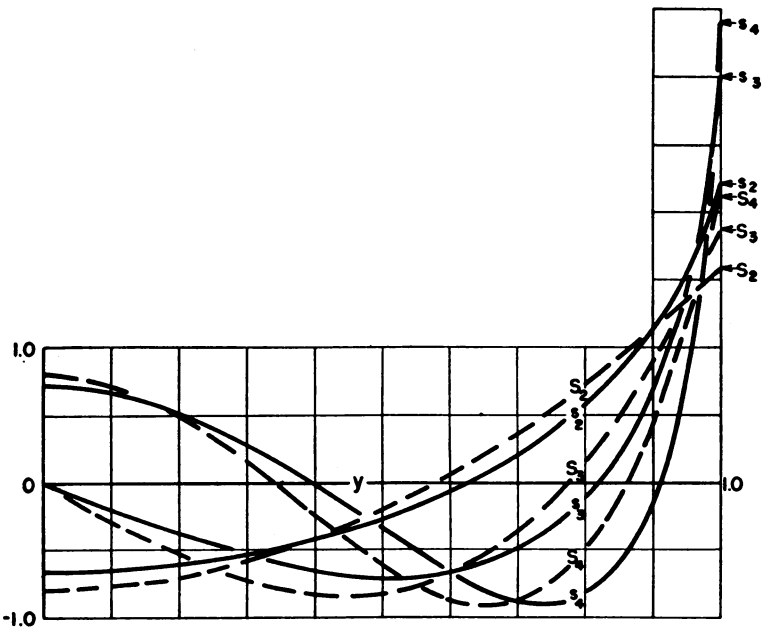

FIG. 2. The first three orthonormal boundary normal tractions. Full lines:

$$s_n(y) \equiv G^{\circ}_{n,yy} \equiv \sum_{i=2}^{n} a_{ni} J^{\circ}_{i,yy} \; ;$$

dashed lines:

$$S_n(y) \equiv \sum_{i=2}^{n} A_{ni} f_i'' .$$

3. The method of analytic continuation. This method of approach was discovered in a somewhat accidental manner. In [4] we investigated the problem of the stress state in an *infinite* strip $-\infty \leq x \leq +\infty$, $-1 \leq y \leq +1$, occasioned by the temperature distribution

$$\vartheta(x, y) = \begin{cases} T & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x > 0 \end{cases} \tag{32}$$

or, what amounts essentially to the same thing, by the edge tractions

$$\tau^* = 0, \qquad \sigma_y^* = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \tag{33}$$

and found, incidental to the solution of the thermal stress problem, also the solution $\frac{1}{2} K_0(x, y)$ of the *semi-infinite* strip for the mixed edge conditions

$$\sigma_x^\circ = 0, \qquad dv^\circ/dy = 1/2. \tag{34}$$

Similarly, solution in [7] of the thermal stress problem of the *infinite* strip for

$$\vartheta(x, y) = \begin{cases} -T'x & x < 0 \\ 0 & x > 0 \end{cases} \tag{35}$$

or, what amounts essentially to the same thing, for the traction distribution

$$\tau^* = 0, \qquad \sigma_y^* = \begin{cases} |x| & x < 0 \\ 0 & x > 0 \end{cases} \tag{36}$$

provided also solution $J_2/6$ of the *semi-infinite* strip problem for the mixed edge conditions

$$\tau^\circ = 0, \qquad u^\circ = (3y^2 - 1)/12. \tag{37}$$

It was then immediately obvious that all end problems of the semi-infinite strip should be reducible to problems of the (doubly) infinite strip. But instead of seeking determination, from a pair of integral equations, of complicated unknown tractions σ_y^*, τ^* applied to the left half, $x < 0$, of the horizontal edges of an infinite strip, in terms of the given values at $x = 0$ (dv°/dy = given, $\sigma_x^\circ = 0$; or u° = given, $\tau^\circ = 0$), one should be able to arrive at the tractions σ_y^*, τ^* by inspection. This is the very program that was carried out in [4], [7] and [2], and led to the determination of the stress functions K_0, J_2, and K_1, J_3. The singularities of these four functions at $x = -\infty$ are, however, not quite severe enough to illustrate fully the general approach. For this reason we determine below the stress function K_4 of the (doubly) infinite strip, appropriate to the boundary conditions

$$\begin{aligned} &\text{for} \quad x > 0: \qquad \sigma_y^* = \tau^* = 0 \\ &\text{for} \quad x = 0: \qquad \sigma_x^\circ = 0, \qquad dv^\circ/dy = 1 - 8y^2 + 7y^4. \end{aligned} \tag{38}$$

We shall find that K_4 has the form

$$K_4 = \frac{63}{2} K_{4\sigma} - 42K_{4\tau} - 34K_{2\sigma} - 6K_{2\tau}, \tag{39}$$

where the functions $K_{2e}, \cdots, K_{4r}$ of x and y are defined, in terms of

$$\begin{aligned} \Delta_e A_e &= sh\lambda + \lambda ch\lambda, & \Delta_e B_e &= -\lambda sh\lambda, \\ \Delta_e C_e &= sh\lambda, & \Delta_e D_e &= -ch\lambda, \end{aligned} \tag{40}$$

(see the Appendix for Δ_e) as follows:[6]

$$\begin{aligned} K_{2e} = -\frac{2^4}{\pi}\int_0^\infty \left[Ach\lambda y + Bysh\lambda y - \frac{1}{2}\right]\frac{\sin\lambda x}{\lambda^5}\,d\lambda \\ + 4\left\{\frac{x^4}{4!} - \frac{2}{\pi}\int_0^\infty \left(\sin\lambda x - \lambda x + \frac{\lambda^3 x^3}{3!}\right)\frac{d\lambda}{\lambda^5}\right\}, \end{aligned} \tag{41a}$$

$$\begin{aligned} K_{2r} = \frac{2^3}{\pi}\int_0^\infty \left[Cch\lambda y + Dysh\lambda y - \frac{1}{4}(1-y^2)\right]\frac{\sin\lambda x}{\lambda^3}\,d\lambda \\ + (1-y^2)\left\{\frac{x^2}{2!} + \frac{2}{\pi}\int_0^\infty (\sin\lambda x - \lambda x)\frac{d\lambda}{\lambda^3}\right\}, \end{aligned} \tag{41b}$$

$$\begin{aligned} K_{4e} = \frac{2^6}{\pi}\int_0^\infty \left[Ach\lambda y + Bysh\lambda y - \frac{1}{2} + \frac{\lambda^4}{48}(1-y^2)^2\right]\frac{\sin\lambda x}{\lambda^7}\,d\lambda \\ + 16\left\{\frac{x^6}{6!} + \frac{2}{\pi}\int_0^\infty \left(\sin\lambda x - \lambda x + \frac{\lambda^3 x^3}{3!} - \frac{\lambda^5 x^5}{5!}\right)\frac{d\lambda}{\lambda^7}\right\} \\ - \frac{2}{3}(1-y^2)^2\left\{\frac{x^2}{2!} + \frac{2}{\pi}\int_0^\infty (\sin\lambda x - \lambda x)\frac{d\lambda}{\lambda^3}\right\}, \end{aligned} \tag{41c}$$

$$\begin{aligned} K_{4r} = \frac{2^5}{\pi}\int_0^\infty \left[Cch\lambda y + Dysh\lambda y - \frac{1}{4}(1-y^2) + \frac{\lambda^2}{24}(1-y^2)^2\right]\frac{\sin\lambda x}{\lambda^5}\,d\lambda \\ - 4(1-y^2)\left\{\frac{x^4}{4!} - \frac{2}{\pi}\int_0^\infty \left(\sin\lambda x - \lambda x + \frac{\lambda^3 x^3}{3!}\right)\frac{d\lambda}{\lambda^5}\right\} \\ - \frac{2}{3}(1-y^2)^2\left\{\frac{x^2}{2!} + \frac{2}{\pi}\int_0^\infty (\sin\lambda x - \lambda x)\frac{d\lambda}{\lambda^3}\right\}. \end{aligned} \tag{41d}$$

We furthermore abbreviate

$$\begin{aligned} \mathcal{K}_{2e} &= K_{2e} - x^2y^2/2, & \mathcal{K}_{2r} &= K_{2r} + x^4/6, \\ \mathcal{K}_{4e} &= K_{4e} - \frac{2x^4y^2 - 4x^2y^4}{9} + \frac{2x^4 - 12x^2y^2}{9}, \\ \mathcal{K}_{4r} &= K_{4r} - \frac{x^6 - 15x^2y^4}{60} - \frac{x^2y^2}{6}. \end{aligned} \tag{42}$$

We start the analysis by considering, for the time being, the following problem. Find the stress function K of the infinite strip subject to boundary tractions

[6] In what follows we omit the subscripts e of A, B, C, D for the sake of simpler notation. The coefficients A_0, B_0, C_0, D_0 which arise in the K_n, J_n expressions when n is odd, are given in [2], Eq. (13).

$$\tau^* = 0, \qquad \sigma_y^* = \begin{cases} 2^5 x^4/4! & x < 0 \\ 0 & x > 0 \end{cases} \tag{43}$$

or what amounts to the same, find $K(x, y)$ so that[7]

$$\begin{aligned} K^*_{,xy} &= 0, \\ 2^{-4} K^*_{,xx} &= \frac{x^4}{4!} - \frac{2}{\pi}\int_0^\infty \left(\sin \lambda x - \lambda x + \frac{\lambda^3 x^3}{3!}\right)\frac{d\lambda}{\lambda^5}. \end{aligned} \tag{44}$$

It is easy to see that insofar as the first term in the integrand of (44) is concerned

$$\frac{64}{\pi}\int_0^\infty \left[A ch\lambda y + B sh\lambda y\right]\frac{\sin \lambda x}{\lambda^7}\, d\lambda \tag{45}$$

is the solution, $A(\lambda)$, $B(\lambda)$ being given by (40). But (45) is not a satisfactory expression since it diverges at $\lambda = 0$, moreover no terms are given in (45) which provide the x, $x^3/3!$ terms of the integrand of (44). We first take care of the singularity of the expression (45). Noting that

$$\begin{aligned} A ch\lambda y + B sh\lambda y = \frac{1}{2} + \lambda^4(1 - y^2)^2\Bigg[&-\frac{1}{2\cdot 4!} + \frac{\lambda^2}{6!}(3 - y^2) \\ &- \frac{\lambda^4}{6\cdot 8!}(41 - 66y^2 + 9y^4) + \cdots\Bigg], \end{aligned} \tag{46}$$

we subtract out the singular part from (45), as shown in the expression (41c). These subtracted out terms are nonbiharmonic, so we must add them back in, and this is done in the braces { } of $K_{4\sigma}$. To compensate furthermore for the singularities introduced by the terms $\sin \lambda x/\lambda^7$ etc. into the braces, we add, under the integral signs of these braces, such terms λx, $\lambda^3 x^3/3!$, etc., that the singularity be removed. The addition of $x^6/6!$ and $x^2/2!$ in the braces outside of the integral sign is finally made in order to insure the condition

$$K^*_{4\sigma,xx} = 0, \qquad K^*_{4\sigma,xy} = 0 \quad \text{for} \quad x > 0. \tag{47}$$

For $x < 0$ there follows then from (41c)

$$K^*_{4\sigma,xy} = 0, \qquad K^*_{4\sigma,xx} = 2^5 x^4/4! \tag{48}$$

Thus, we succeeded in constructing a function, $K_{4\sigma}$, which satisfies the boundary conditions (43), (44). Unfortunately, $K_{4\sigma}$ suffers from the defect that it is (because of the presence of terms in the braces which are outside of the integral sign) not biharmonic. This defect can be corrected by addition of suitable terms, as in (42c), to make it a biharmonic function $\mathcal{K}_{4\sigma}$; but then the conditions (47), (48) become (mildly) violated.

So we start out on a new tack. We consider the problem of finding a stress function $K_{4\tau}$ which gives

$$\sigma_y^* = 0, \qquad \tau^* = \begin{cases} 2x^3 y & x < 0 \\ 0 & x > 0 \end{cases} \tag{49}$$

[7]Compare Eqs. (11) of [2].

or, what amounts to the same thing, finding a biharmonic function which assumes the boundary values

$$K^*_{4\tau,xx} = 0,$$
$$K^*_{4\tau,xy} = 8y\left\{\frac{x^3}{3!} - \frac{2}{\pi}\int_0^\infty \left(\cos\lambda x - 1 + \frac{x^2\lambda^2}{2}\right)\frac{d\lambda}{\lambda^4}\right\}. \tag{50}$$

Proceeding as before, we are first lead to the C and D terms in $K_{4\tau}$, and then, noting that

$$Cch\lambda y + Dysh\lambda y = \frac{1}{4}(1 - y^2) + \frac{\lambda^2}{4}(1 - y^2)^2\left[-\frac{1}{3!} - \frac{\lambda^2}{3\cdot 5!}(1 + 3y^2) + \cdots\right] \tag{51}$$

to the complete expression $K_{4\tau}$ of (41d) which satisfies the boundary conditions

$$\begin{aligned} &\text{for } x > 0: && K^*_{4\tau,xx} = K^*_{4\tau,xy} = 0 \\ &\text{for } x < 0: && K^*_{4\tau,xx} = 0, \qquad K^*_{4\tau,xy} = 8x^3y/3 \end{aligned} \tag{52}$$

but is nonbiharmonic, and the modified expression $\mathcal{K}_{4\tau}$ of (42d) which does not satisfy the condition (52) but is biharmonic. Note, however, that by taking the combination

$$K_{4\sigma} - \frac{4}{3}K_{4\tau} \tag{53}$$

we eliminate the nonbiharmonicity of the 6th degree terms x^6, x^4y^2, x^2y^4 in the combination, and by adding on to (53) a suitable combination of $K_{2\sigma}$, $K_{2\tau}$ (in the present instance, $8K_{2\tau}/9$) we eliminate the nonbiharmonicity also of the 4th degree terms. Thus,

$$\mathcal{K} \equiv K_{4\sigma} - \frac{4}{3}K_{4\tau} + \frac{8}{9}K_{2\tau} \tag{54}$$

is biharmonic and it satisfies the boundary conditions[8]

$$\mathcal{K}^*_{,xx} = \begin{cases} 0 & x > 0 \\ 2^5x^4/4! & x < 0, \end{cases} \tag{55a,b}$$

$$\mathcal{K}^*_{,xy} = \begin{cases} 0 & x > 0 \\ -32(x^3 + x)/9 & x < 0, \end{cases} \tag{55c,d}$$

$$\mathcal{K}^\circ_{,yy} \equiv \sigma^\circ_x = 0, \qquad \mathcal{K}^\circ_{,xx} \equiv dv^\circ/dy = \frac{2}{9}(1 - y^2)(5 - y^2). \tag{56a,b}$$

Our last step is a rather minor one. We take the combination

$$K_4 = \frac{63}{2}\mathcal{K} - 34K_2 \qquad (K_2 = K_{2\sigma} + K_{2\tau}) \tag{57}$$

as stated earlier, in Eq. (39). In this fashion we modify our tractions to

$$K^*_{4,xx} = \begin{cases} 0 & x > 0 \\ 42x^4 - 136x^2 & x < 0, \end{cases} \tag{58a}$$

[8]The boundary conditions (55b,d) are just as suitable for our purposes as were the proposed conditions (43). Equations (43) were not an objective, but merely a starting point. The objective is determination of some biharmonic function $K(x, y)$ which satisfies (55a,c), (56a) and leads to some 4th degree polynomial of the type (56b).

$$K^*_{4,xy} = \begin{cases} 0 & x > 0 \\ -112x^3 + 24x & x < 0, \end{cases} \tag{58b}$$

$$K^\circ_{4,yy} = 0, \qquad K^\circ_{4,xx} = 1 - 8y^2 + 7y^4, \tag{58c,d}$$

and thus achieve that the dv°/dy distribution belonging to K_4 is orthogonal to the dv°/dy distribution belonging to K_2. This facilitates expansion of a given dv°/dy into $K^\circ_{n,xx}$ functions. Higher K_n functions are constructed similarly.

We have thus found that the mixed edge value problem of the semi-infinite strip

$$\sigma^\circ_x = 0, \qquad dv^\circ/dy = n\text{th degree polynomial in } y \tag{59}$$

may be converted into the problem of determining suitable distributions σ^*_y, τ^* which are 0 for $x > 0$, and for $x < 0$ they are n-degree and $(n - 1)$-degree polynomials, respectively, when n = even, and $(n - 1)$-degree and n-degree polynomials, respectively, when n = odd. Furthermore, the stress functions of these σ^*_y, τ^* distributions may be derived in a systematic, though very tedious, manner. So one really does not have to solve for them, but merely construct their expressions. Nevertheless, for large n the Fourier integral representations of $K_n(x, y)$, $J_n(x, y)$ become unmanageably cumbersome. For the region $x \geq 0$ the braces of the $K_{n\sigma}$, $K_{n\tau}$, $J_{n\sigma}$, $J_{n\tau}$ expressions [see Eqs. (41)] vanish, and what remains may be converted by contour integration into rather "simple looking" infinite series, as shown in [4], [7], [2]. These are the expressions displayed in Eqs. (28) and in the first part of Table I[9].

The mixed end problem of the semi-infinite strip

$$\tau^\circ = 0, \; u^\circ = n\text{th degree polynomial in } y \tag{60}$$

leads in a completely similar manner to distributions σ^*_y, τ^* which are zero for $x > 0$, and for $x < 0$ they are $n - 1$, $n - 2$ degree polynomials when n = even, and $n - 2$, $n - 1$ degree polynomials when n = odd. The biharmonic eigenfunction expansions of the stress functions, corresponding to various distributions u°, are displayed in Eq. (29) and in the second part of Table I.

Once we have the integral expressions of K_n, J_n we may also obtain, by differentiation, the boundary values

$$\tau^\circ_n \equiv -K^\circ_{n,xy}, \qquad \sigma^\circ_n \equiv J^\circ_{n,yy}. \tag{61}$$

Note particularly that these stress boundary values involve only terms from the brackets of the type (41) expressions and no terms from the braces. (The terms in the braces are required, as shown in [2], for determining the unknown displacement—$u^\circ(y)$ in the case of K, $dv^\circ(y)/dy$ in the case of J—along edge $x = 0$.) By orthonormalizing the functions τ°_n, σ°_n into functions t_n, s_n, as outlined in the Introduction, we are finally led to the solutions H_n, G_n of the "pure" end problem.

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[9]The minute one tries to use the series in numerical work, the simplicity is replaced by extreme complexity. The writer is therefore inclined to believe that the approximate method of "self-equilibrating polynomials", developed in [3], and well established in regard to simplicity and adequate reliability in [4], [7], [8], [9], [2], [14] is likely to remain the favored technique in the solution of *practical* problems, at least until the time when tables of the functions K_n, J_n and their derivatives become available. (An extension of the self-equilibrating function method to stress problems in polar coordinates is outlined in [13].)

4. The expansion formula of R. C. T. Smith. While in the previous section we carried out the program of solving the mixed and the pure end problems of the semi-infinite strip, we did more than was proposed, we obtained, in addition, also the distributions σ_y^*, τ^* which give rise to the boundary values σ_x°, τ°. This latter result, while of profound mathematical interest is, nevertheless, not germane to the original problem. To obtain σ_y^*, τ^*, i.e., to obtain expressions like Eqs. (39), (41), we had to pay a very high price in amount of labor.

The great amount of labor resulted from the necessity for determining tractions at $x = -\infty$ which balance certain infinite resultants of the σ_y^*, τ^* distributions. These contributions at $-\infty$ were brought into rather sharp focus in [2][10]. However, in our immediate objective we are not interested in what happens at $x = -\infty$, nor, indeed, are we interested in what happens at $x < 0$. We are interested only in what happens to the right of $x = 0$. It is, therefore, very desirable to find an alternate method, which bypasses the determination of σ_y^*, τ^*, and leads directly to the eigenfunction expansions of the K_n, J_n, H_n, G_n sets. Such a direct route is, indeed, provided in the excellent work of R. C. T. Smith [10][11].

Smith in his paper has shown (using a different notation) that the boundary value problem of the semi-infinite strip

$$\begin{aligned} &\nabla^4\Phi = 0, \qquad \Phi^*_{,xx} = \Phi^*_{,xy} = 0, \\ &\Phi^\circ_{,xx} = \text{given}, \qquad \Phi^\circ_{,yy} = \text{given}, \end{aligned} \tag{62}$$

may be solved in the form of a series in the Fadle-Papkovitch functions (27),

$$\Phi = \Re \sum C_k \Phi_k , \tag{63}$$

where

$$C_k = \begin{cases} (z_k \sin^3 z_k/2 \cos^3 z_k) \displaystyle\int & (k = \text{even}), \\ -(z_k \cos^3 z_k/2 \sin^3 z_k) \displaystyle\int & (k = \text{odd}), \end{cases} \tag{64}$$

$$\int \equiv \int_{-1}^{+1} [\Phi^\circ_{k,yy} + 2z_k^2\Phi^\circ_k , z_k^2\Phi^\circ_k] \cdot \begin{bmatrix} -\Phi^\circ_{,yy} \\ \Phi^\circ_{,xx} + 2\Phi^\circ_{,yy} \end{bmatrix} dy. \tag{65}$$

In particular, for the cases of

$$\sigma_x^\circ = 0, \qquad \Phi^\circ_{,xx} = dv^\circ/dy = \text{given}, \tag{66a}$$

$$\tau^\circ = 0, \qquad \Phi^\circ_{,yy} = u^\circ = \text{given}, \tag{66b}$$

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[10]Ref. [2] gives many details and side lights which could not be compressed into the present paper.

[11]The author is indebted to Professor Eric Reissner for calling attention to Smith's work. The papers [4], [7], and the first version of [2] were prepared without knowledge of Smith's paper. The original program, and many of the calculations of the present paper were also carried out without the benefit of familiarity with Smith's results. However, the functions J_4, J_5, J_6, K_3, K_5, K_6 were obtained on the basis of Smith's expansion formula.

formula (65) reduces to

$$\int = z_k^2 \int_{-1}^{+1} \Phi_k^\circ (dv^\circ/dy)\, dy, \qquad \int = -\int_{-1}^{+1} \Phi_{k,yy}^\circ u^\circ\, dy, \tag{67a,b}$$

respectively.

For the case (66a), (67a) the function Φ of (63) is the stress function (we refer to it as function K); for the case (66b), (67b) the x-derivative of Φ

$$J = \Phi_{,x} = \mathcal{R} \sum C_k \Phi_{k,x} \tag{68}$$

is the stress function. Letting

$$l^{(i)} \equiv l(l-1) \cdots (l-i+1) \tag{69}$$

and noting

$$\begin{aligned} \int_{-1}^{+1} \Phi_k^\circ y^l\, dy &= \frac{1}{(l+2)(l+1)} \int_{-1}^{+1} \Phi_{k,yy}^\circ y^{l+2}\, dy \\ &= -\frac{4 \cos z_k}{z_k^{l+4}} \{z_k^l - 2l^{(2)} z_k^{l-2} + 3l^{(4)} z_k^{l-4} - 4l^{(6)} z_k^{l-6} + \cdots \\ &\qquad - \cos^2 z_k [l z_k^{l-2} - 2l^{(3)} z_k^{l-4} + 3l^{(5)} z_k^{l-6} - \cdots]\} \end{aligned} \tag{70a}$$

for k, l = even,

$$\begin{aligned} \int_{-1}^{+1} \Phi_k^\circ y^l\, dy &= \frac{1}{(l+2)(l+1)} \int_{-1}^{+1} \Phi_{k,yy}^\circ y^{l+2}\, dy \\ &= -\frac{4}{z_k^{l+4} \cos z_k} \{z_k^{l+1} - 2l^{(2)} z_k^{l-1} + 3l^{(4)} z_k^{l-3} - \cdots \\ &\qquad - \sin^2 z_k [l z_k^{l-1} - 2l^{(3)} z_k^{l-3} + 3l^{(5)} z_k^{l-5} - \cdots]\} \end{aligned} \tag{70b}$$

for k, l = odd, we determine the integrals listed in Table V and thereby also the coefficients C_k of expansions (63), (68), when dv°/dy, u° are powers of y.

Note that the expansions obtained will, in general, not converge for $x = 0$, since the conditions of self-equilibration are not, in general, satisfied by the distributions y^l. However, the expansions are merely the building blocks which make up the complete self-equilibrating stress functions of Tables Ia,b; the latter expansions, as Smith has shown, do converge. Smith's procedure therefore permits complete solution of the two *mixed* end problems. In order to go beyond the mixed problem and resolve the *pure* end problem, we have to determine the edge values of the τ_n°, σ_n° distributions, in the form given in Table II. However, even this particular representation can be arrived at by an extension of the Smith technique (without prior determination of the distributions σ_y^*, τ^* which give rise to it) by merely retracing the steps employed in the usual application of the residue theorem. Starting out with, e.g.,

$$\begin{aligned} -\tau_2^\circ = K_{2,xy}^\circ &= \mathcal{R} \sum_k \frac{4(2 + \cos^2 z_k) \sin^2 z_k}{z_k^2 \cos^3 z_k} \left[\left(1 + \frac{\cot z_k}{z_k}\right) \sin z_k y + y \cot z_k \cos z_k y\right] \\ &\equiv \mathcal{R} \sum_k \frac{\mathcal{A}(z_k) \sin z_k y + \mathcal{B}(z_k) y \cos z_k y}{4 \cos^2 z_k}, \end{aligned} \tag{71}$$

$$\mathcal{A}(z_k) = -\frac{16}{z_k^2}(3 \cos z_k + z_k \sin z_k), \qquad \mathcal{B}(z_k) = \frac{16}{z_k^2}(2 \sin z_k - z_k \cos z_k),$$

TABLE V. Integrals of $\Phi_k^\circ y^l$. (Subscript k of z is omitted for the sake of simpler notation.) The upper group pertains to k = even, the lower group to k = odd. The integrals vanish when k + l = odd.

l	$z^2 \int_{-1}^{1} \Phi_k^\circ y^l \, dy = \frac{z^2}{(l+2)(l+1)} \int_{-1}^{1} \Phi_{k,yy}^\circ y^{l+2} \, dy$
0	$-4 \cos z/z^2$
2	$(-4 \cos z/z^4)[z^2 - 4 - 2 \cos^2 z]$
4	$(-4 \cos z/z^6)[z^4 - 24z^2 + 72 - \cos^2 z(4z^2 - 48)]$
6	$(-4 \cos z/z^8)[z^6 - 60z^4 + 1080z^2 - 2880 - \cos^2 z(6z^4 - 240z^2 + 2160)]$
8	$(-4 \cos z/z^{10})[z^8 - 112z^6 + 5040z^4 - 80640z^2 + 201600 - \cos^2 z(8z^6 - 672z^4 + 20160z^2 - 161280)]$
1	$(-4/z^3 \cos z)[z^2 - \sin^2 z]$
3	$(-4/z^5 \cos z)[z^4 - 12z^2 - \sin^2 z(3z^2 - 12)]$
5	$(-4/z^7 \cos z)[z^6 - 40z^4 + 360z^2 - \sin^2 z(5z^4 - 120z^2 + 360)]$
7	$(-4/z^9 \cos z)[z^8 - 84z^6 + 2520z^4 - 20160z^2 - \sin^2 z(7z^6 - 420z^4 + 7560z^2 - 20160)]$
9	$(-4/z^{11} \cos z)[z^{10} - 144z^8 + 9072z^6 - 241920z^4 + 1814400z^2 - \sin^2 z(9z^8 - 1008z^6 + 45360z^4 - 725760z^2 + 1814400)]$

we may write, on noting

$$\frac{d}{dz_k}(\sin 2z_k + 2z_k) = 2(1 + \cos 2z_k) = 4 \cos^2 z_k \,, \tag{72}$$

the integral representation $(0 < \epsilon < \mathcal{R}(z_2))$

$$-\tau_2^\circ = \frac{1/2}{2\pi i} \int_{\epsilon+i\infty}^{\epsilon-i\infty} \frac{\mathcal{A}(z) \sin zy + \mathcal{B}(z) y \cos zy}{\sin 2z + 2z} \, dz. \tag{73}$$

Introducing

$$\lambda = -iz \tag{74}$$

we finally convert (73) to

$$-\tau_2^\circ = \frac{1}{\pi} \int_0^\infty \left[\frac{\mathcal{U}(\lambda) ch\lambda y + \mathcal{V}(\lambda) y sh\lambda y}{sh2\lambda + 2\lambda} + \mathcal{C}(\lambda, y) \right] d\lambda, \tag{75}$$

where $\mathcal{C}(\lambda, y)$ is so determined that the infinities of the $\mathcal{U}$, $\mathcal{V}$ terms are compensated. This leads us back to our τ_2^0 expression in Table II. (This is the way the edge distributions σ_4°, σ_5°, σ_6°, τ_3°, τ_5°, τ_6° of Table II were determined.)

It may be added that the approach of Sec. 3, while evidently superfluous in the final establishment of Tables I to V, is by no means superfluous in the deeper insight it has given for the solution of the end problem. In fact, it was the very search for the

distributions σ_y^*, τ^* which give prescribed σ_x°, τ° that provided the guiding idea which lead to the solution of the pure end problem of the semi-infinite strip[12].

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Appendix. The functions $\Delta(\lambda)$, $k(\lambda, y)$, $j(\lambda, y)$[13].

$$\Delta_e = sh2\lambda + 2\lambda, \qquad \Delta_0 = sh2\lambda - 2\lambda,$$

$$\Delta_e k_{\sigma e} = ch\lambda sh\lambda y - sh\lambda \cdot ych\lambda y,$$

$$\Delta_e k_{\tau e} = (\lambda sh\lambda - ch\lambda)sh\lambda y - \lambda ch\lambda \cdot ych\lambda y + \tfrac{1}{2}y\Delta_e ,$$

$$\Delta_0 k_{\sigma 0} = \lambda^2 sh\lambda ch\lambda y - \lambda^2 ch\lambda \cdot ysh\lambda y - \tfrac{3}{4}(1 - y^2)\Delta_0 ,$$

$$\Delta_0 k_{\tau 0} = (sh\lambda - \lambda ch\lambda)ch\lambda y + \lambda sh\lambda \cdot ysh\lambda y + \tfrac{1}{4}(1 - 3y^2)\Delta_0 ,$$

$$\Delta_e j_{\sigma e} = (\lambda ch\lambda - sh\lambda)ch\lambda y - \lambda sh\lambda \cdot ysh\lambda y,$$

$$\Delta_e j_{\tau e} = (\lambda^2 sh\lambda - 2\lambda ch\lambda)ch\lambda y - \lambda^2 ch\lambda \cdot ysh\lambda y + \tfrac{1}{2}\Delta_e ,$$

$$\Delta_0 j_{\sigma 0} = (\lambda^3 sh\lambda - \lambda^2 ch\lambda)sh\lambda y - \lambda^3 ch\lambda \cdot ych\lambda y + \tfrac{3}{2}y\Delta_0 ,$$

$$\Delta_0 j_{\tau 0} = (2\lambda sh\lambda - \lambda^2 ch\lambda)sh\lambda y + \lambda^2 sh\lambda \cdot ych\lambda y - \tfrac{3}{2}y\Delta_0 .$$

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[12]There exists a second type of pure end problem which is of interest, namely, the one where on the edge $x = 0$ both displacements u° and v° are specified. Since this group of problems is usually characterized by infinite stresses in the corners $(x, y) = (0, \pm 1)$, the present scheme of infinite expansions cannot be adopted without first isolating the infinity or taking other precautionary measures, see [11], [12].

[13]The writer shall be glad to pass on, to those interested, power series expansions of the $k(\lambda, y)$, $j(\lambda, y)$ functions (these are needed for evaluating the $\lambda = 0$ values of the integrands of Table II) as well as tabulated values of the functions for $\lambda = 0, .4, .8, \ldots, 6.4$, at $y = 0, .2, .4, .6, .8, .9, .95, 1.0$.