A NOTE ON POISSON'S INTEGRAL*

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Let D be a convex open set with boundary B. Let f be a function defined at the points of B and continuous at these points. This note concerns a mean value formula which extends f to be a continuous function in $D + B$. At first discussion is restricted to the case that D is a bounded set either in space or in the plane.

A line drawn through a point p of D will intersect B in exactly two points, say q_1 and q_2, as is illustrated in Fig. 1. The value of the function f at these points will be

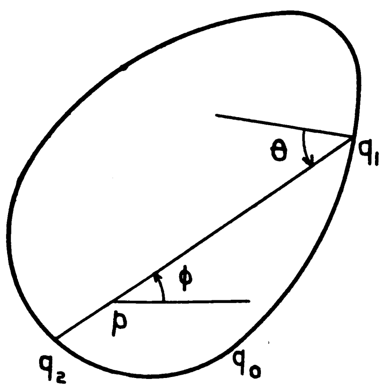

FIG. 1.

denoted by f_1 and f_2. A linear function g is defined on this line so that g takes on the value f_1 at q_1 and the value f_2 at q_2. The value of g at a given point p is seen to be a continuous function of p and of the direction of the line through p. With p held fixed, let $F(p)$ denote the average of g for all directions of the line. Thus $F(p)$ is a continuous function for p in D. (It is of interest to note that if f is the boundary value of a linear function h, then $F \equiv h$ in D.)

Let q_0 be a point on B where B is strictly convex. It is desired to show that if $p \to q_0$, then $g \to f_0$ uniformly with respect to the direction of the line through p. If this is not true, there is a sequence of points p_n such that $p_n \to q_0$ and $| g - f_0 | > c_1$ for a positive constant c_1. By strict convexity $| q_1 - q_0 || q_2 - q_0 | \to 0$. Without loss of generality it may be assumed that $q_1 \to q_0$. If there is a positive constant c_2 such that $| q_2 - q_1 | > c_2$, then by continuity and the definition of g, it is seen that $g \to f_0$. If there is no such c_2, then for some infinite subsequence both $q_1 \to q_0$ and $q_2 \to q_0$ and so $g \to f_0$. In either case there is a contradiction, and so g must converge uniformly to f_0. It follows immediately that $F(p) \to f(q_0)$ as $p \to q_0$. This statement is seen to hold at an arbitrary point of the boundary by slightly modifying the above proof.

Let r_1 be the distance from p to q_1 and let r_2 be the distance from p to q_2. Thus

$$g = (r_1 f_2 + r_2 f_1)/(r_1 + r_2). \tag{1}$$

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In the two-dimensional case

$$F(p) = \frac{1}{2\pi} \int_0^{2\pi} g \, d\phi = \frac{1}{\pi} \int_0^{2\pi} \frac{r_2 f_1}{r_1 + r_2} \, d\phi. \tag{2}$$

Here ϕ denotes the angle the line pq_1 makes with a fixed line. If the boundary is sufficiently smooth, (2) can be transformed into a line integral. Let θ be the angle formed by the normal to B at q_1 and the line pq_1. Then $r_1 \, d\phi = \cos \theta \, ds$ where ds denotes the element of arc length of B. Thus

$$F(p) = \frac{1}{\pi} \int_0^c \frac{\cos \theta}{r_1} f_1 \, ds - \frac{1}{\pi} \int_0^c \frac{\cos \theta}{r_1 + r_2} f_1 \, ds, \tag{3}$$

where c is the length of B. It is clear that the first integral is harmonic in D. In the case that B is a circle of radius a, it is apparent that $r_1 + r_2 = 2a \cos \theta$. Thus the second integral is constant and so F is harmonic if B is a circle.

In three dimensions formulas (2) and (3) become

$$F(p) = \frac{1}{2\pi} \iint \frac{r_2 f_1}{r_1 + r_2} \, d\Omega \tag{2'}$$

and

$$F(p) = \frac{1}{2\pi} \iint \frac{\cos \theta}{r_1^2} f_1 \, dS - \frac{1}{2\pi} \iint \frac{\cos \theta}{(r_1 + r_2)} \frac{f_1}{r_1} \, dS. \tag{3'}$$

Here $d\Omega$ is the element of solid angle and dS is the element of surface area. The first integral in (3′) is seen to be harmonic. If B is a sphere of radius a, then again $r_1 + r_2 = 2a \cos \theta$ and it is seen that the second integral in (3′) is harmonic.

Thus in the case of the circle and the sphere the mean value formula yields a harmonic function. Since the Dirichlet problem has a unique solution, it follows that the mean value formula is equivalent to Poisson's integral. The formula may also be regarded as an extension of the Gauss mean value relation. There is no difficulty in extending the mean value formula to hold for infinite regions. This yields relations equivalent to Poisson's formula for the half-plane and the half-space.

For regions of the plane not too far from circular shape it is to be expected that the mean value formula (2) would give an approximate solution of the Dirichlet problem. Furthermore, the integral is suitable for approximation by a discrete sum. This is the method of *the linear rosette* proposed by M. M. Frocht in his book *Photoelasticity*, vol. II, John Wiley, New York, 1948. Frocht carries out some examples of the approximation method and makes comparison with the exact solution.

Again consider a region of the plane. An analytic function is defined by

$$W(p) = \frac{1}{\pi} \int_0^c \frac{e^{-i\theta}}{r_1} f_1 \, ds. \tag{4}$$

Let $W = U + iV$ then

$$U = \frac{1}{\pi} \int_0^{2\pi} f_1 \, d\phi \tag{5}$$

and

$$V = -\frac{1}{\pi} \int_0^{2\pi} \tan \theta f_1 \, d\phi. \tag{6}$$

It is seen that U is identical with the first integral in (3). Thus for a circle, U differs from F by a constant, and so V is the harmonic conjugate of F.

For regions not too far from circular it is to be expected that $U + C$ where C is a constant would approximate the solution of the Dirichlet problem. Of course $U + C$ would not take on the correct boundary values. By the maximum principle it results that the greatest error would be on the boundary; hence the error could be determined.

Since the error in the above procedure is harmonic, a second approximation could be set up, etc. This would lead to the *method of the arithmetic mean* devised by Neumann to solve the Dirichlet problem in convex regions.

ON SOME EFFECTS OF VELOCITY DISTRIBUTION IN ELECTRON STREAMS

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Although the details of the derivation are different Equations (83), (73) and (67) derived earlier by J. K. Knipp (see: "Klystrons and Microwave Triodes", M.I.T. Radiation Laboratory Series, Vol. 7, McGraw-Hill, Chapter 5) are somewhat more general than the corresponding Equations (11), (14) and (43a) of my paper. This was not indicated in my above paper due to an oversight. The author would like to thank Professor J. K. Knipp for kindly drawing his attention to this matter.