heated. In any case, it is clear that the most rapid convergence is obtained for small x and y and comparatively large t. Although the first term appears formidable, it turns out that in many instances the arctan and log terms are negligible. For example, if $k = 10^3$, $a = 2$, and $xy \leq 4/10^6$, the arctan and log may be neglected if an error within $\pm 1/10$ is satisfactory. Since the series is alternating, the error involved is less, in absolute value, than the first term not employed in the calculation. Thus, once k and a are fixed, one might check terms, beginning with the third, i.e., $ka^2xy/96\pi t^3\,[a^2/3 + x^2 + y^2]$, in order to obtain a range for x, y, and t so that a given error tolerance is not exceeded. From the form of each term, an inequality involving the product, xy, is simplest to handle and furnishes a good check for x and y small. As a usual occurrence the calculation of three or four terms gives sufficient accuracy for applied purposes provided a reasonable balance is maintained among the variables.

VISCO-ELASTIC STRESS ANALYSIS*

By J. R. M. RADOK (*Brown University*)

1. Introduction. In his paper on stress analysis in visco-elastic bodies [1]** E. H. Lee bases his reasoning on the concept of an associated elastic problem to which a visco-elastic problem reduces after removal of its time dependence by application of the Laplace transform. Thus Lee's method requires the application of the Laplace transform to the boundary conditions as well as to the basic equations and it might be expected that it is restricted to problems whose boundary conditions admit such an operation. As a result, for example, the problem of indentation of a half-space by a curved punch could not be solved by this method, since at any one point of the boundary at different times stresses or displacements are specified.

It is the purpose of this paper to extend the applicability of Lee's method to problems of the above type and to show that the apparent restriction is due to the process by which Lee deduced his method, in particular, due to the concept of the associated elastic problem.

At the same time, the Laplace transform method will be restated in a different form which may be called the method of functional equations. This method is completely equivalent to Lee's method, since both these methods coincide, if the functional equations are solved by operational methods. However, the extension of the applicability of the Laplace transform method to the wider range of problems requires the functional equation approach for its justification.

2. The method of functional equations. The basic, quasi-static equations governing the linear theories of isotropic, elastic or visco-elastic media, referred to orthogonal, rectilinear coordinates x_k, may be written in the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = 0, \tag{2.1}$$

$$P^p s_{ij} = Q^q e_{ij}\,, \qquad R^r \sigma_{ii} = S^s \epsilon_{ii}\,, \qquad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.2}$$

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**Numbers in square brackets refer to the bibliography at the end of the paper.

where σ_{ij}, ϵ_{ij}, u_i and X_i are the stress, strain, displacement and body force components, s_{ij} and e_{ij} are the stress and strain deviators, defined by

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}, \qquad e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij}, \tag{2.3}$$

and p^p, Q^q, R^r, S^s are linear differential time operators with constant coefficients

$$P^p = \sum_0^p p_n \frac{\partial^n}{\partial t^n}, \qquad Q^q = \sum_0^q q_n \frac{\partial^n}{\partial t^n}, \qquad R^r = \sum_0^r r_n \frac{\partial^n}{\partial t^n},$$
$$S^s = \sum_0^s s_n \frac{\partial^n}{\partial t^n}. \tag{2.4}$$

In the particular case of an elastic material, the orders p, q, r, s of the operators in (2.2) are all equal to zero and hence

$$\frac{q_0}{p_0} = 2\mu, \qquad \frac{s_0}{r_0} = 3k, \tag{2.5}$$

where μ and k are the shear and bulk moduli repectively.

Any problems in the theories of elasticity and visco-elasticity will be completely specified, if the above equations are accompanied by boundary conditions which mostly take either the form of given tractions T_i

$$\sigma_{ij}n_j = T_i \qquad \text{on } L \tag{2.6}$$

or of given displacements f_i

$$u_i = f_i \qquad \text{on } L \tag{2.7}$$

or of a combination of these conditions, where L is the boundary of the body and n_j are the direction cosines of the outward normal to L. In addition, in general, due to the time derivatives in the stress-strain relations, initial conditions have to be satisfied the number of which depends on the orders of the operators P^p,Q^q, R^r, S^s. In the case of an elastic material no initial conditions occur, since $p = q = r = s = 0$.

There are different ways in which time dependence may enter into the solutions of the basic equations (2.1), (2.2). The given tractions T_i and displacements f_i may be functions of the time, the sections of the boundary to which the conditions (2.6) or (2.7) apply may vary in time or a combination of these conditions may occur. In general, the solutions will thus depend on the time t as well as on the space coordinates x_k and the material constants p_n, q_n, t_n, s_n.

First of all, it will now be shown that every elastic solution of a given boundary value problem, for given initial conditions, uniquely determines a visco-elastic solution. The proof of this statement is effectively contained in Lee's paper [1]. In fact, Lee uses this result in his second method when dealing with moving loads.

Assuming for simplicity that the visco-elastic solution is subject to zero initial conditions and that therefore the given surface tractions and displacements, specifying the boundary value problem, vanish for $t < 0$, the time dependence of the basic equations (2.1), (2.2) may be removed by application of the Laplace transform. Denoting by a star the Laplace transforms of the time dependent quantities in these equations, one finds in this way

$$\frac{\partial \sigma_{ij}^*}{\partial x_j} + X_i^* = 0, \tag{2.8}$$

$$P^{*p}s_{ij}^* = Q^{*q}e_{ij}^* , \qquad R^{*r}\sigma_{ii}^* = S^{*s}\epsilon_{ii}^* , \tag{2.9}$$

where for a visco-elastic material the operators P^p, Q^q, R^r, S^s become polynomials in the Laplace transform variable. Since (2.9) also refers to the elastic case when

$$P^{*p} = p_0 , \qquad Q^{*q} = q_0 , \qquad R^{*r} = r_0 , \qquad S^{*s} = s_0 , \tag{2.10}$$

the Laplace transform of a visco-elastic solution may be obtained from that of the elastic solution by means of the substitutions (2.10), when not all of the orders p, q, r, s of the operators are equal to zero. However, in many cases, the elastic solution will permit a rearrangement of the elastic coefficients p_0 , q_0 , r_0 , s_0 , so that the Laplace transform of the visco-elastic solution is equivalent to ordinary differential equations in time with initial conditions.

This result justifies the following alternative procedure for deducing a visco-elastic solution from a given elastic solution. Instead of applying the Laplace transform to the elastic solution, the elastic coefficients p_0 , q_0 , r_0 , s_0 in this solution may be replaced by the operators P^p, Q^q, R^r, S^s. This process leads to functional equations for the stresses, displacements, etc. of the visco-elastic solution. In many cases which involve the elastic constants in a simple algebraic manner, these functional equations will be differential equations in time which may be integrated for given initial conditions.

So far nothing has been said about the boundary conditions, satisfied by the visco-elastic solution, obtained in the above manner from the elastic solution. If the boundary conditions of the elastic solution can be transformed by means of the Laplace transform, i.e., if an associated elastic problem in the sense of Lee exists, both the elastic and visco-elastic solutions satisfy the same boundary conditions.

However, it will now be shown that the above visco-elastic solution always satisfies the same boundary conditions as the elastic solution from which it was obtained. This may be done using the method of differential equations, irrespective of whether the associated elastic problem exists. For this purpose consider $\sigma_{ij}n_j$ or u_i or both of these expressions for the elastic solution at every point of the region occupied by the body, depending on which of these quantities enter into the boundary conditions. The first of these expressions is to be understood in the sense that inside L the direction cosines n_j may be given any desired values. Since σ_{ij} and u_i , referring to the elastic problem, depend on the elastic constants as well as on the space coordinates x_k and the time t, the above quantities will depend on the coefficients p_0 , q_0 , r_0 , s_0 everywhere inside the boundary L, except possibly at certain points or on preferred planes, such as planes of symmetry where shear stresses vanish. However, on the boundary L, where the values of the direction cosines are given, the above expressions, when they enter into the boundary conditions, do not contain elastic constants, since the tractions and displacements are independent of the properties of the material. Thus, the expressions $\sigma_{ij}n_j$ and u_i lead by the method of functional equations to differential equations at points inside the region, occupied by the body, but not on the boundary L, wherever tractions or displacements are prescribed. Hence, the visco-elastic solution satisfies the same given boundary conditions as the elastic solution.

It should be noted that the above result does not mean that the elastic and visco-elastic solutions agree all along the boundary, since, for example, at points where displacements are prescribed, there will be functional equations for the visco-elastic stresses which therefore, in general, will differ from those of the elastic solution.

In the next section, the method of functional equations will be illustrated by two examples; the first which deals with the point force on a half-space has also been considered by Lee [1], since it has a corresponding associated elastic problem, the second, the two-dimensional problem of a varying size circular hole in an infinite body, subject to internal pressure, has no associated elastic problem.

3. Applications. For the purpose of comparison with the Laplace transform method consider the case of a point load on a half-space $z \geq 0$. This problem was also used by Lee [1] as an illustration. The elastic solution is

$$\sigma_r(r, z, t) = \frac{P(t)}{2\pi}\left\{(1 - 2\nu)\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] - 3r^2z(r^2 + z^2)^{-5/2}\right\}. \tag{3.1}$$

By (2.5),

$$1 - 2\nu = \frac{3q_0r_0}{2s_0p_0 + r_0q_0}, \tag{3.2}$$

and hence the corresponding functional equation is

$$\begin{aligned}(2S^sP^p + R^rQ^q)\sigma_r(r, z, t) = \frac{1}{2\pi}\Bigg\{&3Q^qR^rP(t)\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] \\ &- (2S^sP^p + R^rQ^q)P(t)3r^2z(r^2 + z^2)^{-5/2}\Bigg\}.\end{aligned} \tag{3.3}$$

For the Voigt material, considered by Lee,

$$P^p = 1, \qquad Q^q = A\frac{\partial}{\partial t} + B, \qquad R^r = 1, \qquad S^s = C \tag{3.4}$$

and (3.3) gives the differential equation

$$\begin{aligned}\left(2C + B + A\frac{\partial}{\partial t}\right)\sigma_r(r, z, t) = \frac{1}{2\pi}\Bigg\{&3\left(A\frac{\partial}{\partial t} + B\right)P(t)\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] \\ &- \left(2C + B + A\frac{\partial}{\partial t}\right)P(t)3r^2z(r^2 + z^2)^{-5/2}\Bigg\}.\end{aligned} \tag{3.5}$$

Solution of Eq. (3.5) for zero initial conditions and

$$P(t) = \begin{matrix} 0 & \text{for} & t < 0 \\ P_0 & \text{for} & t > 0 \end{matrix}$$

by operational methods leads to the Laplace transform

$$\sigma_r^* = \frac{P_0}{2\pi p}\left\{\frac{3(Ap + B)}{2C + B + Ap}\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] - 3r^2z(r^2 + z^2)^{-5/2}\right\}$$

of the visco-elastic stress σ_r, and hence to the result given by Lee.

The problem of the moving load, likewise considered by Lee, is easily solved in the same manner from (3.5) after appropriate changes on the right hand side of that equation.

As another simple example consider the plane problem of a circular hole in an infinite

plane, subject to uniform pressure $-P(t)$. The elastic solution gives for the radial displacement

$$v_r = \frac{P(t)R^2}{2\mu r}; \tag{3.6}$$

hence, by (2.5), the functional equation is

$$Q^q v_r = \frac{R^2}{r} P^p P(t)$$

and for the same material as before, specified by (3.4), this equation becomes

$$\left(A \frac{\partial}{\partial t} + B\right) v_r = \frac{R^2}{r} P(t). \tag{3.7}$$

Thus, the general solution for v_r is

$$v_r = \frac{e^{-Bt/A} R^2}{Ar} \int e^{Bt/A} P(t)\, dt + De^{-Bt/A}, \tag{3.8}$$

where D is an arbitrary constant to be determined from the single initial condition required in this case.

Now let also the radius R of the hole vary with time. The elastic solution (3.6) still applies and the boundary condition is now

$$\sigma_r = -\Pi(t) \quad \text{for} \quad r = R(t).$$

This boundary condition cannot be made time independent by application of the Laplace transform. Nevertheless, by the results of Sec. 2, a visco-elastic solution is given by (3.8), where now R^2 must be included under the integral sign.

4. Conclusions. It has been shown that the elastic and visco-elastic solutions of a given boundary value problem are related to each other through functional equations for the quantities specifying the solutions. If the elastic solution of the boundary value problem is known, these functional equations can be obtained by expressing the elastic constants in this solution in terms of the differential time operators in the general isotropic visco-elastic stress-strain law. In many cases these functional equations turn out to be ordinary differential equations in time whose solutions for given initial conditions make up a visco-elastic solution which satisfies the same boundary conditions as the elastic solution.

If these functional equations are solved by use of the Laplace transform, the later stages of obtaining the visco-elastic solution coincide with the Laplace transform method of Lee [1]. The deduction of Lee's method was based on the concept of an associated elastic problem, requiring the application of the Laplace transform to the boundary conditions as well as to the basic equations. The present paper establishes the independence of the Laplace transform method from the associated elastic problem and hence widens its range of applicability.

Reference

1. E. H. Lee, *Stress analysis in visco-elastic bodies*, Quart. Appl. Math. **13**, No. 2, (July 1955)