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THE ANISOTROPIC TENSORS*

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1. Introduction. We consider an nth order Cartesian tensor with components $a_{i_1 i_2 \cdots i_n}$ ($i_p = 1, 2, 3$) in the rectangular Cartesian coordinate system x_i ($i = 1, 2, 3$) and components $a^*_{i_1 i_2 \cdots i_n}$ in the rectangular Cartesian coordinate system x^*_i, where x_i and x^*_i are related by the orthogonal transformation

$$x^*_i = s_{ij} x_j , \tag{1.1}$$

with

$$s_{ij} s_{ik} = \delta_{jk} , \tag{1.2}$$

where δ_{jk} is the Kronecker delta.

Then,

$$a^*_{i_1 i_2 \cdots i_n} = s_{i_1 j_1} s_{i_2 j_2} \cdots s_{i_n j_n} a_{j_1 j_2 \cdots j_n} . \tag{1.3}$$

If

$$a^*_{i_1 i_2 \cdots i_n} = a_{i_1 i_2 \cdots i_n} , \tag{1.4}$$

for all s_{ij} satisfying (1.2), then $a_{i_1 i_2 \cdots i_n}$ are said to be the components of an isotropic tensor. If the relation (1.4) is valid only for a subgroup $\{\mathbf{T}\}$ of the group of transformations defined by (1.1) and (1.2) we shall describe $a_{i_1 i_2 \cdots i_n}$ as an *anisotropic tensor*. $a_{i_1 i_2 \cdots i_n}$ may then be described as *invariant under the group of transformations*. The isotropic tensor is, of course, invariant under the orthogonal group.

If (1.3) and (1.4) are valid for a transformation s_{ij}, then it follows that

$$a_{i_1 i_2 \cdots i_n} = s_{i_1 j_1} s_{i_2 j_2} \cdots s_{i_n j_n} a_{j_1 j_2 \cdots j_n} . \tag{1.5}$$

If s_{ij} satisfies the relation (1.2), then we readily see from (1.5) that

$$\begin{aligned} a_{i_1 i_2 \cdots i_n} &= s_{i_1 j_1} s_{i_2 j_2} \cdots s_{i_n j_n} a_{j_1 j_2 \cdots j_n} \\ &= s_{j_1 i_1} s_{j_2 i_2} \cdots s_{j_n i_n} a_{j_1 j_2 \cdots j_n} . \end{aligned} \tag{1.6}$$

It is shown in Sec. 3 that any tensor which is invariant under the group of transformations $\{\mathbf{T}\}$ may be expressed as the sum of a number of terms, formed from the outer products of a finite set of tensors, with scalar† coefficients. This finite set of tensors,

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†Throughout this paper, we shall employ the term scalar in the following sense. If corresponding quantities φ and φ^* are defined in the coordinate system x_i and in each coordinate system x^*_i, into which x_i is transformed by the group of transformations $\{\mathbf{T}\}$, then if $\varphi^* = \varphi$, we shall say that φ is scalar with respect to the transformation group $\{\mathbf{T}\}$ or, more briefly, scalar.

each of which is invariant under the group of transformations $\{\mathbf{T}\}$, is described as a *tensor basis* for the group of transformations $\{\mathbf{T}\}$.

It is also shown in Sec. 3 how such a tensor basis can be found for any group of transformations $\{\mathbf{T}\}$ for which we can determine a polynomial basis for polynomials in the components of an arbitrary number of vectors, which are form-invariant under the group $\{\mathbf{T}\}$. In Secs. 4, 5 and 6 examples of such tensor bases are obtained for the monoclinic-domatic and rhombic-pyramidal classes of crystal symmetry and for the case of transverse isotropy.

The results in this paper are obtained for three-dimensional space. However, it is immediately evident that similar methods can be used to determine the isotropic or anisotropic tensors for a space of arbitrary dimensions.

2. General considerations. Let P be any scalar polynomial in the components of n vectors $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$, which is form-invariant under a sub-group $\{\mathbf{T}\}$ of the group of orthogonal transformations defined by (1.1) and (1.2). Then, if

$$u_i^{*(r)} = t_{ij}u_j^{(r)} \qquad (r = 1, 2, \cdots, n), \tag{2.1}$$

where t_{ij} is any transformation of $\{\mathbf{T}\}$, we have

$$P(u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}) = P(u_i^{*(1)}, u_i^{*(2)}, \cdots, u_i^{*(n)}). \tag{2.2}$$

There exists a finite polynomial basis for the polynomials P. Let $I_1, I_2, \cdots, I_N$ denote this polynomial basis. Then,

$$P(u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}) = Q(I_1, I_2, \cdots, I_N), \tag{2.3}$$

where Q is a polynomial in the indicated variables.

Let us consider the scalar polynomial

$$P = a_{i_1 i_2 \cdots i_n} u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_n}^{(n)} \tag{2.4}$$

in the components of the n vectors $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$, the coefficients $a_{i_1 i_2 \cdots i_n}$ in which satisfy the relation

$$\begin{aligned} a_{i_1 i_2 \cdots i_n} &= t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_n j_n} a_{j_1 j_2 \cdots j_n} \\ &= t_{j_1 i_1} t_{j_2 i_2} \cdots t_{j_n i_n} a_{j_1 j_2 \cdots j_n} \end{aligned} \tag{2.5}$$

for every transformation t_{ij} of the group $\{\mathbf{T}\}$.

It is readily seen that P is form-invariant under the transformations of the group $\{\mathbf{T}\}$. From (2.1), we have

$$a_{i_1 i_2 \cdots i_n} u_{i_1}^{*(1)} u_{i_2}^{*(2)} \cdots u_{i_n}^{*(n)} = a_{i_1 i_2 \cdots i_n} t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_n j_n} u_{j_1}^{(1)} u_{j_2}^{(2)} \cdots u_{j_n}^{(n)}. \tag{2.6}$$

From (2.6) and (2.5), we obtain the relation

$$a_{i_1 i_2 \cdots i_n} u_{i_1}^{*(1)} u_{i_2}^{*(2)} \cdots u_{i_n}^{*(n)} = a_{i_1 i_2 \cdots i_n} u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_n}^{(n)}, \tag{2.7}$$

stating the form-invariance of P.

Since they satisfy the relations (2.5), the 3^n quantities $a_{i_1 i_2 \cdots i_n}$ may be regarded as the components of a tensor, invariant under the group of transformations $\{\mathbf{T}\}$, in the coordinate system x_i and in the coordinate systems into which x_i is transformed by the transformations of the group $\{\mathbf{T}\}$. We thus obtain the result: a scalar polynomial in the components of n vectors, linear in each of the vectors, in which the coefficients are tensors invariant under the transformations of the group $\{\mathbf{T}\}$, is form-invariant under the transformations of the group $\{\mathbf{T}\}$.

This conclusion can be readily generalized to yield the result that any scalar polynomial in the components of any number of tensors of any rank, in which the coefficients are tensors invariant under the transformation group $\{\mathbf{T}\}$, is form-invariant under the transformations of the group $\{\mathbf{T}\}$.

Now let us consider a polynomial P in the components of n vectors $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$, of the form

$$P = b_{i_1 i_2 \cdots i_n} u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_n}^{(n)}, \tag{2.8}$$

which is form-invariant under the group of transformations $\{\mathbf{T}\}$. Then,

$$b_{i_1 i_2 \cdots i_n} u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_n}^{(n)} = b_{i_1 i_2 \cdots i_n} u_{i_1}^{*(1)} u_{i_2}^{*(2)} \cdots u_{i_n}^{*(n)}. \tag{2.9}$$

Employing the relation (2.1) in (2.9), we obtain

$$b_{i_1 i_2 \cdots i_n} u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_n}^{(n)} = b_{i_1 i_2 \cdots i_n} t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_n j_n} u_{j_1}^{(1)} u_{j_2}^{(2)} \cdots u_{j_n}^{(n)}. \tag{2.10}$$

Whence,

$$b_{i_1 i_2 \cdots i_n} = b_{i_1 i_2 \cdots i_n} t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_n j_n}. \tag{2.11}$$

We thus see that the coefficients in a homogeneous scalar polynomial of degree n in the components of n vectors, which is linear in each of the vectors and form-invariant under the transformations of the group $\{\mathbf{T}\}$, are the components of a tensor of rank n, invariant under the group of transformations $\{\mathbf{T}\}$.

This conclusion can be readily generalized to yield the result that any scalar polynomial in the components of any number of tensors of any rank, which is form-invariant under the transformations of the group $\{\mathbf{T}\}$, has coefficients which are tensors, invariant under the group of transformations $\{\mathbf{T}\}$.

3. The tensor basis. Since the polynomial P defined by (2.8) is form-invariant under the group of transformations $\{\mathbf{T}\}$, it must be expressible as a polynomial in the elements of a polynomial basis for polynomials in the elements of n vectors which are form-invariant under $\{\mathbf{T}\}$. Also, since P is linear in each of these vectors, the elements of the polynomial basis which are non-linear in any of the vectors need not be considered. Let $J_1, J_2, \cdots, J_R$ be the elements of the polynomial basis which are linear or of degree zero in each of the vectors. Let us suppose that

$$P = \Sigma A_{pq \cdots t} J_p J_q \cdots J_t. \tag{3.1}$$

Then we readily see that

$$\begin{aligned} b_{i_1 i_2 \cdots i_n} &= \frac{\partial^n P}{\partial u_{i_1}^{(1)} \partial u_{i_2}^{(2)} \cdots \partial u_{i_n}^{(n)}} \\ &= \Sigma A_{pq \cdots t} \frac{\partial^n (J_p J_q \cdots J_t)}{\partial u_{i_1}^{(1)} \partial u_{i_2}^{(2)} \cdots \partial u_{i_n}^{(n)}}. \end{aligned} \tag{3.2}$$

The product $J_p J_q \cdots J_t$ occurring in each of the terms under the summation sign in (3.1) is of degree n in the elements of the n vectors $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$ and is linear in each of the vectors. No two of the factors $J_p, J_q, \cdots, J_t$ occurring in a single product involve elements of the same vector. Let us assume that J_p is formed from the vectors $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(p)}$ only, that J_q is formed from the vectors $u_i^{(p+1)}, u_i^{(p+2)}, \cdots, u_i^{(q)}$ only and J_t is formed from the vectors $u_i^{(s+1)}, u_i^{(s+2)}, \cdots, u_i^{(t)}$ only. Then Eq. (3.2) may be re-written as

$$b_{i_1 i_2 \cdots i_n} = \Sigma A_{pq \cdots t} \frac{\partial^p J_p}{\partial u_{i_1}^{(1)} \partial u_{i_2}^{(2)} \cdots \partial u_{i_p}^{(p)}} \frac{\partial^{q-p} J_q}{\partial u_{i_{p+1}}^{(p+1)} \partial u_{i_{p+2}}^{(p+2)} \cdots \partial u_{i_q}^{(q)}} \cdots \frac{\partial^{t-s} J_t}{\partial u_{i_{s+1}}^{(s+1)} \partial u_{i_{s+2}}^{(s+2)} \cdots \partial u_{i_t}^{(t)}} \tag{3.3}$$

Since the quantities $A_{pq \cdots t}$ must be scalars under the transformations $\{\mathbf{T}\}$, we see that $b_{i_1 i_2 \cdots i_n}$ must be given as the sum of terms formed from the outer products of a number of tensors of the type $\partial^p J_p / \partial u_{i_1}^{(1)} \partial u_{i_2}^{(2)} \cdots \partial u_{i_p}^{(p)}$, obtained by differentiating the elements $J_1, J_2, \cdots, J_R$ of the polynomial basis with respect to the vectors from which they are formed. The coefficients of these terms are scalars with respect to the transformation group $\{\mathbf{T}\}$.

4. The monoclinic system—domatic class. Monoclinic symmetry may be described with relation to three preferred directions in space. We denote these directions by the unit vectors $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$. The vectors $\mathbf{h}_2$ and $\mathbf{h}_3$ are not at right-angles and the vector $\mathbf{h}_1$ is perpendicular to the plane defined by $\mathbf{h}_2$ and $\mathbf{h}_3$.

Let us choose as reference frame a rectangular Cartesian coordinate system x_i, the x_1 axis of which coincides in direction with the vector $\mathbf{h}_1$. The axes x_2 and x_3 may be in arbitrary perpendicular directions in the $\mathbf{h}_2\mathbf{h}_3$ plane. Then, the group of transformations $\{\mathbf{T}\}$ associated with the monoclinic-domatic symmetry class consists of the identity transformation $\mathbf{T}_1$ and the transformation $\mathbf{T}_2$ defined by

$$\mathbf{T}_2 = \begin{bmatrix} -1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix}. \tag{4.1}$$

Let us consider n vectors with components $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$ in the coordinate system x_i. Any polynomial

$$P(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \cdots, u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$

is unaltered by the transformation $\mathbf{T}_1$ and is transformed into

$$P(-u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, -u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \cdots, -u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$

by the transformation $\mathbf{T}_2$. The necessary and sufficient condition that P be form-invariant under the transformation $\mathbf{T}_2$ is that it be expressible as a polynomial in

$$u_2^{(r)}, u_3^{(r)}, u_1^{(r)} u_1^{(s)} \qquad (r, s = 1, 2, \cdots, n). \tag{4.2}$$

This result follows immediately from the first main theorem of classical invariant theory†, according to which a polynomial basis for polynomials in ξ_r, η_r $(r = 1, 2, \cdots, n)$, which are form-invariant under interchange of ξ_r and η_r, is given by $\xi_r + \eta_r$ and $\xi_r \eta_s + \xi_s \eta_r$. Taking

$$(\xi_1, \xi_2, \cdots, \xi_n) = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, \cdots, u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$

†See, for example, H. Weyl, *The classical groups, their invariants and representations*, Princeton Univ. Press, Princeton, N. J., 1946, p. 36 et seq.

and

$$(\eta_1, \eta_2, \cdots, \eta_n) = (-u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, \cdots, -u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$

and neglecting redundant elements, we obtain the result (4.2).

The expressions (4.2) form a polynomial basis for polynomials in the components of the n vectors $u_i^{(r)}$ $(r = 1, 2, \cdots, n)$ which are form-invariant under the transformations characterizing the symmetry class considered. From the results of the previous section, we see that the quantities $\partial u_2^{(r)}/\partial u_i^{(r)}$, $\partial u_3^{(r)}/\partial u_i^{(r)}$ and $\partial^2 (u_1^{(r)}u_1^{(s)})/\partial u_i^{(r)}\partial u_j^{(s)}$ $(r \neq s)$ define a basic set of anisotropic tensors for the monoclinic-domatic class in terms of which an arbitrary anisotropic tensor for this class can be expressed. We note that $\partial u_2^{(r)}/\partial u_i^{(r)}$ defines the vector with components

$$\delta_{2i} = (0, 1, 0) = (\alpha_i) \quad \text{(say)} \tag{4.3}$$

in the coordinate system x_i, that $\partial u_3^{(r)}/\partial u_i^{(r)}$ defines the vector with components

$$\delta_{3i} = (0, 0, 1) = (\beta_i) \quad \text{(say)} \tag{4.4}$$

in the coordinate system x_i and $\partial^2(u_1^{(r)}u_1^{(s)})/\partial u_i^{(r)}\partial u_j^{(s)}$ $(r \neq s)$ defines the tensor of rank 2 with components

$$\begin{bmatrix} 1, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix} = (\gamma_{ij}) \quad \text{(say)} \tag{4.5}$$

in the coordinate system x_i. Further, any tensor with components $a_{ij\cdots k}$ in the coordinate system x_i, which is invariant under the transformations of the monoclinic-domatic class, is expressible as the sum of terms formed from outer products of α_i, β_i and γ_{ij} with scalar coefficients which are invariant under the transformations of the class.

5. The rhombic crystal system—rhombic-pyramidal class. Rhombic symmetry may be described with relation to three preferred directions in space, defined by the mutually perpendicular unit vectors $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$. We choose as our reference system the rectangular Cartesian coordinate system x_i, the axes of which coincide in direction with the vectors $\mathbf{h}_i$. The group of transformations $\{\mathbf{T}\}$ associated with the rhombic-pyramidal class is composed of the identity transformation $\mathbf{T}_1$ and the transformations $\mathbf{T}_3$, $\mathbf{T}_4$ and $\mathbf{T}_5$ defined by

$$\mathbf{T}_3 = \begin{bmatrix} 1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & 1 \end{bmatrix}, \quad \mathbf{T}_4 = \begin{bmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{T}_5 = \begin{bmatrix} 1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & -1 \end{bmatrix}. \tag{5.1}$$

We can determine, in a manner similar to that adopted in Sec. 4, a polynomial basis for polynomials P in the components of n vectors which are form-invariant under the transformations (5.1). If these vectors have components $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$ in the coordinate system x_i, then invariance under the transformation $\mathbf{T}_3$ implies that P must be expressible as a polynomial P' (say) in the quantities $u_1^{(r)}$, $u_3^{(r)}$, $u_2^{(r)} u_2^{(s)}$ $(r, s = 1, 2, \cdots, n)$. Thus,

$$P = P'(u_1^{(r)}, u_3^{(r)}, u_2^{(r)}u_2^{(s)}). \tag{5.2}$$

Invariance of P' under the transformation $\mathbf{T}_4$ imposes on P' the limitation

$$P'(u_1^{(r)}, u_3^{(r)}, u_2^{(r)}u_2^{(s)}) = P'(u_1^{(r)}, -u_3^{(r)}, u_2^{(r)}u_2^{(s)}). \tag{5.3}$$

It is readily seen that P' must be expressible as a polynomial in the quantities $u_1^{(r)}$, $u_2^{(r)}\ u_2^{(s)}$ and $u_3^{(r)}\ u_3^{(s)}$. Such a polynomial is obviously invariant under the transformation $\mathbf{T}_5$ and hence under all the transformations of the group $\{\mathbf{T}\}$.

Following the procedure described in Sec. 3, we see that $\partial u_1^{(r)}/\partial u_i^{(r)}$ ($= \alpha_i$, say), $\partial^2(u_2^{(r)}\ u_2^{(s)})/\partial u_i^{(r)}\ \partial u_j^{(s)}$ $(r \neq s)$ ($= \beta_{ij}$, say), $\partial^2(u_3^{(r)}\ u_3^{(s)})/\partial u_i^{(r)}\ \partial u_j^{(s)}$ $(r \neq s)$ ($= \gamma_{ij}$, say) define a basic set of anisotropic tensors for the rhombic-pyramidal class. We have

$$(\alpha_i) = (\delta_{1i}) = (1, 0, 0)$$

$$(\beta_{ij}) = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 0 \end{bmatrix} \quad \text{and} \quad (\gamma_{ij}) = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 1 \end{bmatrix}. \tag{5.4}$$

6. Transverse isotropy. Transverse isotropy may be described with relation to a single preferred direction in space defined by a unit vector $\mathbf{h}$. We choose as our reference frame a rectangular Cartesian coordinate system x_i , the x_3 axis of which coincides in direction with the vector $\mathbf{h}$. Then, the group of transformations $\{\mathbf{T}\}$ associated with transverse isotropy with respect to the direction of $\mathbf{h}$ is composed of the transformation $\mathbf{T}_2$ defined by (4.1), the transformation $\mathbf{T}_\omega$ defined by

$$\mathbf{T}_\omega = \begin{bmatrix} \cos\omega, & \sin\omega, & 0 \\ -\sin\omega, & \cos\omega, & 0 \\ 0\ , & 0\ , & 1 \end{bmatrix} \qquad (0 \leq \omega < 2\pi), \tag{6.1}$$

and the transformation $\mathbf{T}_2\mathbf{T}_\omega$.

We can determine a polynomial basis for polynomials P in the components of n vectors which are linear in each of the vectors and form-invariant under the transformations $\mathbf{T}_2$, $\mathbf{T}_\omega$ and $\mathbf{T}_2\mathbf{T}_\omega$. Let $u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n)}$ be the components of the n vectors in the coordinate system x_i . Then, the limitations imposed on P by the condition that it be form-invariant under the transformations $\mathbf{T}_2$, $\mathbf{T}_\omega$ and $\mathbf{T}_2\mathbf{T}_\omega$ are

$$\begin{aligned} P(u_1^{(r)}, u_2^{(r)}, u_3^{(r)}) &= P(u_1^{(r)} \cos\omega + u_2^{(r)} \sin\omega,\ - u_1^{(r)} \sin\omega + u_2^{(r)} \cos\omega, u_3^{(r)}) \\ &= P(-u_1^{(r)}\ , u_2^{(r)}\ , u_3^{(r)}\ ). \end{aligned} \tag{6.2}$$

It is seen that the dependence of P on $u_3^{(r)}$ gives rise to no restrictions on the form of P and, therefore, a polynomial basis for P is formed by $u_3^{(r)}$ $(r = 1, 2, \cdots, n)$ and the polynomial basis for polynomials P' which satisfy the conditions

$$\begin{aligned} P'(u_1^{(r)}, u_2^{(r)}) &= P'(u_1^{(r)} \cos\omega + u_2^{(r)} \sin\omega,\ - u_1^{(r)} \sin\omega + u_2^{(r)} \cos\omega) \\ &= P'(-u_1^{(r)}, u_2^{(r)}), \end{aligned} \tag{6.3}$$

and are linear in the two-dimensional vectors $u_\alpha^{(r)}$ $(\alpha = 1, 2)$. The relations (6.3) imply that P' is form-invariant under the group of two-dimensional orthogonal transformations, proper and improper. Now, the quantities $u_\alpha^{(r)}\ u_\alpha^{(s)}$ $(r, s = 1, 2, \cdots, n; \alpha = 1, 2)$ form a polynomial basis for polynomials in the components of n two-dimensional vectors $u_\alpha^{(r)}$

which are form-invariant under the full orthogonal group†. Since P' is linear in the vectors, we can omit those elements $u_\alpha^{(r)} u_\alpha^{(s)}$ for which $r = s$. We thus have the result that the quantities

$$u_3^{(r)}, u_\alpha^{(r)} u_\alpha^{(s)} \qquad (r, s = 1, 2, \cdots, n; r \neq s) \tag{6.4}$$

form a polynomial basis for polynomials in the components of n vectors $u_i^{(r)}$ which are linear in each of the vectors and form-invariant under the transformations associated with transverse isotropy about the x_3 axis.

As in the previous sections, we immediately see that a basic set of anisotropic tensors for transverse isotropy about the x_3 axis is defined by $\partial u_3^{(r)}/\partial u_i^{(r)}$ $(= \alpha_i$, say) and $\partial(u_\alpha^{(r)} u_\alpha^{(s)}/\partial u_i^{(r)} \partial u_j^{(s)}$ $(= \beta_{ij})$ $(r \neq s; \alpha = 1, 2; i, j = 1, 2, 3)$. We obtain immediately

$$(\alpha_i) = (\delta_{3i}) = (0, 0, 1)$$

and

$$(\beta_{ij}) = (\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j}) = \begin{bmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 0 \end{bmatrix}. \tag{6.5}$$

ON A PRINCIPLE OF RECIPROCITY BETWEEN HIGH- AND LOW-FREQUENCY PROBLEMS CONCERNING LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER*

By AUREL WINTNER (*The Johns Hopkins University*)

1. The following considerations, in which nomenclature and point of view will be the same as in two earlier notes[1], deal with the oscillatory case of the differential equation

$$x'' + f(t)x = 0, \tag{1}$$

in which $f(t)$ is a real-valued, continuous function for large positive t, and $x(t)$ is any real-valued solution distinct from the trivial solution $x(t) \equiv 0$.

If (1) is oscillatory when $f(t) = f_1(t)$, and if $f_1(t) \leqq f_2(t)$, then (1) is oscillatory when $f(t) = f_2(t)$. This follows from Sturm's comparison theorem, which implies also the following fact: If (1) is oscillatory and if $d_n = d_n(f) = d_n(f; x)$ denotes the distance $t_{n+1} - t_n$, where $t_n = t_n(f; x)$ is the nth zero of a solution $x = x(t) = x_f(t)$ of (1), then $\limsup d_n(f_1)/d_n(f_2) < 1$, where $n \to \infty$, holds whenever $f_2(t) - f_1(t)$ exceeds a positive lower bound as $t \to \infty$.

By choosing $f_1(t) = \omega^2$, where ω is a positive constant, and letting $\omega \to \infty$ or $\omega \to 0$, it follows that $d_n(f) \to 0$ as $n \to \infty$ if

$$f(t) \to \infty \quad \text{as} \quad t \to \infty, \tag{2}$$

†See, for example, H. Weyl, loc. cit. p. 52 et. seq.

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[1]A. Wintner, Quart. Appl. Math. 7, 115-117 (1949) and 13, 192-195 (1955). These two papers will be referred to as [1] and [2] respectively.