ON THE ANALYTIC FUNCTIONS OF ORDER n[1] (APPLICATION TO THE PLANE ELASTICITY)

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1. In the course of an investigation on a problem of plane elasticity we have been led to consider the so-called "analytic functions of order n". These functions, introduced for the first time by P. Burgatti [2][2], are, by definition, expressions of the form

$$\Omega_n(z, \bar{z}) = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z), \tag{1}$$

where $f_\nu(z)$ $(\nu = 0, 1, \cdots, n-1)$ are arbitrary analytic functions of the complex variable $z = x + i\,y$ in a domain D and $\bar{z}$ is the imaginary conjugate[3] of z. The positive integer n indicates the order of the function. The properties of analytic functions of order n seem to be in close connection with those of polyharmonic functions of two variables. On the other hand, the latter, especially the biharmonic functions occur in many problems of physics and mechanics; and therefore, their study is particularly important for applications. In what follows, keeping in view this point, we shall establish certain properties of analytic functions of order n and indicate some applications to two-dimensional elasticity.

An immediate property of the analytic functions of order n is the following.

The product of any finite number of analytic functions of order $n_1, n_2, \cdots, n_m$; $\Omega_{n_1}, \Omega_{n_2}, \cdots, \Omega_{n_m}$ defined respectively in domains $D_1, D_2, \cdots, D_m$ is an analytic function $\Omega_N = \Omega_{n_1}\Omega_{n_2}\cdots\Omega_{n_m}$ of order $N = n_1 + n_2 + \cdots + n_m - (m-1)$ defined in the domain D' which is the common part of $D_1, D_2, \cdots, D_m$.

We shall mention two special cases: (i) the product of an analytic function of order n, Ω_n by an analytic function of order 1, Ω_1, i.e. by an analytic function in the ordinary sense $F(z)$, is an analytic function $\Omega_n^* = F(z)\Omega_n$ of order n; (ii) the mth power of Ω_n is the analytic function $(\Omega_n)^m$ of order $m(n-1)+1$.

The above results follow very readily from the form of expression (1).

2. Connection with real polyharmonic functions. A real polyharmonic function of order h, H of the variables x, y is, by definition, a solution of the equation $(\partial^2/\partial x^2 + \partial^2/\partial y^2)^h H = 0$ or, in terms of the variables z and $\bar{z}$, $\partial^{2h}H/\partial\bar{z}^h\partial z^h = 0$. Burgatti showed that the real and imaginary parts P and Q of an analytic function of order n, Ω_n satisfy the above equations, and called them "conjugate polyharmonic functions"[4]. These

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[2]Numbers in brackets refer to the bibliography at the end of the paper.

[3]In the following any quantity $\bar{x}$ will represent the imaginary conjugate of x.

[4]This designation is justified by the fact that Almansi's representations [1], [2] for two such polyharmonic functions contain two sets of harmonic functions which are conjugate two by two.

functions can be written in the following form

$$\begin{cases} P = \frac{1}{2}(\Omega_n + \bar{\Omega}_n) = \frac{1}{2} \sum_{\nu=0}^{n-1} [\bar{z}^\nu f_\nu(z) + z^\nu \bar{f}_\nu(\bar{z})], \\ Q = \frac{1}{2i} (\Omega_n - \bar{\Omega}_n) = \frac{1}{2i} \sum_{\nu=0}^{n-1} [\bar{z}^\nu f_\nu(z) - z^\nu \bar{f}_\nu(\bar{z})]. \end{cases} \tag{2}$$

Burgatti's definition is concerned with the conjugate polyharmonic functions of order n which are the real and imaginary parts of an analytic function of the same order n, Ω_n. However, it should be remarked that the number n, i.e. the order of the analytic function Ω_n, is generally different from the order of harmonicity n' of the polyharmonic functions which constitute its real and imaginary parts. It is also clear that the number n' is less than or equal to n, and that n' is the highest power plus 1 of the product $z\,\bar{z}$ existing in the terms of Ω_n. For later reference, we shall call these numbers n and n', "order of analyticity" and "order of harmonicity" respectively of the function Ω_n. As examples we can cite the functions

$$\sum_{\nu=0}^{n-1} a_\nu \bar{z}^\nu, \qquad \sum_{\nu=0}^{n-1} a_\nu \bar{z}^\nu z^\nu$$

which are both of order of analyticity n, and have as order of harmonicity 1 and n respectively. The coefficients a_ν are arbitrary complex constants.

3. Differential relations between two conjugate polyharmonic functions. Let P and Q be two conjugate polyharmonic functions of order m which we assume to be the real and imaginary parts of an analytic function of the same order m, Ω_m; that is, $\Omega_m = P + iQ$. The equation $\partial^m \Omega_m / \partial \bar{z}^m = 0$ or $(\partial/\partial x + i\partial/\partial y)^m (P + iQ) = 0$ is equivalent to (1). If $\delta_1^{(m)}$ and $\delta_2^{(m)}$ designate respectively the real and imaginary parts of the operator $(\partial/\partial x + i\partial/\partial y)^m$, then we have

$$\delta_1^{(m)} + i\delta_2^{(m)} = (\partial/\partial x + i\partial/\partial y)^m. \tag{3}$$

The expansion of the right-hand side of (3) by the binomial formula furnishes for $\delta_1^{(m)}$ and $\delta_2^{(m)}$ the following expressions:

$$\delta_1^{(m)} = \sum_{\nu=0}^{N} (-1)^\nu \binom{m}{2\nu} \partial^m / \partial x^{m-2\nu}\, \partial y^{2\nu}$$

[$N = (m - 1)/2$ if m is odd and $N = m/2$ if m is even],

$$\delta_2^{(m)} = \sum_{\nu=0}^{N} (-1)^\nu \binom{m}{2\nu + 1} \partial^m / \partial x^{m-(2\nu+1)}\, \partial y^{2\nu+1}$$

[$N = (m - 1)/2$ if m is odd and $N = (m - 2)/2$ if m is even], where $\binom{m}{k}$ is the number of combinations of m distinct objects taken k at a time and $\binom{m}{0}$ is assumed to be equal to 1. Then writing $(\delta_1^{(m)} + i\delta_2^{(m)})(P + iQ) = 0$, we obtain the partial differential equations

$$\delta_1^{(m)} P = \delta_2^{(m)} Q, \qquad \delta_1^{(m)} P = -\delta_2^{(m)} Q, \tag{4}$$

which constitute for two arbitrary polyharmonic functions P and Q of the same order m, a necessary condition that they be conjugate or, in other words, that they be the real and imaginary parts of an analytic function of order m. In this respect the relations (4) correspond to those of Cauchy-Riemann for two conjugate harmonic functions[5].

4. Real analytic functions of order n. We shall next establish the general form of the real analytic functions of order n which occur in certain applications. The problem is to determine functions $g_\nu(z)$ in such a way that the expression

$$\omega_n = \sum_{\nu=0}^{n} \bar{z}^\nu g_\nu(z)$$

be real. It is known *a priori* that the functions $g_\nu(z)$ are polynomials in z of degree $n-1$ with suitable coefficients. The latter can be calculated in the following way

$$\begin{aligned}\omega_n = {} & \bar{z}^{n-1}(a_1^1 z^{n-1} + a_1^2 z^{n-2} + \cdots + a_1^{n-1} z + a_1^n) \\ & + \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ & + \bar{z}^{n-\delta}(a_\delta^1 z^{n-1} + a_\delta^2 z^{n-2} + \cdots + a_\delta^n) \\ & + \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ & + \bar{z}(a_{n-1}^1 z^{n-1} + a_{n-1}^2 z^{n-2} + \cdots + a_{n-1}^n) \\ & + (a_n^1 z^{n-1} + a_n^2 z^{n-2} + \cdots + a_n^n),\end{aligned}$$

or, more compactly,

$$\omega_n = \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} a_\nu^\mu \bar{z}^{n-\nu} z^{n-\mu}, \tag{5}$$

where a_k^k is real and $a_k^i = a_i^k$. The expression (5) is the general form of a real analytic function of order n; it can be observed that this expression is a special polyharmonic function and that the order of analyticity and the order of harmonicity are necessarily equal for such a function.

5. Definition of the transformation T_k. Application to the analytic functions of order n. Let $p(z)$ and $q(z)$ be two polynomials in z both of the same degree k and having no common factor,

$$p(z) = \sum_{\nu=0}^{k} a_\nu z^\nu, \qquad q(z) = \sum_{\nu=0}^{k} b_\nu z^\nu,$$

and replace in an analytic function of order n,

$$\Omega_n = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z)$$

z and $\bar{z}$ by p/q and $\bar{p}/\bar{q}$ respectively. This substitution will be called a "transformation T_k". The effect of this transformation on Ω_n is expressed by

Theorem 1. *If* $\Omega_n(z, \bar{z}) = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z)$ *is an analytic function having as order of analyticity and order of harmonicity the same number* n, *then the expression* $(\bar{p})^{n-1}\,\Omega_n(p/q, \bar{p}/\bar{q})$ *is an analytic function having as its two orders the same number* $k(n-1)+1$.

Proof. When in $\Omega_n(z, \bar{z})$ the variables z and $\bar{z}$ are replaced by p/q and $\bar{p}/\bar{q}$, respectively

[5] One has also $\nabla^{2m} = \delta_1^{(2m)} + \delta_2^{(2m)}$, for $\nabla^{2m} = (\partial/\partial x + i\partial/\partial y)^m\,(\partial/\partial x - i\partial/\partial y)^m = (\delta_1^{(m)} + i\delta_2^{(m)})(\delta_1^{(m)} - i\delta_2^{(m)}) = \delta_1^{(2m)} + \delta_2^{(2m)}$; and the condition that H be a polyharmonic function of order m takes the form $(\delta_1^{(2m)} + \delta_2^{(2m)})\,H = 0$.

this function becomes

$$\Omega_n(p/q, \bar{p}/\bar{q}) = \sum_{\nu=0}^{n-1} (\bar{p}/\bar{q})^\nu f_\nu(p/q)$$

$$= 1/(\bar{q})^{n-1} \sum_{\nu=0}^{n-1} \bar{p}^\nu \bar{q}^{n-(\nu+1)} f_\nu(p/q);$$

hence we obtain

$$(\bar{q})^{n-1}\Omega_n(p/q, \bar{p}/\bar{q}) = \sum_{\nu=0}^{n-1} \bar{p}^\nu \bar{q}^{n-(\nu+1)} f_\nu(p/q). \tag{6}$$

In (6) the product $\bar{p}^\nu \bar{q}^{n-(\nu+1)}$ is a polynomial in $\bar{z}$ of degree $k(n - 1)$ and $f_\nu(p/q)$ is a function $f_\nu^*(z)$ of z, then the right-hand side of (6) is of the form

$$\sum_{\lambda=0}^{k(n-1)} \bar{z}^\lambda F_\lambda(z) \tag{7}$$

which represents an analytic function of order of analyticity $k(n - 1) + 1$. Now writing $F_\lambda(z)/z^\lambda = G(z)$, (7) becomes

$$\sum_{\nu=0}^{k(n-1)} \bar{z}^\lambda z^\lambda G(z). \tag{8}$$

From (8) it follows that the new analytic function also has the number $k\ (n - 1) + 1$ as well as order of harmonicity. Hence the proof is complete.

Special case. The transformation T_k in general changes the two orders of Ω_n ; these orders remain unchanged if and only if $k = 1$, so that the linear transformation T_1 , $(az + b)/(cz + d)$ where a, b, c, d are arbitrary complex constants, gives from Ω_n another analytic function having the same number n as both orders.

6. Effects of the transformation T_k on a polyharmonic function. Let us now consider a real polyharmonic function of order n of two variables and suppose that it constitutes the real or imaginary part, say real part, of an analytic function of the same order n. The effect of the transformation T_k on such a function is expressed by the following theorem

Theorem 2. If $H(z, \bar{z})$ is a real polyharmonic function of order n, then $(q\bar{q})^{n-1} H(p/q, \bar{p}/\bar{q})$ is a polyharmonic function of order $k(n - 1) + 1$.

Proof. Writing that $H(z, \bar{z})$ is the real part of an analytic function of order n, Ω_n , we have

$$H(z, \bar{z}) = \tfrac{1}{2}(\Omega_n + \overline{\Omega}_n); \tag{9}$$

then replacing in (9) the variables z and $\bar{z}$ respectively by p/q and $\bar{p}/\bar{q}$, and multiplying both members by $(q\bar{q})^{n-1}$, we obtain

$$(q\bar{q})^{n-1}H(p/q, \bar{p}/\bar{q}) = \tfrac{1}{2}\{(\bar{q}^{n-1}\Omega_n(p/q, \bar{p}/\bar{q}))q^{n-1} + (q^{n-1}\overline{\Omega}_n(\bar{p}/\bar{q}, p/q))\bar{q}^{n-1}\}. \tag{10}$$

By Theorem 1, the term $(\bar{q})^{n-1}\,\Omega_n(p/q, \bar{p}/\bar{q})$ in the right-hand side of (10) is an analytic function having as both orders the number $k(n - 1) + 1$. The multiplication of this function by q^{n-1} gives another analytic function of the same order. On the other hand, the term $q^{n-1}\ \overline{\Omega}_n(\bar{p}/\bar{q},\ p/q)\ (\bar{q})^{n-1}$ is the imaginary conjugate of the preceding analytic function. It follows that the right-hand side of (10) represents a polyharmonic function of order n. This completes the proof.

The special case mentioned above for Theorem 1 holds also for Theorem 2; that is, the linear transformation $(az + b)/(cz + d)$ conserves the order of polyharmonic functions. Theorem 1 may be considered, in the case of two variables, as a generalization of the similar theorems previously established by Lord Kelvin [5], Levi-Civita [4], P. Burgatti [2] and Nicolesco [8].

7. Nets of principal lines attached to analytic functions. Let ρ and -2θ be the absolute value and argument of an analytic function of order n, Ω_n, and write

$$\Omega_n = \rho \bar{e}^{2i\theta} = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z);$$

the values of ρ and θ are given by $\rho = (\Omega_n \bar{\Omega}_n)^{\frac{1}{2}}$ and

$$\theta = \tfrac{1}{2} \text{ arc } tgi \, \frac{\Omega_n - \bar{\Omega}_n}{\Omega_n + \bar{\Omega}_n} = \tfrac{1}{4} i \log \frac{\bar{\Omega}_n}{\Omega_n}. \tag{11}$$

If the function Ω_n contains real factors (constant or not), then the value of its argument is evidently independent of those factors and is always equal to that of the complex factor which Ω_n contains. Suppose, for instance, that Ω_n is the product of an analytic function of order r, Ω_r, which contains no real factor, and a real analytic function of order s, ω_s; it is clear, by Eq. (1), that $r = n - (s - 1)$ and that $\Omega_n = \omega_s . \Omega_{n-(s-1)}$. Since ω_s is real, we have $\arg \Omega_n = \arg \Omega_{n-(s-1)}$. In the following, the order of harmonicity h of $\Omega_{n-(s-1)}$, that is the order of harmonicity of the complex factor contained in Ω_n, will also be called "order of the argument of Ω_n", and we shall use the notation $-2\theta_h$ to designate an argument of order h.

The application of the transformation T_k to the argument of an analytic function Ω_n gives a simple property which may be observed as a corollary of one of the preceding theorems.

If $-2\theta_h\ (z, \bar{z})$ is the argument of an analytic function of order h, Ω_h, then $-2\theta_h$ $(p/q, \bar{p}/\bar{q})$ is the argument $-2\theta_l$ of an analytic function whose order of harmonicity is less than or equal to $l = k(h - 1) + 1$. The proof of this result is easily obtained when Theorem 2 is applied to both terms of the first expression of θ in (11), or when Theorem 1 is used for the second expression in (11).

We now consider the family of curves (C_h) in the plane (x, y) whose direction angle with the x-axis at each point is equal to θ, -2θ being the argument of an analytic function Ω_n. The curves (C_h) with their orthogonal trajectories constitute an orthogonal net of plane curves which we call "net of principal lines attached to Ω_n". As an example, when $h = 1$, we have an isometric net of plane curves. We shall be concerned with certain properties of nets of principal lines which can easily be deduced from those of analytic functions of order n.

Theorem 3. *Let (C_h) be a net of principal lines attached to an analytic function of order n, Ω_n; it is possible to determine real analytic functions ω_m of arbitrary order m such that the function $\omega_m . \Omega_n$ admits (C_h) as a net of principal lines.*

Proof. Let $\Omega_n = \rho e^{-2i\theta_h} = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z)$ be the given analytic function and let

$$\omega_m = \sum_{\nu=1}^{m} \sum_{\mu=1}^{m} a_\nu^k \bar{z}^{m-\nu} z^{m-\mu}$$

be a real analytic function of order m; the argument of the analytic function of order

$n + m - 1$, $\Omega_{n+m-1} = \omega_m \Omega_n$ will be $\arg \Omega_{n+m-1} = \arg \Omega_n = -2\theta_k$; if (C_k) is assumed to be a net of principal lines attached to an analytic function of order $N = n + m - 1$, then

$$\Omega_N = \omega_m \cdot \Omega_n = \Omega_n \cdot \sum_{\nu=1}^{m} \sum_{\mu=1}^{m} \bar{z}^{m-\nu} z^{m-\mu}.$$

Special case. When $n = 1$, we have an isometric net (C_1), and Ω_N is of the form

$$\Omega_N = F(z) \sum_{\nu=1}^{m} \sum_{\mu=1}^{m} a_\nu^\mu \bar{z}^{m-\nu} z^{m-\mu}, \tag{12}$$

where $F(z)$ is an ordinary analytic function whose argument is the harmonic angle -2θ.

We shall end this section with the extension of a property of the ordinary analytic functions to the analytic functions of order n. This will be the subject of the following problem.

Problem. Consider two analytic functions of the same order. Determine the conditions under which the arguments of these functions are two symmetric angles with respect to the x-axis. Let

$$\Omega_{n+1} = \sum_{\mu=0}^{n} \bar{z}^\mu f_\mu(z), \qquad \Omega^*_{n+1} = \sum_{\nu=0}^{n} \bar{z}^\nu f^*_\nu(z)$$

be two analytic functions both of order $n + 1$, and let

$$-\frac{1}{2i}[\log(\bar{\Omega}_{n+1}/\Omega_{q+1})], \qquad -\frac{1}{2i}[\log(\bar{\Omega}^*_{n+1}/\Omega^*_{n+1})] \tag{13}$$

be their arguments. The condition that the angles (13) be symmetric with respect to the x-axis is equivalent to $\bar{\Omega}_{n+1} \,.\, \Omega^*_{n+1} = \Omega_{n+1} \,.\, \Omega^*_{n+1} =$ a real quantity. Hence, we have to form the product $\Omega_{n+1}\, \Omega^*_{n+1}$ which will be a real analytic function of order $2n + 1$, ω_{2n+1} (Sects. 1 and 4). Therefore,

$$\omega_{2n+1} = \sum_{\mu=0}^{n} \sum_{\nu=0}^{n} \bar{z}^{\mu+\nu} f_\mu(z) f^*_\nu(z) = \sum_{\mu=1}^{2n+1} \sum_{\nu=1}^{2n+1} a_\nu^\mu \bar{z}^{2n-\nu+1} z^{2n-\mu+1}. \tag{14}$$

Equating now the coefficients of $\bar{z}^m (m = 0, 1, \cdots, 2n)$ in the two members of (14), we have

$$\left\{\begin{array}{ll} \displaystyle\sum_{\mu=0}^{m} f_\mu f^*_{m-\mu} = \sum_{\mu=1}^{2n+1} a^\mu_{2n-m+1} z^{2n-\mu+1}, & \text{if} \quad n \geqq m \geqq 0, \\ \text{or} & \\ \displaystyle\sum_{\mu=m-n}^{n} f_\mu f^*_{m-\mu} = \sum_{\mu=1}^{2n+1} a^\mu_{2n-m+1} z^{2n-\mu+1}, & \text{if} \quad 2n \geqq m > n. \end{array}\right. \tag{15}$$

When m varies from 0 to $2n$, (15) furnishes a system of $2n + 1$ equations between $2n + 2$ unknown functions f_μ and f^*_ν $(\mu, \nu = 0, 1, \cdots, n)$. If one takes, for instance, f_μ/f_0 and f^*_ν/f^*_0 as unknowns of the problem, the number of the unknowns reduces to $2n$ for $2n + 1$ equations. Suppose that the unknowns f^*_ν/f^*_0 are eliminated from the system (15), the n unknowns f_μ/f_0 will remain for n equations. On the other hand, Eq. (15) are obviously symmetric with respect to f_μ and f^*_ν ; then either f_μ/f_0 or f^*_ν/f^*_0 will satisfy the same conditions. We do not want to enter into the detail of this discussion. A solution of the problem is known *a priori*; it consists of the isometric case, i.e. when the angle θ

is a harmonic function. In order to indicate an example of a non-isometric case, we shall give the following functions

$$\begin{aligned} \Omega_{n+1} &= F(z) \sum_{k=1}^{n+1} \sum_{j=1}^{n+1} C_{kj}\, \bar{z}^{n-k+1} z^{n-j+1}, \\ \Omega^*_{n+1} &= 1/F(z) \sum_{k=1}^{n+1} \sum_{j=1}^{n+1} \overline{C_{kj}}\, \bar{z}^{n-j+1} z^{n-k+1}, \end{aligned} \tag{16}$$

where the coefficients C_{kj} are arbitrary complex constants.

8. Applications to plane elasticity. Since the stress deviator of a two-dimensional elastic field is an analytic function of order 2, $\Omega_2 = \bar{z}f_1(z) + f_0(z)$, the above developments on the analytic functions of order n may be used, as far as the geometry of the elastic field is concerned, for the problems of two-dimensional elasticity. The net of the lines of principal stress constitutes a net of principal lines attached to the analytic function Ω_2 which represents the stress deviator of our elastic field; therefore, the theorem established for the nets attached to analytic functions of order n can be applied to the case of plane elasticity.

A. We consider first the special case of Theorem 1 for which $k = 1$. Thus we have:

If $\Omega_2 = \bar{z}f_1(z) + f_0(z)$ is the stress deviator of a two-dimensional elastic field in a domain D, then $(\bar{c}\bar{z} + \bar{d}).\ \Omega_2[(az + b)/(cz + d),\ (\bar{a}\bar{z} + \bar{b})/(\bar{c}\bar{z} + \bar{d})]$ is the stress deviator of another two-dimensional elastic field in the domain D' into which D is carried by the transformation T_1.

B. Considering now the special case of transformation T_1 in the property concerning the invariance of the argument of an analytic function of order n, we can state:

Let $\theta(z, \bar{z})$ be the direction angle of a plane elastic stress field defined in a domain D; the angle

$$\theta\left(\frac{az + b}{cz + d},\quad \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}\right)$$

is the direction angle of another plane elastic stress field defined in the domain D' into which D is carried by the transformation T_1.

C. Let us now take Theorem 3 in the special case of isometric nets. For plane elasticity this theorem takes the following form which states, as a special case of the general property, a well-known result.

Let (C_1) be an isometric net of plane curves defined by means of the argument of an analytic function $\Omega_1 = F(z)$; it is possible to determine real analytic functions of order 2, ω_2, such that the plane elastic stress field defined by the analytic function $\Omega_2 = \omega_2\, \Omega_1$ accepts (C_1) as a net of lines of principal stress.

The corresponding analytic function Ω_2 of order 2 is determined by Eq. (12); in fact, taking $m = 2$, $N = 2$ in this formula we obtain

$$\Omega_2 = F(z)(a_1^1\bar{z}z + a_2^1 z + a_1^2\bar{z} + a_2^2), \tag{17}$$

where the coefficients have the same meaning as in Sect. 4. This question has previously been discussed in different ways by P. F. Nemenyi [6], U. Wegner [11], W. Prager [9], S. Süray [10], R. Legendre [3] and P. F. Nemenyi and A. W. Saenz [7].

D. The general problem discussed above has been solved in the special case of plane elasticity by Nemenyi and Saenz [7]; these authors also gave an example of a non-isometric case of this problem. A more general example may be deduced from the expressions (16) if one takes $n = 1$:

$$\Omega_2 = F(z)(a_{11}\bar{z}z + a_{12}\bar{z} + a_{12}z + a_{22}),$$

$$\Omega_2^* = 1/F(z)(\bar{a}_{11}z\bar{z} + \bar{a}_{12}\,z + \bar{a}_{22}),$$

where the coefficients are arbitrary complex constants.

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