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THE TORSION OF A HOLLOW SQUARE*

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Summary. SEAC‡ is used to give an approximate solution to the torsion problem for a hollow square, with accurate determination of lower and upper bounds for torsional rigidity.

1. The torsion problem for a multiply connected cross section. To solve the torsion problem for a beam of multiply connected cross section, we seek a function $\psi(x, y)$ which is harmonic in the domain A occupied by the material, and such that

$$\begin{aligned} \psi &= \tfrac{1}{2}r^2 \text{ on the outer boundary } B_0 , \\ \psi &= \tfrac{1}{2}r^2 + C_\lambda \text{ on the inner boundaries } B_\lambda , \end{aligned} \tag{1.1}$$

the constants C_λ being chosen so that for each inner boundary we have

$$\int_{B_\lambda} \frac{\partial \psi}{\partial n} \, dB_\lambda = 0. \tag{1.2}$$

Here $r^2 = x^2 + y^2$, where x, y are rectangular Cartesian coordinates in the plane of the cross section.

The components of shearing stress are

$$\mu\alpha \frac{\partial}{\partial y} (\psi - \tfrac{1}{2}r^2), \qquad -\mu\alpha \frac{\partial}{\partial x} (\psi - \tfrac{1}{2}r^2), \tag{1.3}$$

where μ is the rigidity and α the twist per unit length. The lines of stress are given by the equation

$$\psi - \tfrac{1}{2}r^2 = \text{const.}, \tag{1.4}$$

this family of curves including, of course, the curves B_0 , B_λ .

The warping of the section is given by $\phi(x, y)$, which is the harmonic function conjugate to ψ.

If we denote the torque by $\mu\alpha\Gamma$, then

$$\Gamma = I - \int \left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 \right] dA, \tag{1.5}$$

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when I is the polar moment of inertia of the cross section,

$$I = \int (x^2 + y^2)\, dA. \tag{1.6}$$

2. The torsion of a hollow square. The method used below for solving approximately the torsion problem for a hollow square cross section is essentially the hypercircle method, originally developed by Prager and Synge[1] for more general problems in elasticity.

Details of the method will be found in a forthcoming book,[2] together with numerical results obtained by a desk calculator. The present paper is concerned with the application of SEAC to these calculations, and only essential formulas will be given to make the computations intelligible.

The method is applicable in principle to any cross section, and is particularly suited to a cross section which is a hollow rectangle or a hollow square. The only case treated here in detail is the case of a hollow square bounded externally by a square B_0 of side s and internally by a square B_1 of side $\frac{1}{2}s$, as shown in Fig. 1.

The square is triangulated as shown, a good approximation demanding a fine triangulation. We speak of approximation of order n when the outer side is divided into n equal parts. On account of symmetry, only an octant is needed in the calculations; Fig. 1 shows the triangulated octant MNZY for $n = 16$.

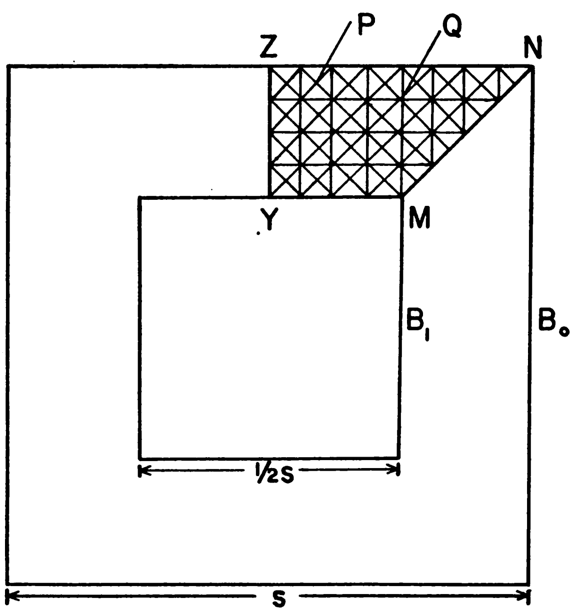

FIG. 1. Triangulation for the approximation $n = 16$.

Each junction in the triangulation is a "station." There are two types of station, type P at the centers of the small squares and type Q at the corners of these small squares.

[1] W. Prager and J. L. Synge, Quart. Appl. Math. 5 (1947), 241-269.

[2] J. L. Synge, *The hypercircle in mathematical physics*, Cambridge University Press.

To each station we attach a "weight" w. These weights uniquely determine a function $f(x, y)$ in the domain A of the cross section, by the rule that $f(x, y) = w$ at each station and $f(x, y)$ is a linear function of x and y in each triangle.

The general plan of approximation is to find two sets of weights, say w' and w'', such that if $f'(x, y)$ and $f''(x, y)$ are the corresponding functions, defined as above, then the following approximations hold:

$$\begin{aligned} f'(x, y) &\sim \psi - \tfrac{1}{2}r^2, \\ f''(x, y) &\sim \phi. \end{aligned} \tag{2.1}$$

This plan is slightly modified to permit dimensionless calculations, "reduced weights" b', b'' being introduced.

3. Approximation for lines of stress, and lower bound for torsional rigidity. To approximate $\psi - \frac{1}{2}r^2$ we have to solve the following difference equations:

$$\begin{aligned} b'(P) &= \tfrac{1}{4} \textstyle\sum b'(Q) - d(P)\, b'(R) + 1, \\ b'(Q) &= \tfrac{1}{4} \textstyle\sum b'(P) - d(Q)\, b'(R) + 2, \\ b'(R) &= -\frac{4}{3n^2} \sum_{MN} [b'(P) + b'(Q)] + \frac{7n}{12}, \\ b'(Q) &= 0 \text{ for } Q \text{ on } B_0 \text{ or } B_1 . \end{aligned} \tag{3.1}$$

Here $b'(P)$, $b'(Q)$ are reduced weights at stations P, Q. The interpretation of the first line of (3.1) is as follows: The $\sum$ is a sum over the four stations Q which are neighbors of P. $d(P) = 1$ if P is on MN, and otherwise $d(P) = 0$. $b'(R)$ is an additional reduced weight, introduced to allow for multiple-connectivity. The interpretation of the second line of (3.1) is similar. Note that if Q is on B_0 or B_1 it lacks some of its four neighbors; in that case fictitious neighbors are supplied with zero weights. In the third line of (3.1) $\sum_{MN}$ is summation along the diagonal MN of Fig. 1.

The weights $b'(P)$, $b'(Q)$ are symmetric with respect to the lines of symmetry of the hollow square.

The approximation to $\psi - \frac{1}{2}r^2$ is given by

$$\psi - \tfrac{1}{2}r^2 \sim \frac{1}{6}\frac{s^2}{n^2} F'(x, y) + \frac{1}{3}\frac{s^2}{n} b'(R)G(x, y), \tag{3.2}$$

where $F'(x, y)$ is a function which is linear in each triangle and takes, at the stations P, Q, the values $b'(P)$, $b'(Q)$, while $G(x, y)$ is a function which is linear in each of the four triangles in which the outer square is divided by its diagonals, with $G = 1$ at the center of the squares, and $G = 0$ on B_0.

If we denote the torque by $\mu\alpha\Gamma$, then a lower bound for torsional rigidity is given by

$$\frac{\Gamma}{s^4} \geq \frac{1}{9n^4} [\textstyle\sum b'(P) + 2 \sum b'(Q)] + \frac{7}{36n} b'(R), \tag{3.3}$$

these summations extending over the whole domain A, and not merely the octant MNZY.

4. Approximation for warping function, and upper bound for torsional rigidity. To approximate ϕ we have to solve

$$b''(P) = \tfrac{1}{4} \sum b''(Q),$$
$$b''(Q) = \tfrac{1}{4} \sum b''(P) + m(Q). \tag{4.1}$$

Here each summation is over four neighbors. If neighbors are lacking, i.e. when Q is on B_0 or B_1, fictitious neighbors are to be added by reflection in B_0 or B_1, the images having the same weights as the actual neighbors. The number $m(Q)$ is given by

$$m(Q) = 0 \text{ if } Q \text{ is inside the domain } A,$$
$$m(Q) = 2x/s \text{ if } Q \text{ is on } B_0 \text{ (the outer boundary)},$$
$$m(Q) = -2x/s \text{ if } Q \text{ is on } B_1 \text{ (the inner boundary)}. \tag{4.2}$$

Here x is the abscissa of Q, the origin of coordinates being at the center of the squares.

In this case the weights are skew-symmetric; therefore

$$b''(P) = 0, \; b''(Q) = 0, \tag{4.3}$$

if P or Q is on a line of symmetry of the section.

The approximation to the warping function ϕ is given by

$$\phi \sim \frac{s^2}{4n} F''(x, y), \tag{4.4}$$

where $F''(x, y)$ is linear in each triangle and has the values $b''(P)$, $b''(Q)$ at the stations P, Q.

An upper bound for the torsional rigidity is given by

$$\frac{\Gamma}{s^4} \leq \frac{1}{n^2} \left[\frac{5n^2}{32} - \sum_{ZN} m(Q)b''(Q) - \sum_{YM} m(Q)b''(Q) \right], \tag{4.5}$$

the summations being along ZN and YM in Fig. 1.

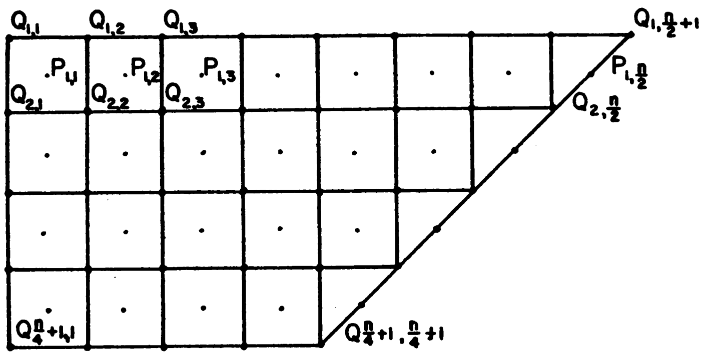

FIG. 2. Enumeration of stations for the approximation n (a multiple of 4)

5. Plan for SEAC calculations. For the approximation of order n, the stations were numbered as indicated in Fig. 2.[1] Then, using the letters P, Q, R for the reduced

[1]The procedure may be applied with little additional complication, to the case where the ratio of the sides s, s' of the outer and inner squares is not 2:1. If it is possible to triangulate with n squares along s and n' squares along s', then we make the following notational changes in Fig. 2:

$$Q_{(n/4)+1,1} \rightarrow Q_{(n-n')/2+1,1}$$
$$Q_{(n/4)+1,(n/4)+1} \rightarrow Q_{(n-n')/2+1,(n'/2)+1} .$$

weights b', we have, as in §3,

$$Q(x, y) = Q(y, x) = Q(-x, y) = Q(x, -y) = Q(-x, -y), \tag{5.1}$$

$$P(x, y) = P(y, x) = P(-x, y) = P(x, -y) = P(-x, -y), \tag{5.2}$$

$$R = -\frac{4}{3n^2} \sum_{MN} (P + Q) + \frac{7n}{12}, \tag{5.3}$$

$$P_{ij} = \tfrac{1}{4}(Q_{ij} + Q_{i,j+1} + Q_{i+1,j} + Q_{i+1,j+1}) + 1 - d(P_{ij})R, \tag{5.4}$$

$$Q_{ij} = \tfrac{1}{4}(P_{i-1,j-1} + P_{i-1,j} + P_{i,j-1} + P_{ij}) + 2 - d(Q_{ij})R, \tag{5.5}$$

$$d(X_{ij}) = \begin{cases} 1 \text{ if } X_{ij} \text{ sits on diagonal,} \\ 0 \text{ otherwise} \end{cases} \tag{5.6}$$

$$Q_{1j} = Q_{(n/4)+1,j} = 0. \tag{5.7}$$

The lower bound for Γ is given by

$$\left(\frac{\Gamma}{s^4}\right)_{LB} = \frac{1}{9n^4}\left\{\sum_{ALL} (P + 2Q) + \frac{7}{36n} R\right\}. \tag{5.8}$$

Using P, Q for the reduced weights b'', we have as in §4,

$$Q(x, y) = -Q(-x, y) = -Q(x, -y) = Q(-x, -y) = -Q(y, x), \tag{5.9}$$

$$P(x, y) = -P(-x, y) = -P(x, -y) = P(-x, -y) = -P(y, x). \tag{5.10}$$

For interior points,

$$P_{ij} = \tfrac{1}{4}(Q_{ij} + Q_{i,j+1} + Q_{i+1,j} + Q_{i+1,j+1}), \tag{5.11}$$

$$Q_{ij} = \tfrac{1}{4}(P_{i-1,j-1} + P_{i-1,j} + P_{i,j-1} + P_{i,j}); \tag{5.12}$$

on the diagonal

$$P_{ij} = Q_{ij} = 0; \tag{5.13}$$

on the outer boundary B_0,

$$Q_{1j} = \tfrac{1}{2}(P_{1,j-1} + P_{1,j}) + \frac{2(j-1)}{n}, \tag{5.14}$$

$$Q_{11} = Q_{1,(n/2)+1} = 0;$$

on the inner boundary B_1,

$$Q_{(n/4)+1,j} = \tfrac{1}{2}(P_{(n/4),j-1} + P_{(n/4),j}) - \frac{2(j-1)}{n}, \tag{5.15}$$

$$Q_{(n/4)+1,1} = Q_{(n/4)+1,(n/4)+1} = 0;$$

on the vertical axis,

$$Q_{i1} = 0. \tag{5.16}$$

The upper bound for Γ is given by

$$\left(\frac{\Gamma}{s^4}\right)_{UB} = \frac{5}{32} - \frac{1}{n^2}\left\{\sum_{k=1}^{(n/2)+1} Q_{1k}m(Q_{1k}) + \sum_{k=1}^{(n/4)+1} Q_{(n/4)+1,k}m(Q_{(n/4)+1,k})\right\}. \tag{5.17}$$

The iteration scheme for the lower bound started with all $P_{ij} = Q_{ij} = 0$ and the following cycle describes one complete iteration:

1) Set $i = 1$
2) Compute all P in row i by formula (5.4)
3) Have we done last row of P?

   No, go to step 4.

   Yes, go to step 7.
4) Replace i by $i + 1$
5) Compute all Q in row i by formula (5.5)
6) Return to step 2
7) Compute R by formula (5.3)
8) Return to step 1

After every T iterations [where T (approximately 20) is chosen at the start of a problem] the approximations for some chosen P, for some chosen Q, for R and also the

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	32	32	32	31	31	30	29	27	26	24	21	19	15	12	8	3	-2	-8	-14	-21	-30	-40	-52	
63	63	62	62	61	60	58	56	54	51	47	43	38	32	26	18	9	0	-11	-23	-37	-53	-71	-92	
89	89	88	87	86	84	81	78	74	69	63	56	48	39	28	16	2	-13	-30	-49	-71	-96	-125		
114	113	113	112	110	108	105	101	96	91	84	75	65	53	40	25	8	-12	-33	-57	-84	-115	-150		
134	134	133	131	129	125	121	116	110	102	92	81	67	51	34	13	-9	-34	-63	-94	-130	-170			
153	153	152	150	148	145	140	135	128	120	109	97	82	64	44	22	-4	-33	-64	-100	-140	-184			
168	167	166	164	160	156	151	144	135	124	111	95	76	54	30	2	-30	-65	-104	-147	-196				
181	181	179	177	175	171	165	158	149	138	125	108	89	66	39	9	-25	-62	-104	-151	-202				
190	189	188	185	181	176	170	161	150	137	120	100	76	48	17	-19	-59	-103	-153	-207					
198	197	196	194	190	186	179	171	161	148	131	111	87	59	26	-11	-53	-100	-151	-208					
201	200	198	196	191	186	178	169	156	140	121	97	68	34	-4	-47	-96	-149	-208						
203	203	201	199	195	190	184	175	163	148	130	106	78	44	5	-40	-89	-144	-205						
201	200	198	195	191	185	177	167	153	136	114	86	53	13	-32	-83	-139	-201							
197	197	196	193	190	185	178	169	157	141	120	94	62	23	-23	-74	-132	-195							
190	189	187	184	180	175	167	156	143	124	100	69	31	-14	-66	-125	-189								
181	180	179	177	174	169	163	154	142	126	105	77	41	-4	-56	-115	-181								
168	167	165	163	159	154	147	137	124	106	81	48	6	-46	-106	-173									
153	152	151	150	147	143	137	130	119	105	84	55	16	-35	-95	-162									
134	133	132	130	127	123	118	110	99	83	59	25	-23	-83	-152										
113	113	112	111	109	106	102	97	89	77	60	33	-9	-68	-138										
89	89	88	87	85	82	78	73	66	54	36	4	-50	-123											
63	62	62	61	60	59	57	54	49	43	33	15	-27	-101											
32	32	32	31	31	30	29	27	24	20	13	-2	-68												
0	0	0	0	0	0	0	0	0	0	0	0	0												

TABLE I. Computation of lower bound in approximation $n = 48$. The numbers are solutions of Eq. (3.1), or equivalently (5.1) — (5.7), divided by 10 and rounded off.
$R = 30.13171$

sum of the absolute values of the residuals are printed. When this sum becomes sufficiently small, all values are printed.

The iteration scheme for the upper bound is similar to that of the lower bound.

6. Numerical results. The equations (3.1), or equivalently (5.1)-(5.7), were solved as indicated in §5, in the approximations $n = 8$, $n = 16$, $n = 32$, $n = 48$. This last was the highest approximation feasible on SEAC, without resorting to tape storage.

The solutions for $n = 48$ rounded off to a maximum of three significant figures, are exhibited in Table I. To within a constant factor, these numbers are the values of the function $F'(x, y)$ [cf. (3.2)] at the stations. Addition of the last term in (3.2) to the numbers in Table I yields the numbers in Table II, which therefore represent, to within a constant factor, the stress function $(\psi - \frac{1}{2}r^2)$. A plot of lines of stress could easily be prepared from Table II.

Similarly, the equations (4.1), or equivalently (5.9)-(5.16), were solved for the same approximations. The solutions for $n = 48$, rounded off to a maximum of three

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
92	92	92	92	91	91	90	89	87	86	84	81	79	75	72	68	63	58	52	46	39	30	20	8	
183	183	182	182	181	180	178	176	174	171	167	163	158	152	146	138	129	120	109	97	83	63	49	28	
270	270	269	268	267	265	262	259	255	250	244	237	229	220	209	197	183	168	151	132	110	85	56		
355	354	354	353	351	349	346	342	337	332	325	316	306	294	281	266	249	229	208	184	157	126	91		
435	435	434	432	430	426	422	417	411	403	393	382	368	352	335	314	292	267	238	207	171	131			
515	515	514	512	510	507	502	497	490	482	471	459	444	426	406	384	358	329	298	262	222	178			
590	589	588	586	582	578	573	566	557	546	533	517	498	476	452	424	392	357	318	275	226				
663	663	661	659	657	653	647	640	631	620	607	590	571	548	521	491	457	420	378	331	280				
732	731	730	727	723	718	712	703	692	679	662	642	618	590	559	523	483	439	389	335					
801	800	799	797	793	789	782	774	764	751	734	714	690	662	629	592	550	503	452	395					
864	863	861	859	854	849	841	832	819	803	784	760	731	697	659	616	567	514	455						
926	926	924	922	918	913	907	898	886	871	853	829	801	767	728	683	634	579	518						
984	983	981	978	974	968	960	950	936	919	897	869	836	796	751	700	644	582							
1041	1041	1040	1037	1034	1029	1022	1013	1001	985	964	938	906	867	821	770	712	649							
1094	1093	1091	1088	1084	1079	1071	1060	1047	1028	1004	973	935	890	838	779	715								
1145	1144	1143	1141	1138	1133	1127	1118	1106	1090	1069	1041	1005	960	908	849	783								
1192	1191	1189	1187	1183	1178	1171	1161	1148	1130	1105	1072	1030	978	918	851									
1238	1237	1236	1235	1232	1228	1222	1215	1204	1190	1169	1140	1101	1050	990	923									
1279	1278	1277	1275	1272	1268	1263	1255	1244	1228	1204	1170	1122	1062	993										
1318	1318	1317	1316	1314	1311	1307	1302	1294	1282	1265	1238	1196	1137	1067										
1355	1355	1354	1353	1351	1348	1344	1339	1332	1320	1302	1270	1216	1143											
1389	1388	1388	1387	1386	1385	1383	1380	1375	1369	1359	1341	1299	1225											
1418	1418	1418	1417	1417	1416	1415	1413	1410	1406	1399	1384	1318												
1450	1450	1450	1450	1450	1450	1450	1450	1450	1450	1450	1450	1450												

TABLE II. Computation of lower bound in approximation $n = 48$. The numbers represent approximate values of

$$[(6 \times 48)^2/s^2]\,(\psi - \tfrac{1}{2}r^2)$$

significant figures, are exhibited in Table III. These numbers represent, to within a constant factor, the values of the warping function ϕ, and from these numbers a plot of level lines of warping could easily be prepared.

0	63	126	188	249	309	367	422	475	523	567	606	638	662	677	683	677	658	626	577	510	423	313	174	0
30	88	147	204	261	316	368	418	465	507	544	575	600	616	622	618	602	573	528	467	387	285	158	0	
0	55	110	164	217	268	318	366	410	451	487	517	541	558	566	563	550	524	484	428	354	260	143	0	
26	76	127	176	225	272	317	359	398	433	463	487	504	512	512	501	478	442	391	324	238	131	0		
0	47	94	140	185	229	272	312	349	382	412	435	453	463	464	455	436	403	357	296	217	119	0		
22	65	108	150	191	230	268	303	335	364	387	405	416	419	412	396	367	326	270	198	109	0			
0	39	79	117	155	192	228	261	291	319	342	360	372	376	372	358	334	297	246	181	100	0			
18	54	89	124	158	190	222	250	277	299	317	330	336	334	323	302	269	224	165	91	0				
0	32	64	96	126	157	185	212	237	259	277	290	298	298	290	272	244	204	.50	83	0				
14	43	71	99	126	152	177	200	221	239	253	262	264	258	244	220	184	136	75	0					
0	25	50	74	98	122	145	166	186	203	217	227	231	228	217	197	166	123	68	0					
11	32	53	74	95	115	135	153	169	183	194	200	199	191	175	148	111	62	0						
0	18	36	53	71	88	105	122	137	151	162	169	171	167	154	132	99	56	0						
7	22	36	50	65	79	93	107	120	131	139	144	142	133	116	88	50	0							
0	11	22	33	44	55	67	78	90	101	110	116	118	113	100	77	44	0							
4	11	18	26	34	43	52	62	71	81	89	94	93	84	67	39	0								
0	4	8	12	16	22	28	35	43	52	61	68	72	68	56	33	0								
0	0	1	2	3	6	10	16	24	32	42	49	51	45	28	0									
0	-4	-7	-10	-12	-12	-12	-9	-4	3	13	24	32	32	22	0									
-4	-11	-18	-24	-28	-32	-33	-31	-26	-17	-4	10	18	15	0										
0	-11	-22	-32	-41	-48	-54	-56	-55	-48	-35	-17	0	8	0										
-7	-22	-36	-50	-62	-72	-80	-82	-80	-70	-51	-24	-3	0											
0	-19	-37	-55	-72	-86	-98	-107	-111	-107	-91	-59	-20	0											
-11	-34	-56	-78	-97	-115	-129	-138	-141	-134	-110	-51	0												
0	-27	-54	-79	-104	-127	-147	-163	-173	-175	-164	-127	0												

TABLE III. Computation of upper bound in approximation $n = 48$. The numbers are solutions of Eq.(4.1), or equivalently (5.9) — (5.16), multiplied by 100 and rounded off, they represent approximate values of $100 \times [(4 \times 48)/s^2]\,\phi$.

The bounds on torsional rigidity, and the number of iterations required to give solutions to nine significant figures (starting from zero values) in the several approximations, are shown in Table IV.

TABLE IV

Lower and upper bounds for Γ/s^4

n	Lower Bound	Upper Bound	(U.B.)—(L.B.)	Number of Iterations L.B.	Number of Iterations U.B.
8	.1263 0626	.1320 1773	.0057 1147	50	60
16	.1282 4044	.1300 3396	.0017 9352	180	270
32	.1288 4044	.1294 2496	.0005 8452	680	1070
48	.1289 7704	.1292 8697	.0003 0993	1400	2200

In the approximation n, the number of simultaneous equations to be solved is $3n^2/16$ in the case of the lower bound, and one less in the case of the upper bound. Thus, for $n = 48$, there were 432 and 431 equations respectively.