BOUNDS ON MINIMUM WEIGHT DESIGN*

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Summary. A somewhat limited design procedure for elastic-perfectly plastic structures was developed previously [1]**. It is extended here to provide upper and lower bounds on the minimum weight of three dimensional structures and is specialized to safe one and two dimensional structures in which either direct stresses or bending stresses are negligible. The generalization also includes the influence of body forces. In principle, therefore, such troublesome factors as the weight of the structure itself or centrifugal "forces" may be designed for in a direct manner. Radially symmetric plane stress and plate bending examples are solved to demonstrate direct design procedures.

Introduction. Although design rather than analysis is the real problem in machines and structures, far more research effort is spent on analysis. The reason is primarily the specific nature of the problem posed and the greater possibility of obtaining an unambiguous solution. In what follows the material of construction is idealized as elastic-perfectly plastic. This first approximation to the behavior of real structural metals beyond the elastic range and the accompanying techniques of limit analysis are suitable for problems in which load carrying capacity is of primary importance. In addition to providing more realistic answers for ductile materials, the great advantage of limit analysis over elastic analysis is its relative simplicity. The assumption of perfect plasticity opens the possibility of direct design in the strict sense as opposed to preliminary guess and repeated analysis with the end point not necessarily the best that can be achieved. For problems of such difficulty that direct plastic design is not feasible, it is possible to obtain bounds on an optimum design. Such bounds provide a sound basis of comparison with any proposed structure and indicate the gains, if any, which can be achieved by further refinement.

An economical structure ordinarily will be one of least weight within the restrictions of the procedures of fabrication. Framed steel structures, for example, often are best constructed of beams of constant cross-section between joints. Heyman [2], Foulkes [3], Prager [4] and Livesley [5] have studied minimum weight frames of this type. The present paper dealing with beams, plates, sheets, and space structures is in the spirit of Michell [6] where the minimum weight is sought without regard to problems and costs of manufacture and construction. Prior studies of plates in bending from this point of view have been made by Hopkins and Prager [7] [8], Freiberger and Tekinalp [9].

A start on a broad theory was made [1] by the establishment of a criterion for absolute minimum weight design for structures which are subjected to direct or membrane stresses and for relative minimum weight in the case of beams or plates in transverse bending. These criteria which include that of Michell as a special case suffer from the same disadvantage. They cannot be satisfied in all, and probably not in most, problems which arise. As part of the extension of this earlier work, the matter of existence of

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solutions will be explored briefly, but major emphasis will be on the inclusion of body force and the obtaining of bounds.

2. The problem and its relation to limit analysis. The theorems of limit analysis [10] are concerned with whether a body of elastic-perfectly plastic material will collapse or not under given loads. The term collapse is used here to describe conditions for which plastic flow would occur under constant loads if the accompanying changes in the geometry of the body were disregarded. The boundary conditions are assumed to be of the stress type. That is, each of the components T_i of the surface traction is prescribed except where the corresponding velocity component is prescribed to be zero. The behavior of the elastic-perfectly plastic material is characterized by a positive yield function f which is a function of the stress components σ_{ij} $(i, j = 1, 2, 3)$ referred to rectangular Cartesian coordinates x_i . The material behaves elastically under states of stress for which $f < k^2$, where k is a constant. Plastic flow can occur under states of stress for which $f = k^2$. States of stress for which $f > k^2$ are not permissible.

The yield condition $f(\sigma_{ij}) = k^2$ can be considered to define a surface in stress space in which the components of stress are used as coordinates. Plastic flow can occur under a state of stress represented by a point on this surface. In the same space, a plastic strain *rate* ϵ_{ij}^p can be represented by a vector with components proportional to the components of the plastic strain rate. For a perfectly plastic material the vector representing the plastic strain rate ϵ_{ij}^p associated with a plastic state of stress σ_{ij} is in the direction of the exterior normal to the yield surface at the stress point σ_{ij} . If the yield surface has a singular point where there is not a unique normal, then the strain rate vector must lie in the fan bounded by the normals to the yield surface at adjacent points [11].

The rate D of dissipation of energy per unit volume of the material due to plastic action is given by $\sigma_{ij}\epsilon_{ij}^p$. For a given yield function f, the plastic flow rule enables the dissipation rate D to be expressed as a function of the plastic strain rate only,

$$D = \sigma_{ij}\epsilon_{ij}^p = D(\epsilon_{ij}^p). \tag{1}$$

The term safe statically admissible stress field, will be used to denote a stress field σ_{ij} which satisfies the equations of equilibrium

$$\sigma_{ij,j} + F_i = 0 \tag{2}$$

throughout the volume V of the body, satisfies the boundary conditions

$$\sigma_{ij}n_j = T_i \tag{3}$$

on the surface S where the tractions T_i are prescribed, and which is below yield everywhere in V, $f(\sigma_{ij}) < k^2$. In (2), F_i are the components of the body force per unit volume of the material and in (3), n_j are the components of the unit normal to the surface S. Repeated subscripts in the same term denote summation.

A *velocity* field u_i will be called kinematically admissible if it represents a possible plastic flow in the body and if it vanishes on those portions of the surface of the body where the surface traction is not prescribed.

The following theorems have been formulated [10] for an elastic-perfectly plastic body under surface tractions T_i .

Theorem 1. *If a safe statically admissible stress field* σ_{ij}^S *can be found then the body will not collapse under the loads* T_i .

Theorem 2. Collapse must impend or have taken place previously if a kinematically admissible velocity field u_i^k can be found such that the rate at which work is done by the external forces equals or exceeds the dissipation rate, that is for u_i^k continuous in V

$$\int_V D(\epsilon_{ij}^k)\, dV \leq \int_S T_i u_i^k\, dS + \int_V F_i u_i^k\, dV. \tag{4}$$

In (4), the rate of dissipation D is calculated from the strain rates

$$\epsilon_{ij}^k = \tfrac{1}{2}(u_{i,j}^k + u_{j,i}^k), \tag{5}$$

which are treated as purely plastic.

As the basic problem of design for minimum weight we consider a volume V_T within which a structure loaded by a given force distribution is to be placed. The loads T_i are to be supported by some distribution of a given elastic-perfectly plastic material in the volume V_T. The problem is to determine a structure or distribution of the material which is just at the point of collapse under the loads T_i and which involves minimum volume of the material. We shall assume that the material is homogeneous so that the problem of design for minimum volume coincides with the problem of design for minimum weight.

It is apparent that for the problem to admit a solution it must be possible to design at least one structure in V_T capable of supporting the loads T_i. Also the loads T_i will be reached by some loading program and it is presumed that the minimum volume structure would not collapse at any previous stage of the loading program.

We remark that in design for minimum weight, modifications to the approach below can be made to allow for non-homogeneity if it is prescribed that different materials are to be used in different specified regions of the volume V_T.

3. Bounds on the minimum volume. A design based on any safe statically admissible stress field σ_{ij}^S in V_T for the given material will support the loads T_i. The body formed by filling the volume V_S where σ_{ij}^S is non-zero with the given material will not collapse under the loads T_i, by Theorem 1 above. In general the stresses σ_{ij}^S are not the actual stresses in the body V_S subject to the loads T_i. By definition, the magnitude of the volume V_m of a design with minimum volume is bounded from above by the magnitude of V_S, $V_m < V_S$. In practice it is convenient to use stress fields for which $f(\sigma_{ij}^s) \leq k^2$. For such fields $V_m \leq V_s$.

In order to bound V_m from below the second theorem of limit analysis is applied. Since the minimal design is just at collapse under the loads T_i, then by Theorem 2 for any kinematically admissible continuous velocity field u_i^k defined in V_T,

$$\int_{V_m} D(\epsilon_{ij}^k)\, dV \geq \int_A T_i u_i^k\, dS + \int_{V_m} F_i u_i^k\, dV, \tag{6}$$

where A is the loaded surface. The body forces F_i are assumed to act only if material is present. The equality sign holds in (6) only if u_i^k is a possible collapse mode of the minimum volume design. Inequality (6) can be written

$$\int_{V_m} \Delta(u_i^k)\, dV \geq \int_A T_i u_i^k\, dS, \tag{7}$$

where

$$\Delta(u_i^k) = D(\epsilon_{ij}^k) - F_i u_i^k. \tag{8}$$

A lower bound on V_m follows from (7):

$$V_m \geq V_k = \int_A T_i u_i^k \, dS / \max \Delta(u_i^k), \tag{9}$$

where max Δ (u_i^k) is the maximum value of Δ in V_T which includes V_m. We note that Δ must remain finite in V_T if (9) is to provide a non-zero bound. For a material obeying either the Tresca or von Mises yield condition this requires the velocity components u_i^k to be continuous throughout V_T and to satisfy the condition of incompressibility. For some materials discontinuous velocity fields will be permissible if the material is such that the work done per unit area of the discontinuity surface is zero. Soil which is unable to take tension provides such an example when the discontinuity in velocity is associated with the formation of a tension crack [12].

The lower bounds V_k determined from kinematically admissible velocity fields u_i^k by (9) may prove to be too low to be of value in bracketing V_m. The less restrictive statement (7) may then provide useful information by employing the following procedure. For a given kinematically admissible velocity field u_i^k in V_T, the value of the modified dissipation function Δ (u_i^k) will in general vary throughout V_T. Material is placed in the regions of V_T where Δ takes its largest values until a stage is reached at which material occupies a region V_1 of V_T and

$$\int_{V_1} \Delta(u_i^k) \, dV = \int_A T_i u_i^k \, dS. \tag{10}$$

The surface bounding V_1 will be a surface on which Δ has a constant value. The values of Δ at points in the volume V_1 will not be less than the values Δ assumes at points in V_T not in V_1. It follows that the volume V_m satisfying the inequality (7) cannot have less volume than V_1, $V_m \geq V_1$. Equality between the lower bound V_1 and the lower bound V_k given by (9) can only occur if Δ (u_i^k) is constant in the region V_1 so that in general the bound V_1 is an improvement on the bound V_k.

4. Plane stress. The case of plane stress follows the general approach of the previous section. It is required to design a plane sheet of variable thickness which is just at the point of collapse under loads acting in the plane of the sheet and which has minimum volume. The sheet is formed by distributing a given material at points on or at a small distance from a plane middle surface A_T. The variable thickness of the sheet at any point of A_T is denoted by h. Forces $T_i h$ which are independent of h act in the plane of A_T on specified lines L in the plane A_T. The body forces F_i per unit *volume* of the material also act in the plane of A_T.

The middle plane A_T is taken to lie in the plane $x_3 = 0$ for definiteness. The assumption of plane stress requires the stress components σ_{i3} to vanish. The remaining components $\sigma_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ are functions of x_1 and x_2 only and for equilibrium satisfy the equations

$$(\sigma_{\alpha\beta} h)_{,\beta} + F_\alpha h = 0. \tag{11}$$

The stress boundary conditions on the lines L where the forces $T_\alpha h$ are prescribed require

$$(\sigma_{\alpha\beta} h) n_\beta = T_\alpha h, \tag{12}$$

where n_β are the components of the unit normal to L in the plane $x_3 = 0$.

A kinematically admissible *velocity* field u_i representing a possible plastic flow of a sheet with middle surface A_T is such that u_1 and u_2 are functions of x_1 and x_2 only. The rate D of dissipation of energy per unit *volume* can be found from the velocity components u_1, u_2 since the stresses σ_{i3} are zero,

$$D = \sigma_{\alpha\beta}\epsilon_{\alpha\beta} = D(\epsilon_{\alpha\beta}). \tag{13}$$

As in Sec. 3, a design based on any statically admissible stress field $\sigma^s_{\alpha\beta}$ will provide an upper bound for the minimum volume V_m. The stress field $\sigma^s_{\alpha\beta}$ must satisfy Eqs. (11) and (12) for some distribution of thickness h_s and must be at or below yield, $f(\sigma^s_{\alpha\beta}) \leq k^2$. The volume V_s,

$$V_s = \int_{A_T} h_s \, dA, \tag{14}$$

is an upper bound on V_m.

For any kinematically admissible velocity field defined by the components u^k_1, u^k_2 over A_T it follows from Theorem 2 that since the minimal design of thickness h_m is just at the point of collapse under the loads

$$\int_{A_T} \Delta(u^k_\alpha)h_m \, dA \geq \int_L (T_\alpha h)u^k_\alpha \, dL, \tag{15}$$

where

$$\Delta(u^k_\alpha) = D(\epsilon^k_{\alpha\beta}) - F_\alpha u^k_\alpha . \tag{16}$$

The lower bound V_k on V_m follows from (15) as before,

$$V_m \geq V_k = \int_L (T_\alpha h)u^k_\alpha \, dL / \max \Delta(u^k_\alpha), \tag{17}$$

where max Δ (u^k_α) is the maximum value of Δ in the plane area A_T.

As described in Sec. 3, the inequality (7) can be used to improve the lower bound (9) and similar remarks apply here with respect to inequality (15) and the lower bound (17). However in this case an upper limit may have to be placed upon the thickness h in order to avoid the somewhat unrealistic line flanges or similar reinforcements of finite cross-sectional area and therefore infinite height.

5. Membranes. In analogous fashion to the case of plane stress, the theory of Sec. 3 can be applied to membrane design. The membrane has a prescribed curved middle surface A_T and supports prescribed loads which in this case may act normal to the surface A_T as well as in the tangent planes to A_T.

6. Special stress and velocity fields. A special case arises if it is possible to design a body V_c in V_T which is compatible with a collapse mode u^c_i such that Δ (u^c_i) is constant throughout V_T which includes V_c. In the absence of body force this is the problem treated in the previous paper [1]. As u^c_i is the actual collapse mode for V_c under the loads T_i,

$$\int_{V_c} \Delta(u^c_i) \, dV = \int_A T_i u^c_i \, dS. \tag{18}$$

For any other design V_s in V_T just at the point of collapse or capable of carrying the loads T_i, by Theorem 2 above,

$$\int_{V_s} \Delta(u_i^c)\, dV \geq \int_A T_i u_i^c\, dS. \tag{19}$$

As $\Delta\,(u_i^c)$ is constant throughout V_T, comparison of (18) and (19) shows that $V_c \leq V_s$. It follows that a design compatible with such a collapse state is a minimal design $V_c = V_m$. The term compatible means that there exists an equilibrium state of stress which is consistent with the plastic strain rates and which satisfies the boundary conditions on the surface traction and satisfies the yield condition everywhere in V_c.

A similar result applies to the plane stress and membrane cases. For plane stress, a sheet design V_c of thickness h_c compatible with a collapse state for which Δ given by (16) is constant over the middle surface A_T is a design with minimum volume. An example which uses this result is given later in Sec. 8.

In most cases it will not be possible to design a body compatible with a collapse state Δ = constant. This may be seen by analogy with small strain linear and non-linear elasticity. For a positive yield function $f(\sigma_{ij})$ which is homogeneous of degree n in the stress components σ_{ij} and which defines a smooth yield surface, the plastic strain rates are given by

$$\epsilon_{ij}^P = \lambda \frac{\partial f}{\partial \sigma_{ij}}, \tag{20}$$

where λ is a non-negative scalar function of position. In the absence of body force, the dissipation rate $\Delta\,(u_i)$ or $D(\epsilon_{ij}^P)$ is given by

$$D(\epsilon_{ij}^P) = \sigma_{ij}\epsilon_{ij}^P = n\lambda f. \tag{21}$$

The body is at yield everywhere, $f(\sigma_{ij}) = k^2$. It follows that for D to be constant, λ must be constant throughout V_c. For $n = 2$, relation (20) is then formally equivalent to a linear stress-strain relation between elastic strains ϵ_{ij}^P and stresses σ_{ij}, the function λf playing the role of strain-energy density. In general a solution of the equations of elasticity will not have constant strain-energy density, and therefore in general a collapse state with D constant will not exist for the body V_c. For $n > 2$ the analogy is with non-linear elasticity and the function λf is equivalent to the complement of the strain-energy density.

The result $V_c \leq V_s$ also follows from (18) and (19) if $\Delta\,(u_i^c)$ is constant throughout V_c and smaller elsewhere in V_T. A stronger result is the following.

A design occupying the volume V_c compatible with a collapse mode such that the value of Δ at any point in V_c is not less than the value of Δ elsewhere in V_T is a minimum volume design.

A velocity field which is discontinuous across a surface may be considered as the limiting case of a continuous velocity field. The approach of Sec. 3 and the result of the previous paragraph therefore apply to discontinuous velocity fields. The term involving the dissipation of energy at the discontinuity surface can be taken into account explicitly as in Ref. 9. Discontinuous velocity fields may prove useful but only if the total rate of energy dissipation at the discontinuity surface is finite.

When the formal approach of this section is of no practical value, the more general approach of Sec. 3 must be used to obtain either the minimum design or bounds on its volume.

7. Bending of plates. A thin homogeneous plate in bending considered as a three dimensional body provides an example in which Δ cannot be constant throughout.

Here the results of the previous paper [1] are generalized to include body force. In bending, the strain rates $\epsilon_{\alpha\beta}$ in the plate vary linearly throughout the thickness and the rate of dissipation per unit volume is given by

$$D(\epsilon_{\alpha\beta}) = \sigma_{\alpha\beta}\epsilon_{\alpha\beta} = \frac{2 \mid x_3 \mid}{h} D_0 , \tag{22}$$

where D_0 is the value of D at the plate surfaces $x_3 = \pm h/2$. The modified dissipation rate Δ defined by (8) is therefore given by

$$\Delta(w) = \frac{2 \mid x_3 \mid}{h} D_0 - F_3 w, \tag{23}$$

where w is the rate of (normal) deflection of the plate. For a plate, with middle surface A_c and thickness h_c, which is on the point of collapse in a deflection rate pattern w^c

$$\int_A pw^c \, dA + \int_{A_c} F_3 w^c h_c \, dA = \int_{V_c} D(\epsilon_{\alpha\beta}^c) \, dV = \int_{A_c} \tfrac{1}{2} D_0^c h_c \, dA, \tag{24}$$

where p is the transverse load per unit area. For any other plate, with middle surface A_s and thickness h_s, just at collapse or capable of carrying the load p,

$$\int_A pw^c \, dA + \int_{A_s} F_3 w^c h_s \, dA \leq \int_{A_s} \tfrac{1}{2} D_0^s h_s \, dA. \tag{25}$$

From (22),

$$D_0^s = D_0^c \frac{h_s}{h_c}, \tag{26}$$

and it follows from (24) and (25) that

$$\int_{A_c} (\tfrac{1}{2} D_0^c - F_3 w^c) h_c \, dA \leq \int_{A_s} (\tfrac{1}{2} D_0^c \frac{h_s}{h_c} - F_3 w^c) h_s \, dA. \tag{27}$$

We now suppose that the plate with middle surface A_s is any neighboring plate to the plate A_c in the sense that $h_s = h_c + \delta h$ where $\delta h \ll h_c$. The middle surfaces A_s and A_c are then the same, $A_c = A_s$, and if the second order term in (27) is ignored

$$\int_{A_s} (D_0^c - F_3 w^c) \delta h \, dA \geq 0. \tag{28}$$

It follows that if $D_0^c - F_3 w^c$ is constant (and positive) over A_c,

$$\int_{A_c} h_c \, dA \leq \int_{A_s} h_s \, dA. \tag{29}$$

Thus a design compatible with a deflection rate pattern for which Δ is constant on the surfaces of the plate is a *relative* minimum in the solution of the minimum volume problem.

It is more usual to consider a plate in two dimensions rather than three. The corresponding result in terms of bending moments $M_{\alpha\beta}$[9] and rates of curvature $k_{\alpha\beta}$ can be obtained directly or it can be derived from the work above. The rate of dissipation per unit area of the middle surface is given by

$$M_{\alpha\beta} k_{\alpha\beta} = \int_{-h/2}^{h/2} D \, dx_3 = \tfrac{1}{2} D_0 h. \tag{30}$$

A relative minimum is provided by a design compatible with a deflection rate pattern such that $2M_{\alpha\beta}k_{\alpha\beta}/h - F_3 w$ is constant over the plate.

The results for the non-homogeneous sandwich plate are more definite. By a sandwich plate is meant here a plate which has a light-weight core of constant or variable thickness H between thin identical face sheets each of thickness t which varies from point to point of the middle surface A_T. The core carries no bending stresses and a bending moment across the plate is carried by equal and opposite membrane stresses in the outer faces. The problem is to determine the variable thickness t of the faces so that the plate is just at the point of collapse under the prescribed loads and the face sheets have minimum volume. Since H is considered to be large compared with t, $H \gg t$, the strain rates in the faces of the plate can be considered to be constant throughout the thickness t. It follows from the general result of Sec. 6 that a sandwich plate designed to collapse in a mode w such that Δ is constant in the faces of the plate is an absolute minimum volume design.

8. Plane stress example-rotating disc. In the previous paper [1] the formal approach of Sec. 6 was used to obtain minimum volume designs for a circular annular disc loaded by uniform forces per unit length on the inner and outer edges of the disc. Here an illustration is given of the way in which centrifugal "forces" can be taken into account.

A circular annular disc of inner radius r_i and outer radius r_0 is rotating with constant angular velocity ω about a perpendicular axis through the center of the disc. The outer edge of the disc is loaded by a uniform tensile force T per unit length and the inner edge is stress free. The problem is equivalent to that of a stationary disc with outward radial body force $\rho\omega^2 r$ per unit volume, where ρ is the mass density of the material. The material of the disc is assumed to obey Tresca's yield condition, Fig. 1, with yield stress σ_0.

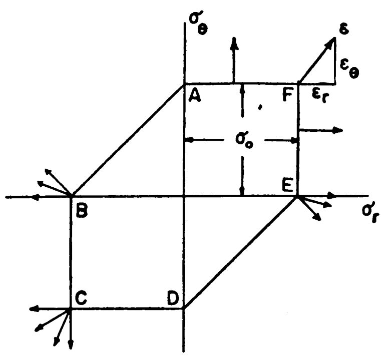

FIG. 1 Tresca's yield condition for plane stress with radial symmetry.

For equilibrium the principal stresses σ_r and σ_θ satisfy

$$\frac{d}{dr}(h\sigma_r) + h\frac{(\sigma_r - \sigma_\theta)}{r} = -\rho\omega^2 rh, \tag{31}$$

where h is the thickness of the disc. In terms of the radial velocity u the strain rates are given by

$$\epsilon_r = du/dr, \qquad \epsilon_\theta = u/r, \tag{32}$$

and for Δ constant,

$$\Delta = \sigma_r \epsilon_r + \sigma_\theta \epsilon_\theta - \rho\omega^2 r u = \text{constant}. \tag{33}$$

The condition Δ = constant restricts the position of the stress point on the yield hexagon of Fig. 1. For example, the stress point cannot lie on the side AF because the normality condition requires $\epsilon_r = 0$, and the condition (33) cannot be satisfied in view of (32). It is readily found that only the stress states represented by points C and F, B and E, can be associated with a velocity field for which Δ is constant for a finite range of r.

In the present problem the stresses are tensile so that the stress points B and C are prohibited. The stress point E is also inadmissible as it would require inward radial velocity at the inner edge $r = r_i$. The remaining stress point F provides the solution. At the stress point F, $\sigma_r = \sigma_\theta = \sigma_0$ and equilibrium requires

$$\frac{dh}{dr} = -\frac{\rho\omega^2}{\sigma_0} rh. \tag{34}$$

The region $r_i < r \leq r_0$ of the disc is at the stress state F so that

$$h = \frac{T}{\sigma_0} \exp (x_0^2 - x^2) \qquad x_i < x \leq x_0 , \tag{35}$$

where $x = (\rho\omega^2/2\sigma_0)^{1/2} r$. In order to complete the design, a line flange must be added at the inner boundary, Fig. 2a, to satisfy the condition of zero stress at $r = r_i$. The line flange is of infinite height but of finite cross-sectional area of amount

$$r_i h_i/(1 - 2x_i^2), \tag{36}$$

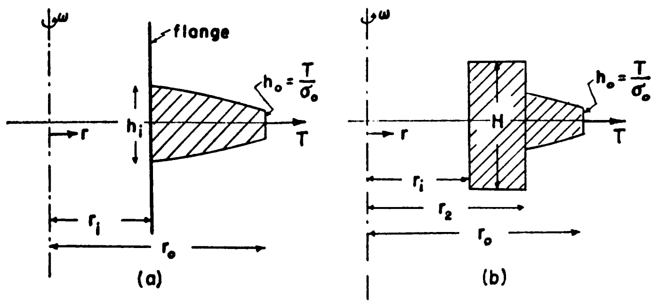

FIG. 2 Minimum volume design for rotating annular disc under exterior tension (a) of unlimited thickness and (b) of restricted thickness.

where h_i is the value of h given by (35) at $x = x_i$. The speed of rotation of the disc is restricted by the condition $x_i^2 < 1/2$, that is $\rho\omega^2 r_i^2 < \sigma_0$.

It remains to associate a velocity field of constant Δ with the design. For the stress point F,

$$\sigma_0 \left(\frac{du}{dr} + \frac{u}{r}\right) - \rho\omega^2 r u = \text{constant} \tag{37}$$

and

$$\epsilon_r = du/dr \geq 0, \qquad \epsilon_\theta = u/r \geq 0. \tag{38}$$

The line flange is at the stress point A so that $du/dr \leq 0$ at $r = r_i$. Taking $du/dr = 0$ at $r = r_i$ gives with Eq. (37)

$$u = \frac{A}{x}[\exp(x^2 - x_i^2)/(1 - 2x_i^2) - 1], \tag{39}$$

where A is a positive constant. The restrictions (38) are satisfied for $r \geq r_i$ provided that $x_i^2 < 1/2$.

The volume V_m of the minimum volume design is given by

$$\frac{V_m \rho\omega^2}{2\pi T} = \exp(x_0^2 - x_i^2)/(1 - 2x_i^2) - 1, \tag{40}$$

and is an increasing function of x_i.

The somewhat unrealistic line flange can be avoided by placing an upper limit H on the thickness h, Fig. 2b. The full thickness H is taken for $r_i \leq r < r_2$ and the stress state in this region is represented by points on the side AF of the yield hexagon, $\sigma_\theta = \sigma_0$, $0 \leq \sigma_r < \sigma_0$. The edge $r = r_2$ of the full thickness region is found from the condition that $(\sigma_r h)$ is continuous at $r = r_2$. The collapse mode must be such that $du/dr = 0$ for $r_i \leq r < r_2$, so that u is constant in this region, $u = B$, and

$$\Delta = B(\sigma_0/r - \rho\omega^2 r) \qquad r_i \leq r < r_2. \tag{41}$$

Thus Δ is a decreasing function of r for $r_i \leq r < r_2$. For $r_2 < r \leq r_0$ the radial velocity is determined by (37), (38) and the condition $du/dr = 0$ at $r = r_2$,

$$u = \frac{C}{x}[\exp(x^2 - x_2^2)/(1 - 2x_2^2) - 1] \qquad x \geq x_2. \tag{42}$$

The positive constant C is chosen so that u is continuous at $r = r_2$. The value of Δ is constant for $r \geq r_2$ and equal to the value given by (41) at $r = r_2$. A lower limit must be placed on H and an upper limit on the speed of rotation ω for the design to be possible. The design is clearly the minimum volume design but this can also be seen from the following. The design is compatible with a collapse mode for which the value of Δ in the body is not less than the value of Δ elsewhere in the region where material may be placed and is therefore a minimum volume design.

9. Sandwich plate example. The problem of this section illustrates the direct design approach of Sec. 6. A simply supported circular sandwich plate of constant core thickness H is loaded by a uniform pressure p over its upper surface, which is horizontal. It is required to find the variable thickness t of the two identical face sheets so that the plate is just at the point of collapse under the pressure p and the face sheets have minimum volume. The weight per unit volume of the face sheets is denoted by γ and the body force is taken into account in the analysis. We first outline the design procedure for a material with a general yield function and then consider the particular case of the Tresca yield condition.

We write the yield condition on the radial bending moment M and the circumferential bending moment N in the form

$$F(M, N) = M_0^2, \tag{43}$$

where M_0 is given by

$$M_0 = \sigma_0 H t \tag{44}$$

and σ_0 is the yield stress of the material in tension or compression. The yield function F is homogeneous of order two in the moments M and N and will be assumed to define a smooth curve in the M, N plane. The von Mises yield criterion for example provides the special form $F = M^2 - MN + N^2 = M_0^2$. For equilibrium the moments satisfy the equation

$$\frac{d^2}{dr^2}(rM) - \frac{dN}{dr} + pr + 2\gamma tr = 0, \tag{45}$$

where r measures distance from the center of the plate. Measuring the deflection rate w in the downward direction, the radial and circumferential rates of curvature κ, λ are given by

$$\kappa = -\frac{d^2w}{dr^2}, \qquad \lambda = -\frac{1}{r}\frac{dw}{dr}. \tag{46}$$

The moment-rate of curvature relation associated with the yield condition (43) requires

$$\kappa/\lambda = \frac{\partial F}{\partial M}\Big/\frac{\partial F}{\partial N}. \tag{47}$$

From the discussion in Sec. 7, a design compatible with a collapse mode w such that the modified dissipation function Δ (w) is constant throughout the plate is a minimum volume design. The radial and circumferential strain rates in the upper face of the plate are $-\frac{1}{2}H\kappa$ and $-\frac{1}{2}H\lambda$ respectively and are equal in magnitude and opposite in sign in the lower face. We therefore seek a deflection rate w such that

$$\Delta(w) = D\left(\frac{1}{2}H\frac{d^2w}{dr^2}, \frac{1}{2}\frac{H}{r}\frac{dw}{dr}\right) - \gamma w = \text{constant}, \tag{48}$$

where the form of the dissipation function D is known from the yield condition of the material. At the simply supported edge $r = r_0$ of the plate the deflection rate is zero and at the center $dw/dr = 0$ since λ is finite there. Also for equilibrium $M = N$ at the center of the plate so that $\kappa = \lambda$ at $r = 0$ since the yield condition (43) is symmetrical in M and N for an isotropic material. In general, Eq. (48) will be non-linear and analytical solution of (48) subject to the conditions on w and its derivatives will be difficult. A possible numerical approach is the following. Numerical values are arbitrarily assigned to the positive value of w and the common negative value of d^2w/dr^2 and $(1/r)dw/dr$ at the center,

$$(w)_{r=0} = a > 0, \qquad \left(\frac{d^2w}{dr^2}\right)_{r=0} = \left(\frac{1}{r}\frac{dw}{dr}\right)_{r=0} = -b < 0. \tag{49}$$

The constant on the right hand side of (48) is then determined by substitution of the values (49). With this value of the constant, Eq. (48) is integrated numerically in the range $0 < r \leq r_0$ to give a deflection rate which will be denoted by W. If W_0 is the value of W at $r = r_0$, then the deflection rate w given by

$$w = W - W_0 \tag{50}$$

satisfies Eq. (48) with a different constant on the right hand side and satisfies all the conditions on w and its derivatives.

When the collapse mode w such that Δ (w) = constant is known, the rates of curvature

κ, λ can be found from (46). The yield condition (43) and the normality condition (47) then determine M and N in terms of M_0 and hence in terms of the thickness t. The thickness t must be chosen so that M and N satisfy the equilibrium equation (45). At the edge of the plate $M = 0$ so t is taken to be zero at $r = r_0$. Also the resulting equation for t will in general be singular at $r = 0$ and the solution must be such that t is finite at $r = 0$. Thus, in principle, a minimum volume design can be obtained for a material with the general yield condition (43).

The solution for the Tresca yield criterion, which does not define a smooth yield curve, is particularly simple. For the problem considered here the curvature rates κ, λ are non-negative and the condition Δ = constant requires

$$\Delta(w) = -\frac{\sigma_0 H}{2}\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) - \gamma w = \text{constant}. \tag{51}$$

It follows that the appropriate solution is

$$w = C[J_0(\alpha r) - J_0(\alpha r_0)], \tag{52}$$

where C is a positive constant, $\alpha^2 = 2\gamma/\sigma_0 H$, and J_0 is the Bessel function of the first kind of order zero. The restriction that κ and λ be non-negative requires that $\alpha r_0 < 1.84$ approximately. When $\alpha r_0 > 1.84$ the weight of the material will predominate over its strength.

It remains to determine the thickness t of the faces. As both κ and λ are positive, the Tresca yield criterion requires the bending moments M and N to be positive and equal to the fully plastic moment $M_0 = \sigma_0 Ht$. Substitution in the equilibrium equation (45) gives

$$\frac{d^2t}{dr^2} + \frac{1}{r}\frac{dt}{dr} + \alpha^2 t = -\frac{p}{\sigma_0 H}. \tag{53}$$

The appropriate solution of this equation with $t = 0$ at $r = r_0$ where M is zero and t finite at $r = 0$ is

$$t = \frac{p}{\sigma_0 H\alpha^2}\left[\frac{J_0(\alpha r)}{J_0(\alpha r_0)} - 1\right]. \tag{54}$$

As M, N are positive, αr_0 must be less than the first zero of J_0, that is $\alpha r_0 < 2.4$ approximately. For $\alpha r_0 < 1.84$, the design (54) is compatible with the collapse mode (52) for which Δ = constant and is therefore a minimum volume design The design (54) could have been obtained directly from moment considerations alone

As the weight per unit volume γ of the material tends to zero, the design (54) tends to the design for zero body force

$$t = \frac{1}{4}\frac{p}{\alpha_0 H}(r_0^2 - r^2), \tag{55}$$

obtained previously for the von Mises yield condition in Ref. 9.

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