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ONE-DIMENSIONAL DIFFUSION WITH THE DIFFUSION COEFFICIENT A LINEAR FUNCTION OF CONCENTRATION: REDUCTION TO AN EQUATION OF THE FIRST ORDER*

By D. H. PARSONS (*University of Reading, England*)

Abstract. The problem considered by Stokes[1], of one-dimensional diffusion from an initially sharp boundary between two semi-infinite columns of liquid, the diffusion constant being a linear function of concentration, is discussed. It is shown how the differential equation may be reduced to an equation of the first order. Some properties of the solution are investigated, and the method of obtaining numerical solutions is considered.

The differential equation to be satisfied is

$$2\frac{d}{dy}\left(D\frac{dc}{dy}\right) + y\frac{dc}{dy} = 0, \tag{1}$$

with boundary conditions

$$c \to c_1 \text{ when } y \to -\infty,$$

$$c \to c_2 \text{ when } y \to \infty,$$

where c is the concentration, c_1 and c_2 the initial concentrations of the two columns, D the diffusion coefficient, and $y = xt^{-\frac{1}{2}}$, where x is the distance from the boundary in the direction of diffusion, and t is the time (see Stokes, loc. cit.). Let D^* be the mean concentration, and let b = ratio of diffusion coefficient at concentration c_1 to that at c_2 ($b \neq 1$). Then, since we assume that D is a linear function of c, we may write

$$D = D^*\left[1 + \frac{1-b}{1+b}\left\{\frac{\frac{1}{2}(c_1+c_2) - c}{\frac{1}{2}(c_1 - c_2)}\right\}\right], \tag{2}$$

and, making this substitution in (1), and also writing $y = 2D^{*\frac{1}{2}}.\, z$, the equation to be satisfied becomes

$$\left[1 + \frac{1-b}{1+b}\left\{\frac{\frac{1}{2}(c_1+c_2) - c}{\frac{1}{2}(c_1 - c_2)}\right\}\right]\frac{d^2c}{dz^2} - \frac{1-b}{1+b}\,\frac{1}{\frac{1}{2}(c_1-c_2)}\left(\frac{dc}{dz}\right)^2 + 2z\frac{dc}{dz} = 0, \tag{3}$$

with the conditions $c \to c_1$ when $z \to -\infty$, $c \to c_2$ when $z \to \infty$.

Let now

$$z = \frac{1}{(1+b)^{1/2}}\, ue^{-w} \tag{4}$$

$$c = c_2 + \frac{c_1 - c_2}{1-b}(1 - e^{-2w}). \tag{5}$$

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[1]R. H. Stokes, Transactions of the Faraday Society, No. 358, Vol. 48, Part 10, October, 1952.

Then, substituting these expressions in (3), the equation to be satisfied reduces to

$$d^2w/du^2 + u\,dw/du - 2(1 + u^2)(dw/du)^2 + u(3 + u^2)(dw/du)^3 = 0, \tag{6}$$

or finally, writing $p = -\,dw/du$,

$$dp/du + up + 2(1 + u^2)p^2 + u(3 + u^2)p^3 = 0, \tag{7}$$

an equation of the first order. It is clear from (5) that $c = c_1$ when $w = \log (b^{-\frac{1}{2}})$ and $c = c_2$ when $w = 0$. Consequently, from (4) and (5) the new boundary conditions are

$$w \to \log (b^{-\frac{1}{2}}) \text{ when } u \to -\infty,$$

$$w \to 0 \text{ when } u \to \infty.$$

We may remark here for reference later on, that we have

$$\begin{aligned} \frac{-1}{c_1 - c_2}\frac{dc}{dz} &= \frac{-2\,(1 + b)^{1/2}}{1 - b}\,e^{-w}\,\frac{dw/du}{1 - u\,dw/du} \\ &= \frac{2\,(1 + b)^{1/2}}{1 - b}\,e^{-w}\,\frac{p}{1 + up} \end{aligned} \tag{8}$$

Now the equation (7) is an ordinary differential equation of the first order and the normal form. Its solutions depend upon one arbitrary constant, which may conveniently be taken to be the value of p when $u = 0$. It follows from the form of (7) that p cannot be zero for any real u. For if $p = 0$ when $u = u_0$, say, then from the form of (7) this integral is certainly analytic at the point $u = u_0$, and is therefore simply the integral $p \equiv 0$. p would therefore be identically zero for all u, which makes w, and hence c, constant throughout, a trivial solution which we reject. We may thus suppose p positive throughout: for a solution for which p is negative throughout is obtained by writing $-p'$ for p and $-u'$ for u, which does not change the form of (7). Suppose, therefore, that $p = \lambda > 0$ when $u = 0$. Then the following results, the proofs of which are given in the appendix, may be established:

(i) Every point of the real axis of u is a regular point of the integral.

(ii) When u increases from zero to ∞, p decreases strictly, and $p \to 0$ as $u \to \infty$, while up increases strictly from zero to a single maximum at a point in the range $0 < u < 1$, this maximum value being $< 1/3$, and then decreases strictly, and $\to 0$ as $u \to \infty$.

(iii) When u decreases from zero to $-\infty$, p first increases strictly to a maximum at a point in the range $-1 \leq u < 0$, then decreases strictly, and $\to 0$ when $u \to -\infty$. But, according to the value of λ, either up decreases strictly from zero to a single minimum value, this value being > -1, at some point for which $u < -1$, then increases strictly, and $\to 0$ as $u \to -\infty$; or else up decreases strictly whenever $u \leq 0$, and $\to -1$ as $u \to -\infty$. For all small enough λ, the former circumstance obtains.

(iv) $\int_0^\infty p\,du$ and $\int_0^\infty up\,du$ are convergent, uniformly with respect to λ, in any positive range of λ.

(v) $\int_{-\infty}^0 p\,du$ and $\int_{-\infty}^0 up\,du$ are convergent uniformly with respect to λ, if λ lies in a range such that $up \to 0$ when $u \to -\infty$.

Suppose then that we select a positive value of λ, and calculate numerically the solution of (7) for which $p = \lambda$ when $u = 0$. It is essential that the calculations be carried on far enough, in the negative direction, to make sure that up passes its (negative) minimum value, and thereafter increases towards zero; for otherwise we cannot be sure that $| up |$ tends to zero and not to unity as $u \to -\infty$. If this condition be not satisfied, we simply start again with a smaller value of λ: for it is proved in the appendix that for all λ up to a positive limit, $up \to 0$ as $u \to -\infty$. With this understanding, we have $dw/du = -p$, and hence $w = A - \int_0^u p\,du$. But since we require that $w \to 0$ when $u \to \infty$, and since the integral is convergent, we see at once that this gives

$$w = \int_u^\infty p\,du, \tag{9}$$

and thus w is known to any desired degree of accuracy, from the calculated solution of (7). This latter being, by hypothesis, such that $\int_{-\infty}^u p\,du$ is convergent when $u \to -\infty$, the second boundary condition shows that

$$b = \exp\left(-2\int_{-\infty}^\infty p\,du\right), \qquad (b < 1 \text{ since } p > 0). \tag{10}$$

The constant b is therefore calculated from (10), and clearly depends only on λ. From (4) and (5) we then have the required solution parametrically, in terms of u, in the form

$$z = \frac{1}{(1+b)^{1/2}}\, u \exp\left(-\int_u^\infty p\,du\right), \tag{11}$$

$$\frac{c - c_2}{c_1 - c_2} = \frac{1 - \exp\left(-2\int_u^\infty p\,du\right)}{1 - b}, \tag{12}$$

b being given by (10): and from the form of these expressions and from (10), we see that $c \to c_1$ when $u \to -\infty$ and $z \to -\infty$, $c \to c_2$ when $u \to \infty$, $z \to \infty$, so that the original boundary conditions are satisfied, and we have the required solution of (1).

We also have, from (4), (8), (9), and (10)

$$\frac{-1}{c_1 - c_2} z \frac{dc}{dz} = \frac{2}{1-b} e^{-2w} \frac{up}{1+up} = \frac{2}{1-b}\frac{up}{1+up} \exp\left(-2\int_u^\infty p\,du\right) \tag{13}$$

and since $1 + up > 0$ throughout[2], and $up \to 0$ when $u \to \pm\infty$, we see that $z dc/dz \to 0$ when $z \to \pm\infty$.

The family of curves of c against z, or the gradient curves $-(c_1 - c_2)^{-1} dc/dz$ against z, can then be plotted for various values of b, by means of the expressions (11), (12), (13), which give these expressions in terms of the parameter u. The method is firstly to calculate the solutions of (7) for an arbitrary positive λ; then to calculate b from (10); and finally z, c and dc/dz in terms of u, from (11), (12), (13).

The advantages of the foregoing method of procedure are two-fold. First of all, there is the advantage of calculating a solution for a first-order differential equation instead of a second-order one. Secondly, the single arbitrary quantity is the value of p at the point $u = 0$, or $z = y = x = 0$. Thus the calculations are begun at a point corresponding to the boundary between the two columns of liquid, and proceed symmetrically in both directions. Thirdly, the convergence of the integrals required to give b, and w in terms of u, has been shown to be uniform with respect to λ when λ does not exceed a fixed

[2]See Appendix.

limit, H say. And denoting the solution of (7) which takes the value λ when $u = 0$ by $p(u, \lambda)$, it is proved in the appendix that $p(u, \lambda) < p(u, \mu)$ when $\lambda < \eta$. Then if $0 < u_1 < u_2$ and $u_3 < u_4 < 0$, we have

$$0 < \int_{u_1}^{u_2} p(u, \lambda)\, du < \int_{u_1}^{u_2} p(u, H)\, du,$$

and

$$0 < \int_{u_3}^{u_4} p(u, \lambda)\, du < \int_{u_3}^{u_4} p(u, H)\, du.$$

It follows that, when calculating the infinite integrals which occur in (10)-(13), if we carry the calculations far enough in each direction to attain the required degree of accuracy for $\lambda = H$, then carrying the calculations the same distance will certainly give the required accuracy for any smaller value of λ.

I am very much indebted to Professor Stokes who drew my attention to this interesting problem.

Appendix

We require to prove the properties (i)-(v), listed above, of that solution of (7) which takes the value $\lambda > 0$ when $u = 0$.

(i) The equation (7) has the fixed singularities $u = 0$, $u = \pm 3^{\frac{1}{2}}i$. The integral such that $p = \lambda > 0$ when $u = 0$ is analytic when $u = 0$. Consequently, it follows from a result of Goursat[3], that a point $u = u_0$ on the real axis can only be a singularity if $p \to \infty$ when $u \to u_0$. We have shown that $p > 0$ throughout. But from (7) it follows that for all large enough positive p, $dp/du < 0$ when $u_0 > 0$ and $| u - u_0 |$ is small, and $dp/du > 0$ when $u_0 < 0$ and $| u - u_0 |$ is small. This contradicts either of the hypotheses that $u_0 > 0, p \to +\infty$ when $u \to u_0$ from below, or that $u_0 < 0, p \to +\infty$ when $u \to u_0$ from above. It follows that every point of the real axis is a regular point of the integral.

(ii) From (7), when $u > 0$ and $p > 0$, $dp/du < 0$, and p decreases strictly. But $p > 0$ throughout, therefore $p \to \alpha \geq 0$ when $u \to +\infty$. If $\alpha > 0$, clearly $dp/du \to -\infty$, contradicting the hypothesis that $p \to \alpha$. Hence $p \to 0$ as $u \to \infty$.

Next, let $v = up$. Then, using (7), we have, after simplifying,

$$u\, dv/du = v(1 + v)\{(1 - u^2) - (3 + u^2)v\}. \tag{14}$$

When $u > 0$, $v > 0$, and therefore, from (14), when $u \geq 1$, $dv/du < 0$. It follows that when $u \to \infty$, $v \to \beta \geq 0$. But if $\beta > 0$, $dv/du \sim -\beta(1 + \beta)^2 u \to -\infty$, contradicting $v \to \beta$. Therefore, $v \to 0$ as $u \to \infty$. To show that v has only one maximum, occurring when $0 < u < 1$, we first observe that when $u = 0$, $dv/du = p_0 = \lambda > 0$. When $dv/du = 0$ and $u > 0$, we have, from (14), $d^2v/du^2 = -2v(1 + v)^2 < 0$. Hence there is only one maximum value. At this value, from (14), we see that

$$v_{\max} = (1 - u^2)/(3 + u^2) = 1/3 - 4u^2/3(1 + u^2) < 1/3.$$

Thus proposition (ii) is established.

(iii) When $u < 0$, we may write the equation (7) in the form

$$dp/du = -\frac{p}{u(3 + u^2)} [\{u(3 + u^2)p + (1 + u^2)\}^2 + (u^2 - 1)], \tag{15}$$

[3]E. Goursat, "Mathematical Analysis", Vol. II, Part II, sect. 67, pp. 182-184 (Ginn and Co., 1945).

and from this we see that when $u < -1$, $dp/du > 0$. Since $p > 0$, it follows that $p \to \gamma \geq 0$ when $u \to -\infty$. If $\gamma > 0$, $dp/du \to \infty$ as $u \to -\infty$, contradicting $p \to \gamma$. Therefore $p \to 0$ as $\to -\infty$. Again, when $u = 0$, $dp/du = -2\lambda^2 < 0$. There is thus a maximum value between $u = 0$ and $u = -1$. To show that there is only one maximum, we observe from (15) above that when $dp/du = 0$ and $-1 \leq u < 0$,

$$d^2p/du^2 = -p\{1 + 4up + 3(1 + u^2)p^2\}$$

$$= -\frac{p}{3(1 + u^2)}[\{3(1 + u^2)p + 2u\}^2 + 3 - u^2] < 0:$$

and the first part of the proposition is proved.

To establish the second part, we again write $v = up$, so that v again satisfies (14) above, and if $dv/du = 0$ and $u < 0$, we have $d^2v/du^2 = -2v(1 + v)^2$. But when $u < 0$, $v < 0$, and it follows from the form of (14) that $v > -1$ for any $u < 0$: if $v = -1$ when $u = u_0 < 0$, the integral is certainly analytic at u_0, and therefore $v \equiv -1$, $p \equiv -1/u$, a solution which we may reject, since by hypothesis $p = \lambda$ when $u = 0$. Consequently, when $u < 0$, $-1 < v < 0$, and therefore if $dv/du = 0$, $d^2v/du^2 > 0$. Hence there can be at most one negative minimum value of v. But from (14) we see that when $-1 \leq u < 0$, $dv/du > 0$. Hence the minimum, if it exists, must occur for $u < -1$.

Since $dv/du > 0$ when $u = 0$, and since $-1 < v < 0$ when $u < 0$, it follows that when $u \to -\infty$, v either decreases strictly to a limit δ where $-1 \leq \delta < 0$, or else decreases to a minimum and then increases strictly to a limit ϵ, where $-1 < \epsilon \leq 0$. In either case, suppose v tends to a limit η, $-1 < \eta < 0$. Then $dv/du \sim -\eta(1 + \eta)^2 u \to -\infty$ as $u \to -\infty$, contradicting the hypothesis that $v \to \eta$. Hence $\eta = 0$ or else $\eta = -1$. In other words, as u decreases from 0 to $-\infty$, either v decreases to a minimum value between 0 and -1, for some value of u less than -1, and then increases strictly to the limit zero, or else v decreases strictly to the limit -1.

We have yet to show that, for small enough λ, the former event takes place. Since dv/du can have at the most one zero when $u < 0$, and since $dv/du > 0$ when $u = 0$, it follows that if $dv/du < 0$ for any negative u, then certainly v has a minimum, and increases to zero as $u \to -\infty$. But, selecting any fixed negative number N, where $N < -1$, and observing that $p \equiv 0$ is a solution of (7), it is a standard result[4] that a positive number L exists (depending upon N), such that whenever $0 < \lambda \leq L$, the solution of (7) for which $p = \lambda$ when $u = 0$ is such that when

$$u = N, \qquad 0 < p(N) < \frac{1}{-N}\frac{N^2 - 1}{N^2 + 3},$$

and therefore $0 > (3 + N^2)v > 1 - N^2$. From (14) we see, therefore, that whenever $\lambda \leq L$, $dv/du < 0$ when $u = N$: and the result follows.

(iv) Denoting the integral of (7) which takes on the value λ when $u = 0$ by $p(u, \lambda)$, we first observe that if $\lambda < \mu$, then for all u, $p(u, \lambda) < p(u, \mu)$. For by the usual existence theorem, there cannot be two integrals taking a given value for any given value of u, so that $\{p(u, \mu) - p(u, \lambda)\} \neq 0$. But it is impossible for $\{p(u, \mu) - p(u, \lambda)\}$ to be negative for any u: for being continuous, and positive when $u = 0$, it would then be zero for some other u.

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[4]E. Goursat, "Cours d'Analyse Mathematique", tome III, sect. 461, pp. 16-18. (Gauthier-Villars, 1942).

Now, integrating the equation (7) between the limits 0 and u, we have

$$\lambda - p = \int_0^u \{up(1 + up)^2 + (2p^2 + 3up^3)\}\, du. \tag{16}$$

Suppose that $0 < \lambda \leq M$. Then we have proved that, for any λ, $up \to 0$ as $u \to \infty$, and that $p(u, \lambda) < p(u, M)$. Hence $p = o(1/u)$, $p^2 = o(1/u^2)$, $up^3 = o(1/u^3)$: and therefore $\int_0^u (2p^2 + 3up^3)du$ is uniformly convergent as $u \to \infty$. But since $p \to 0$ and $p(u, \lambda) < p(u, M)$ as $u \to \infty$, the left hand side of (16) is uniformly convergent to λ. Therefore, since for all large enough u, $(1 + up) > \frac{1}{2}$, it follows at once that $\int_0^u up\, du$ is uniformly convergent when $u \to \infty$: and the same is therefore true of $\int_0^\infty p\, du$.

(v) We may write (16) in the form

$$\lambda - p = -\int_u^0 \{up(1 + up)^2 + (2p^2 + 3up^3)\}\, du.$$

We have proved that for all small enough λ, say $\lambda \leq H$, $up \to 0$ as $u \to -\infty$. In this event, it follows, by precisely the reasoning of (iv) above, that $\int_{-\infty}^0 up\, du$ and $\int_{-\infty}^0 p\, du$ are uniformly convergent.

AN ELEMENTARY SOLUTION OF TWO STRESS CONCENTRATION PROBLEMS IN THE NEIGHBOURHOOD OF A HOLE*

By W. T. KOITER (*Technische Hogeschool, Delft*)

1. Introduction. The problem of the stress concentration around a central circular hole in a strip in tension (or in bending in its plane) has been investigated extensively. The theoretical solution of this problem has been given by Howland [1]. However, Howland's numerical calculations were restricted to hole diameters which do not exceed half the strip width, and his method requires a rapidly increasing numerical effort for larger holes. Experimental investigations have yielded valuable data on the stress concentration factor for large holes. On the basis of these experimental results Heywood [2] conjectures that the stress concentration factor, referred to the nominal stress in the net section, tends to the value 2 if the hole diameter approaches to the strip width. This conjecture is confirmed by the elementary analysis in Sec. 2.

Recently the more complicated problem of a central spherical hole in a cylindrical bar in tension has been investigated by Ling [3]. His numerical evaluation has also been restricted to a hole radius equal to 1/4 and 1/2 times the bar radius. Here again the numerical effort, required in evaluation of the theory, increases rapidly with increasing hole radius. According to Ling "the value 1 for the stress concentration factor K (referred to the nominal stress in the net section) in the limiting case $\lambda = 1$ (where λ is the ratio of hole radius to bar radius) can be visualized readily from physical considerations of the cylinder" [3, p. 391], and the graph of his results [3, Fig. 2] has been completed for the entire range $0 < \lambda < 1$. However, Ling's argument, which has been cited above, is incorrect, as will be shown in Sec. 3. The correct limiting value of the stress concentration

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