THERMO-ELASTIC SIMILARITY LAWS*

BY

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1. Introduction. In a book edited by Hetényi [1], Mindlin and Salvadori have discussed certain similarity laws for thermo-elastic problems. They were, however, primarily concerned with the replacement of thermo-elastic problems by purely elastic problems involving dislocations. In the present paper we are concerned with the conditions which must be satisfied by the physical parameters describing the properties of a body and a scale model (scaled both in linear dimensions and time) in order that the stress in the original body may be determined from that at the corresponding position and time in the scale model by means of the multiplier M (= rigidity modulus of body/rigidity modulus of model). It will be seen that the conditions for the stresses in the body to be an arbitrary constant C times those in the model are then easily obtained. In deriving these conditions much of the mathematical analysis of Mindlin and Salvadori could have been used. However, we have preferred to obtain them by somewhat different methods.

We consider that surface tractions and temperatures are specified on the surface of a body and a scale model, the surface tractions and temperatures on the body being respectively M and Θ_0 (a constant) times those at corresponding points and times in the scale model. The dimensions of the body are assumed to be l times those of the model and any time for the body is τ_0 times the corresponding time for the model.

It is then found that for three-dimensional thermo-elastic problems, the stresses in the original body are M times those in the model provided that

(i) Poisson's ratio σ is the same for the model and the original body;

(ii) β $(= \Theta_0\nu/\mu)$ is the same for the model and original body, where ν is the coefficient of thermal expansion, μ is the rigidity modulus and $\Theta_0 = 1$ for the model; and

(iii) α $(= \tau_0\kappa/l^2)$ is the same for the model and original body, where κ is the thermal diffusivity and $\tau_0 = 1$, $l = 1$ for the model.

If the body is subjected to plane thermo-elastic strain, the stresses in the body are M times those in the model provided that β, σ and α have the same values for the model and the original body. If the surface tractions applied to each closed boundary of the body considered are self-equilibrating, the stresses in the body are M times those in the scale model provided that $\beta(1 - 2\sigma)/(1 - \sigma)$ and α have the same values for the model and the original body.

In a generalized plane stress problem, the average stresses are M times those in the model provided that β, σ and α have the same values for the body and the model. If the surface tractions applied to each closed boundary of the body are self-equilibrating, the conditions become that α and $\beta(1 - 2\sigma)$ have the same values in the model and the original body.

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Finally, we have the case of a body in plane thermo-elastic strain and a scale model in generalized plane thermo-elastic stress, the surface temperatures in the body being Θ_0 times the average surface temperatures in the model, while the surface tractions in the body are M times the average surface tractions in the model, the averages being taken over the thickness of the generalized plane stress model. Then, the stress in the body is M times the average stress at the corresponding position and time in the model, provided that σ, β and α have the same values in the model and the original body. However, if the surface tractions applied to each closed boundary of the body are self-equilibrating, these conditions can be replaced by the weaker conditions

$$\alpha^* = \alpha \text{ and } \beta^*(1 - 2\sigma^*) = \beta(1 - 2\sigma)/(1 - \sigma),$$

where σ^*, β^* and α^* are the values of σ, β and α respectively for the model.

In all the cases considered, it is assumed that no body forces are applied and that, although the surface tractions and temperatures may vary with time, such changes are sufficiently slow for inertial forces to be neglected. In the plane problems, the similarity laws apply, of course, only to the planar stress components.

In each of the cases discussed, the modelling conditions given above are those for which the stresses in the body considered are M times those in the model. Since the governing equations of the problems considered are linear, we can readily derive conditions under which the stresses in the original body shall be any constant C times those in the model. For example, let us consider the case in which the original body is in plane strain and the model is in generalized plane stress, and the surface tractions applied to each closed boundary of the body are self-equilibrating. The condition $\beta^*(1 - 2\sigma^*) = \beta(1 - 2\sigma)/(1 - \sigma)$ implies that

$$\Theta_0 = \frac{M}{N} \frac{(1 - 2\sigma^*)(1 - \sigma)}{1 - 2\sigma},$$

where N is the ratio of the thermal expansion coefficients for the original body and the model. If we now multiply both the surface tractions and temperatures in the scale model by M/C the stresses in the original body will be C times the average stresses in the model and Θ_0 will be changed to

$$\Theta_0 = \frac{C}{N} \frac{(1 - 2\sigma^*)(1 - \sigma)}{1 - 2\sigma}.$$

Similar results may readily be obtained in the other cases considered.

2. Fundamental equations. We use Cartesian tensor notation and denote a rectangular coordinate system by y_i, where Latin indices take the values 1, 2, 3. Components of the displacement, strain, stress and surface traction (or stress vector) are denoted by v_i, ϵ_{ij}, τ_{ij} and F_i respectively and τ is time and θ temperature. If motion in the elastic body is such that inertial terms may be neglected, and if body forces are zero, then

$$\frac{\partial \tau_{ij}}{\partial y_j} = 0. \tag{2.1}$$

The stress-strain relations [2] are

$$\tau_{ij} = 2\mu\left(\epsilon_{ij} + \frac{\sigma}{1 - 2\sigma} \epsilon_{rr}\delta_{ij}\right) - \nu\theta\delta_{ij}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right), \tag{2.2}$$

and the equation of heat conduction is

$$\frac{\partial^2 \theta}{\partial y_i\, \partial y_i} = \frac{1}{\kappa}\frac{\partial \theta}{\partial t}. \tag{2.3}$$

In (2.2) δ_{ij} is a Kronecker delta, μ and σ are the rigidity modulus and Poisson's ratio respectively, and ν is a coefficient of thermal expansion. In (2.3) κ is the thermal diffusivity.

We now express (2.1), (2.2) and (2.3) in non-dimensional form by using the substitutions

$$\begin{aligned} y_i &= lx_i\,, \qquad v_i = lu_i\,, \qquad \tau = \tau_0 t, \qquad \epsilon_{ij} = e_{ij}\,,\\ \tau_{ij} &= \mu t_{ij}\,, \qquad F_i = \mu f_i\,, \qquad \theta = \theta_0 T, \end{aligned} \tag{2.4}$$

where l is a standard length, τ_0 a standard time and Θ_0 a standard temperature. Also, x_i, u_i, t, e_{ij}, t_{ij}, f_i and T are non-dimensional coordinates, displacements, time, strains, stresses, surface tractions (or stress vectors) and temperature respectively. Using (2.4), Eqs. (2.1), (2.2) and (2.3) become

$$\frac{\partial t_{ij}}{\partial x_j} = 0, \tag{2.5}$$

$$\begin{aligned} t_{ij} &= 2\left(e_{ij} + \frac{\sigma}{1 - 2\sigma} e_{rr}\delta_{ij}\right) - \beta T \delta_{ij}\,,\\ e_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \end{aligned} \tag{2.6}$$

and

$$\frac{\partial^2 T}{\partial x_i\, \partial x_i} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.7}$$

respectively, where

$$\alpha = \frac{\tau_0 \kappa}{l^2} \quad \text{and} \quad \beta = \frac{\Theta_0 \nu}{\mu}. \tag{2.8}$$

The stress components t_{ij} may be eliminated from (2.6) and (2.7) to give the three differential equations

$$\frac{\partial^2 u_i}{\partial x_j\, \partial x_j} + \frac{1}{1 - 2\sigma}\frac{\partial^2 u_j}{\partial x_i\, \partial x_i} - \beta\frac{\partial T}{\partial x_j} = 0. \tag{2.9}$$

In addition to the above fundamental equations, we have boundary conditions in which temperature or temperature gradients, displacements or surface tractions—or a mixture of such conditions—are prescribed. For clarity, we restrict our attention to boundary conditions which involve no new parameters, although it would not be difficult to include these.

If F_i is the surface traction per unit area acting on a surface, the normal to which has direction-cosines l_i, then

$$F_i = \tau_{ij} l_j . \tag{2.10}$$

We see that the non-dimensional surface traction f_i is given by

$$f_i = t_{ij} l_j . \tag{2.11}$$

3. Similarity laws in three dimensions. If temperature or temperature gradients are prescribed on the boundaries of the body considered, then we see from (2.7) that the non-dimensional temperature distribution throughout the body depends only on the non-dimensional coordinates x_i, non-dimensional time t, and the non-dimensional parameter α. If we regard this non-dimensional temperature distribution as being the actual temperature distribution in a scale model of the body under consideration, then the temperature distribution in the original body may be obtained by using the scale relations (2.4), provided α is kept constant. If, however, we are concerned only with a steady state temperature distribution, we see from (2.7) that T is then independent of α, so that no restriction on α is necessary. Further, if the boundary conditions on the temperature do not involve a time scale (e.g. if their dependence on time has the form of a step-function), then we can choose the scale-factor τ_0 in such a way that α is kept constant.

If boundary conditions for the body under consideration are given in terms of displacements, then (2.9) shows that the non-dimensional displacement u_i throughout the body depends, in general, on σ and β. Again, if the boundary conditions are given in terms of surface tractions (i.e. in terms of stress components), we see, from (2.5), (2.6) and (2.11) that the non-dimensional displacement components u_i and stress components t_{ij} throughout the body depend, in general, on σ and β. Thus, if we regard these non-dimensional displacement and stress distributions as being the actual displacement and stress distributions in a scale model of the body under consideration, then the displacement and stress distributions in the original body can, in general, be calculated from those in the model, by means of Eqs. (2.4), only if σ and β have the same values in the model and the original body.

4. Plane strain. The assumptions of plane strain are that

$$u_1 = u_1(x_1, x_2, t), \qquad u_2 = u_2(x_1, x_2, t), \qquad T = T(x_1, x_2, t), \qquad u_3 = 0. \tag{4.1}$$

It follows from (2.5), (2.6) and (2.7) that

$$e_{31} = e_{23} = e_{33} = t_{31} = t_{23} = 0, \tag{4.2}$$

and

$$\frac{\partial t_{\lambda\mu}}{\partial x_\lambda} = 0, \tag{4.3}$$

$$\begin{aligned} t_{\lambda\mu} &= 2\left(e_{\lambda\mu} + \frac{\sigma}{1 - 2\sigma} e_{\rho\rho} \delta_{\lambda\mu}\right) - \beta T \delta_{\lambda\mu} , \\ e_{\lambda\mu} &= \frac{1}{2}\left(\frac{\partial u_\lambda}{\partial x_\mu} + \frac{\partial u_\mu}{\partial x_\lambda}\right), \end{aligned} \tag{4.4}$$

$$\frac{\partial^2 T}{\partial x_\lambda \, \partial x_\lambda} = \frac{1}{\alpha} \frac{\partial T}{\partial t} , \tag{4.5}$$

where Greek subscripts take values 1, 2. With the help of complex variable notations, Eqs. (4.3) and (4.4) can be integrated in terms of complex potentials (see [3] and [4]). We use the notation

$$z = x_1 + ix_2\,, \qquad D = u_1 + iu_2\,, \qquad \Theta = t_{11} + t_{22}\,, \qquad \Phi = t_{11} - t_{22} + 2it_{12}\,, \tag{4.6}$$

so that Eqs. (4.3) and (4.4) are replaced by

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Theta}{\partial \bar{z}} = 0, \tag{4.7}$$

$$\Theta = \frac{2}{1 - 2\sigma}\left(\frac{\partial D}{\partial z} + \frac{\partial \bar{D}}{\partial \bar{z}}\right) - 2\beta T, \qquad \Phi = 4\,\frac{\partial D}{\partial \bar{z}}\,, \tag{4.8}$$

where a bar placed over a quantity denotes the complex conjugate of that quantity.

From (4.7), it can be shown that Θ and Φ must be expressible in the form

$$\Theta = 4\,\frac{\partial^2 \varphi}{\partial z\,\partial \bar{z}}\,, \qquad \Phi = -4\,\frac{\partial^2 \varphi}{\partial \bar{z}\,\partial \bar{z}}\,, \tag{4.9}$$

where φ is a real function (Airy's stress function). The second equations in (4.8) and (4.9) can be integrated to give

$$D + \frac{\partial \varphi}{\partial \bar{z}} = 4(1 - \sigma)f(z), \tag{4.10}$$

where $f(z)$ is an arbitrary function* of z. Using (4.10) and its complex conjugate we may eliminate D and Θ from the first equations in (4.8) and (4.9) to obtain

$$\frac{\partial^2 \varphi}{\partial z\,\partial \bar{z}} = f'(z) + \bar{f}'(\bar{z}) - \frac{\beta(1 - 2\sigma)}{4(1 - \sigma)}\,T. \tag{4.11}$$

Since φ is a real function, two integrations of (4.11) then give

$$\varphi = z\bar{f}(\bar{z}) + \bar{z}f(z) + g(z) + \bar{g}(\bar{z}) - \frac{\beta(1 - 2\sigma)}{1 - \sigma}\,R, \tag{4.12}$$

where $g(z)$ is a second arbitrary function of z and where $R(z, \bar{z}, t)$ is a real particular integral of the equation

$$4\,\frac{\partial^2 R}{\partial z\,\partial \bar{z}} = T. \tag{4.13}$$

From (4.9), (4.10) and (4.12), we now obtain stress and displacement components in the form

$$\begin{aligned} \Theta &= 4[f'(z) + \bar{f}'(\bar{z})] - \frac{\beta(1 - 2\sigma)}{1 - \sigma}\,T, \\ \Phi &= -4[z\bar{f}''(\bar{z}) + \bar{g}''(\bar{z})] + \frac{4\beta(1 - 2\sigma)}{1 - \sigma}\,\frac{\partial^2 R}{\partial \bar{z}\,\partial \bar{z}}\,, \\ D &= (3 - 4\sigma)f(z) - z\bar{f}'(\bar{z}) - \bar{g}'(\bar{z}) + \frac{\beta(1 - 2\sigma)}{1 - \sigma}\,\frac{\partial R}{\partial \bar{z}}. \end{aligned} \tag{4.14}$$

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*$f(z)$ and $g(z)$, defined in (4.12), also depend on t, but this is not shown explicitly.

If X_α are the components of non-dimensional resultant force (per unit length of x_3 axis) due to the stresses acting across an arc AB in the material, in a certain sense, then

$$\begin{aligned} X_1 &= \int_A^B f_1\, ds = \int_A^B (t_{12}\, dx_1 - t_{11}\, dx_2) \\ X_2 &= \int_A^B f_2\, ds = \int_A^B (t_{22}\, dx_1 - t_{12}\, dx_2), \end{aligned} \tag{4.15}$$

where ds is an element of length of the arc AB. Hence, using (4.6),

$$X_1 + iX_2 = \tfrac{1}{2}i \int_A^B (\Theta\, dz - \Phi\, d\bar{z}), \tag{4.16}$$

and with (4.9), this becomes

$$X_1 + iX_2 = 2i\left[\frac{\partial\varphi}{\partial\bar{z}}\right]_A^B, \tag{4.17}$$

where $[\quad]_A^B$ denotes the change of value of the argument as we move along the arc from A to B. From (4.17) and (4.12), we have

$$X_1 + iX_2 = 2i\left[f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) - \frac{\beta(1-2\sigma)}{1-\sigma}\frac{\partial R}{\partial\bar{z}}\right]_A^B. \tag{4.18}$$

5. Single-valued stresses and displacements. We now restrict attention to problems in which the stress and displacement components and the temperature are single-valued and without singularities at every internal point of the body. We also assume at present that the real particular integral R of (4.13) can be chosen to be single-valued, together with its derivatives up to and including the fourth order. This assumption will be justified in the Appendix (Sec. 9). It follows from (4.14) that the contributions of R to the stress and displacement components are single-valued. The conditions for the stress and displacement components to be single-valued therefore reduce to

$$[f'(z) + \bar{f}'(\bar{z})]_A^A = 0, \qquad [z\bar{f}''(\bar{z}) + \bar{g}''(\bar{z})]_A^A = 0$$

and

$$[(3-4\sigma)f(z) - z\bar{f}'(\bar{z}) - \bar{g}'(\bar{z})]_A^A = 0, \tag{5.1}$$

where $[\quad]_A^A$ denotes the change in value of the function inside the brackets on passing once around a contour in the conventional sense which keeps the area enclosed on the left, the contour lying inside the body.

We may differentiate the last of Eqs. (5.1) with respect to z(or $\bar{z}$), inside the bracket, since we have already assumed that the relevant derivatives exist (see [5]), to obtain

$$[(3-4\sigma)f'(z) - \bar{f}'(\bar{z})]_A^A = 0. \tag{5.2}$$

Using this, together with (5.1), we obtain

$$\begin{gathered} [f'(z)]_A^A = 0, \qquad [g''(z)]_A^A = 0, \\ (3-4\sigma)[f(z)]_A^A = [\bar{g}'(\bar{z})]_A^A, \end{gathered} \tag{5.3}$$

as given, e.g. by [2] or [3], for the case when the temperature is constant throughout the body.

If we further restrict attention to problems in which the stress system is such that the resultant force acting on any closed curve in the body (or its boundary, if singular points on the boundary are excluded by small indentations of the contour) is zero, then, from (4.18) we see that the conditions (5.3) may be replaced by

$$[f(z)]_A^A = 0, \qquad [g'(z)]_A^A = 0. \tag{5.4}$$

6. Similarity laws for plane strain. Suppose that the body under consideration is in a state of plane thermo-elastic strain and that the region S occupied by the cross-section of the body is finite and multiply-connected and bounded by one or more smooth non-intersecting contours L_0 , L_1 , $\cdots$, L_n , of which L_0 contains all the others.

In view of (5.4), we shall assume that, for the particular initial and boundary conditions of the problem, $f(z)$ and $g'(z)$ are regular functions of z, for every value of the time t, in the open region S, and continuous in S and on its boundaries, except possibly at points on the boundaries at which isolated forces or couples act and where the singularities are prescribed. We shall also assume that derivatives with respect to time exist up to any required order. Then, from (4.14) and (4.18) it follows that the non-dimensional stress and displacement components are single-valued and continuous throughout the interior of S for all time and that the resultant force acting on any closed contour in S is zero. Moreover, if temperatures and self-equilibrating surface tractions are prescribed on the boundaries so that the right-hand side of (4.18) is given on L_0 , L_1 , $\cdots$, L_n , the solution (4.14) for Θ and Φ of the resulting boundary-value problem involves only the physical parameters α and β $(1 - 2\sigma)/(1 - \sigma)$. It is thus seen that if we regard the non-dimensional stress components given by (4.14) as the actual stress components in a scale model of the body under consideration, then the stress components in the original body can be calculated from those in the scale model by using the scale relations (2.4), provided that α and $\beta(1 - 2\sigma)/(1 - \sigma)$ have the same values in the model and the original body. However, the non-dimensional displacement components D given by (4.14) involve the physical parameters α, σ and β $(1 - 2\sigma)/(1 - \sigma)$ and therefore the displacement components in the original body can, in general, be calculated from those of the scale model only if α, σ and β have the same values in the model and the original body. If the surface tractions on each of the contours L_0 , L_1 , $\cdots$, L_n are not self-equilibrating, then the conditions for the stress and displacement components to be single-valued are given by Eqs. (5.3). Thus, if the surface tractions on L_0 , L_1 , $\cdots$, L_n , given by (4.18), are specified, it is seen that Eqs. (4.14) for Θ and Φ and Eq. (5.3) involve the physical parameters α, $\beta(1 - 2\sigma)/(1 - \sigma)$ and σ. Hence, in order to calculate the stress components of a body from those of a scale model, by means of Eqs. (2.4), α, σ and β must, in general, have the same values in the body and the model.

7. Generalized plane stress. We consider a body bounded by plane faces $x_3 = \pm h$, where h is a non-dimensional constant, and by cylindrical surfaces perpendicular to to these surfaces. The planes $x_3 = \pm h$ are assumed to be free from applied stress and to be insulated so that there is no loss of heat from them. Thus,

$$\frac{\partial T}{\partial x_3} = 0, \qquad t_{31} = t_{23} = t_{33} = 0 \qquad (x_3 = \pm h). \tag{7.1}$$

We assume that the material of the body has Poisson's ratio σ^*, coefficient of thermal expansion ν^*, rigidity modulus μ^* and thermal diffusivity κ^* and α^* and β^* are defined

$$\alpha^* = \frac{\tau_0 \kappa^*}{l^2} \quad \text{and} \quad \beta^* = \frac{\Theta_0 \nu^*}{\mu^*},$$

where l is again a standard length, τ_0 a standard time and Θ_0 a standard temperature and μ^* is taken as the scale factor for the stress components.

From (2.6), replacing σ and β by σ^* and β^*, we have

$$e_{33} = -\frac{\sigma^*}{1 - \sigma^*} e_{\rho\rho} + \frac{1 - 2\sigma^*}{2(1 - \sigma^*)} (t_{33} + \beta^* T) \tag{7.2}$$

and using this to eliminate e_{33} from the remaining equations in (2.6) we find that

$$t_{\lambda\mu} = 2\left[e_{\lambda\mu} + \frac{\sigma^*}{1 - \sigma^*} e_{\rho\rho}\delta_{\lambda\mu}\right] - \frac{\beta^*(1 - 2\sigma^*)T\delta_{\lambda\mu}}{1 - \sigma^*} + \frac{\sigma^* t_{33}\delta_{\lambda\mu}}{1 - \sigma^*},$$

$$t_{\lambda 3} = 2e_{\lambda 3}. \tag{7.3}$$

We now assume that the body is subject to a stress system which is such that the displacements u_1, u_2 are even functions of x_3 and u_3 is an odd function of x_3. We then follow the usual procedure and take average values of Eqs. (2.5) and (7.3), using conditions (7.1). We add the extra assumption that the average value of t_{33} is neglected in comparison with the average values of $t_{\lambda\mu}$ Denoting the average values of $t_{\lambda\mu}$, $e_{\lambda\mu}$, u_α and T by $t^*_{\lambda\mu}$, $e^*_{\lambda\mu}$, u^*_α and T^* respectively, we obtain

$$\frac{\partial t^*_{\lambda\mu}}{\partial x_\mu} = 0, \tag{7.4}$$

$$t^*_{\lambda\mu} = 2\left(e^*_{\lambda\mu} + \frac{\sigma^*}{1 - \sigma^*} e^*_{\rho\rho}\delta_{\lambda\mu}\right) - \frac{\beta^*(1 - 2\sigma^*)}{1 - \sigma^*} T^*\delta_{\lambda\mu},$$

$$e^*_{\lambda\mu} = \frac{1}{2}\left(\frac{\partial u^*_\lambda}{\partial x_\mu} + \frac{\partial u^*_\mu}{\partial x_\lambda}\right) \tag{7.5}$$

and

$$\frac{\partial T^*}{\partial x_\lambda \, \partial x_\lambda} = \frac{1}{\alpha^*} \frac{\partial T^*}{\partial t}. \tag{7.6}$$

If we denote by X^*_α the average values of the components of the non-dimensional resultant force (per unit of x_3-axis) due to the stresses acting across an arc AB of the material lying in the x_1x_2-plane, then we see from (4.15) that

$$X^*_1 = \int_A^B (t^*_{12}\, dx_1 - t^*_{11}\, dx_2), \qquad X^*_2 = \int_A^B (t^*_{22}\, dx_1 - t^*_{12}\, dx_2). \tag{7.7}$$

We note, replacing μ by μ^* in Eqs. (2.4), that the average values of the actual stresses $\tau^*_{\lambda\mu}$, displacements v^*_α and temperature Θ^* can be obtained from the non-dimensional values by means of the relations

$$y_\alpha = l x_\alpha, \qquad v^*_\alpha = l u^*_\alpha, \qquad \tau = \tau_0 t, \qquad \tau^*_{\lambda\mu} = \mu^* t^*_{\lambda\mu},$$

and

$$\Theta^* = \Theta_0 T^*. \tag{7.8}$$

Equations (7.4) to (7.7) are identical in form with Eqs. (4.3), (4.4), (4.5) and (4.15) if we replace

$$t^*_{\lambda\mu}\,, \qquad e^*_{\lambda\mu}\,, \qquad T^*, \qquad u^*_\lambda\,, \qquad \sigma^*/(1-\sigma^*), \qquad \beta^*(1-2\sigma^*)/(1-\sigma^*)$$

and α^* in the former by $t_{\lambda\mu}$, $e_{\lambda\mu}$, T, u_λ, $\sigma/(1-2\sigma)$, β and α respectively. Hence, we can write down the solution of Eqs. (7.4) to (7.7) by analogy with the plane strain problem. The conditions for the average values of the non-dimensional displacement and stress components to be single-valued may similarly be written down by analogy with the case of plane strain.

With the notation

$$\Theta^* = t^*_{11} + t^*_{22}\,, \qquad \Phi^* = t^*_{11} - t^*_{22} + 2it^*_{12} \quad \text{and} \quad D^* = u^*_1 + iu^*_2\,, \tag{7.9}$$

we obtain [see Eqs. (4.14)]

$$\Theta^* = 4[f^{*\prime}(z) + \bar{f}^{*\prime}(\bar{z})] - \beta^*(1-2\sigma^*)T^*,$$

$$\Phi^* = -4[z\bar{f}^{*\prime\prime}(\bar{z}) + \bar{g}^{*\prime\prime}(\bar{z})] + 4\beta^*(1-2\sigma^*)\frac{\partial^2 R^*}{\partial \bar{z}\,\partial \bar{z}}\,,$$

$$D^* = \frac{3-\sigma^*}{1+\sigma^*}f^*(z) - z\bar{f}^{*\prime}(\bar{z}) - \bar{g}^{*\prime}(\bar{z}) + \beta^*(1-2\sigma^*)\frac{\partial R^*}{\partial \bar{z}}\,, \tag{7.10}$$

where $f^*(z)$ and $g^*(z)$ are functions of z which depend on the boundary conditions and $R^*(z, \bar{z}, t)$ is a real particular integral of the equation

$$4\frac{\partial^2 R^*}{\partial z\,\partial \bar{z}} = T^*. \tag{7.11}$$

From (7.7) and (7.10) it is seen that

$$X^*_1 + iX^*_2 = 2i\left[f^*(z) + z\bar{f}^{*\prime}(\bar{z}) + \bar{g}^{*\prime}(\bar{z}) - \beta^*(1-2\sigma^*)\frac{\partial R^*}{\partial \bar{z}}\right]_A^B. \tag{7.12}$$

It can readily be seen, in a manner similar to that adopted in Sec. 5, that the conditions for the average values of the non-dimensional average stress and average displacement components $t^*_{\lambda\mu}$ and u^*_λ to be single-valued are

$$[f^{*\prime}(z)]_A^A = 0, \qquad [g^{*\prime\prime}(z)]_A^A = 0$$

and

$$\frac{3-\sigma^*}{1+\sigma^*}[f^*(z)]_A^A = [\bar{g}^{*\prime}(\bar{z})]_A^A\,. \tag{7.13}$$

In the case when the stress system is such that the resultant average force acting on any closed curve in the body is zero, the conditions (7.13) become

$$[f^*(z)]_A^A = 0, \qquad [g^{*\prime}(z)]_A^A = 0. \tag{7.14}$$

8. Similarity laws for generalized plane stress. We now assume, as in the case of plane strain previously discussed, that the cross-section of the body considered, normal to the x_3-axis, is finite and multiply-connected and bounded by one or more smooth non-intersecting contours L_0, L_1, $\cdots$, L_n, of which L_0 contains all the others, and that the values of the non-dimensional average surface tractions (and hence of X^*_α) and non-dimensional average temperature T^* are given over each of the contours L_0, L_1, $\cdots$, L_n. Since, from Eqs. (7.6) and (7.11), the values of T^* and R^* throughout the body depend only on the physical parameter α^*, the expressions (7.10) and (7.12) for Θ^*, Φ^* and X_α^* involve only the physical parameters α^* and $\beta^*(1 - 2\sigma^*)$. Also, if the surface tractions applied to each of the contours L_0, L_1, $\cdots$, L_n are self-equilibrating, the conditions (7.14) for the average stress and average displacement components to be single-valued do not involve any physical constants. It follows that, in the case when the specified surface tractions applied to each of the contours L_0, L_1, $\cdots$, L_n are self-equilibrating, if the non-dimensional average stress distribution is taken as the actual average stress distribution in a scale model of the body under consideration, then the average stress distribution in the original body can, in general, be calculated from that in the model by employing the relations (7.8), if α^* and $\beta^*(1 - 2\sigma^*)$ have the same values for the model and the original body. In the case when the specified surface tractions applied to each of the contours L_0, L_1, $\cdots$, L_n are not self-equilibrating, since the conditions (7.13) for the stress and displacement components to be single-valued involve σ^*, the average stress distribution in the original body can, in general, be obtained from that in the model by means of the relations (7.8) only if α^*, σ^* and β^* have the same values in the model and the original body.

Now, suppose that we have two bodies A and B with geometrically similar cross-sections bounded by the closed contours L_0, L_1, $\cdots$, L_n, as described above. Let the dimensions of the cross-section of A be l times those of B. The body A is maintained in a state of plane thermo-elastic strain and B is maintained in a state of generalized plane thermo-elastic stress by surface tractions and surface temperatures such that the actual average surface tractions and average surface temperatures in B are equal to the non-dimensional surface tractions and surface temperatures in A. We assume that the physical parameters for the body A are the unstarred parameters defined in Sec. 2, while those for the body B are the starred parameters defined in the present section. Taking $l = 1$, $\tau_0 = 1$, $\mu^* = 1$ and $\Theta_0 = 1$ for the body B, we see that (7.10) and (7.12) give the actual average stress components and surface tractions in B, while T^* represents the actual average temperature in B. Then, comparing Eqs. (7.6), (7.10), (7.11), (7.12) and (7.13) with Eqs. (4.5), (4.14), (4.13), (4.18) and (5.3), and bearing in mind that we have assumed $T = T^*$ and $X_\alpha = X_\alpha^*$ on the surfaces of A and B, we see that provided that

$$\alpha^* = \alpha, \qquad \beta^*(1 - 2\sigma^*) = \frac{\beta(1 - 2\sigma)}{1 - \sigma}, \qquad \frac{3 - \sigma^*}{1 + \sigma^*} = 3 - 4\sigma, \tag{8.1}$$

i.e. provided that

$$\alpha^* = \alpha, \qquad \sigma^* = \frac{\sigma}{1 - \sigma} \quad \text{and} \quad \beta^* = \frac{\beta(1 - 2\sigma)}{1 - 3\sigma}, \tag{8.2}$$

we have

$$\Theta^* = \Theta \quad \text{and} \quad \Phi^* = \Phi, \tag{8.3}$$

so that the stress distribution in A can be calculated from the average stress distribution in B by the scale factors defined in Eqs. (2.4).

If the surface tractions applied to each of the contours L_0, L_1, $\cdots$, L_n are self-equilibrating, then comparing Eqs. (7.6), (7.10), (7.11), (7.12) and (7.14) with Eqs.

(4.5), (4.14), (4.13), (4.18) and (5.4), we see that Eqs. (8.3) are satisfied provided that

$$\alpha^* = \alpha \quad \text{and} \quad \beta^*(1 - 2\sigma^*) = \frac{\beta(1 - 2\sigma)}{1 - \sigma}. \tag{8.4}$$

9. Appendix. The function R. In deriving in Sec. 5 the conditions for the stress and displacement components to be single-valued the assumption was made that the function $R(z, \bar{z}, t)$ satisfying Eq. (4.13) can be chosen to be real and single-valued, together with its derivatives up to the fourth order. This assumption will be justified in the present section.

We deal first with the case of steady heat flow and statical equilibrium for which $T = T_0(z, \bar{z})$, $R = R_0(z, \bar{z})$ and

$$\frac{\partial^2 T_0}{\partial z\, \partial \bar{z}} = 0, \qquad T_0 = 4\,\frac{\partial^2 R_0}{\partial z\, \partial \bar{z}}. \tag{9.1}$$

Since T_0 is real, the first of Eqs. (9.1) may be integrated in the form

$$T_0 = h'(z) + \bar{h}'(\bar{z}), \tag{9.2}$$

where, since T_0 and its derivatives are single-valued inside the region S occupied by the cross-section of the body considered,

$$[h''(z)]_A^A = 0, \qquad [h'(z) + \bar{h}'(\bar{z})]_A^A = 0. \tag{9.3}$$

We now suppose that the region S is the multiply-connected region bounded by contours L_0, L_1, $\cdots$, L_n, as described in Sec. 6. We assume that the temperature distribution in the body is such that $h''(z)$ [which is seen from (9.3) to be single-valued] is a regular function of z in the open region S. From Laurent's theorem it can be seen that $h''(z)$ must be expressible in the form

$$h''(z) = \sum_{k=1}^{n} \frac{a_k}{2\pi(z - z_k)} - \sum \frac{b_k + id_k}{2\pi(z - z_k)^2} + k''(z), \tag{9.4}$$

where $k(z)$ is a regular function of z in S, b_k and d_k are real constants, z_k is a point inside the contour L_k (and therefore outside S) and a_k is a constant. The result (9.4) follows from the fact that if terms of higher degree than the second in $1/(z - z_k)$ are included in the Laurent expansion (9.4) for $h''(z)$ they can be absorbed into the function $k''(z)$ without vitiating the regularity of $k(z)$ in the region S.

By integration of (9.4), we obtain

$$h'(z) = \sum_{k=1}^{n} \frac{a_k}{2\pi} \log (z - z_k) + \sum_{k=1}^{n} \frac{b_k + id_k}{2\pi(z - z_k)} + k'(z), \tag{9.5}$$

$$\begin{aligned} h(z) = \sum_{k=1}^{n} \frac{a_k}{2\pi} (z - z_k) \log (z - z_k) + \sum_{k=1}^{n} \frac{b_k + id_k}{2\pi} \log (z - z_k) \\ + k(z) - \sum_{k=1}^{n} \frac{a_k}{2\pi} (z - z_k). \end{aligned} \tag{9.6}$$

From the second of Eqs. (9.3), it is seen that a_k is a real constant. It is apparent from (9.2) and (9.5) that if $h(z)$ is expressible in the form (9.6) with a_k real, T_0 is a real, single-valued function in S.

Introducing the expression (9.2) into the second of Eqs. (9.1), we obtain

$$4 \frac{\partial^2 R_0}{\partial z \, \partial \bar{z}} = h'(z) + \bar{h}'(\bar{z}). \tag{9.7}$$

Introducing the expression (9.5) for $h'(z)$ into (9.7), it is apparent that (9.7) has a particular integral for R_0 of the form

$$\begin{aligned} 4R_0 = & \sum_{k=1}^{n} \frac{a_k}{2\pi} (z - z_k)(\bar{z} - \bar{z}_k)[\log (z - z_k) + \log (\bar{z} - \bar{z}_k)] \\ & + \sum_{k=1}^{n} \left[\frac{b_k + id_k}{2\pi} \bar{z} + \frac{b_k - id_k}{2\pi} z \right] [\log (z - z_k) + \log (\bar{z} - \bar{z}_k)] + l(z, \bar{z}), \end{aligned} \tag{9.8}$$

where $l(z, \bar{z})$ is a real single-valued function in S, given by

$$l(z, \bar{z}) = \bar{z}k(z) + z\bar{k}(\bar{z}) - \sum_{k=1}^{n} \frac{a_k}{\pi} z\bar{z}. \tag{9.9}$$

Since the expression on the right-hand side of (9.8) is real and it and its derivatives are single-valued, R_0 may be chosen in such a way that it is real and it and its derivatives are single-valued.

When the heat flow is not steady, T and R satisfy the equations

$$4 \frac{\partial^2 T}{\partial z \, \partial \bar{z}} - \frac{1}{\alpha} \frac{\partial T}{\partial t} = 0, \qquad 4 \frac{\partial^2 R}{\partial z \, \partial \bar{z}} = T. \tag{9.10}$$

Eliminating T, we obtain

$$\frac{\partial^2}{\partial z \, \partial \bar{z}} 4 \left(\frac{\partial^2 R}{\partial z \, \partial \bar{z}} - \frac{1}{\alpha} \frac{\partial R}{\partial t} \right) = 0. \tag{9.11}$$

Equation (9.11) is satisfied by

$$4 \frac{\partial^2 R}{\partial z \, \partial \bar{z}} - \frac{1}{\alpha} \frac{\partial R}{\partial t} = Q, \tag{9.12}$$

where Q is any real function which satisfies the equation

$$\frac{\partial^2 Q}{\partial z \, \partial \bar{z}} = 0. \tag{9.13}$$

We make a particular choice of the function Q that it be expressible in the form

$$Q = -\frac{1}{\alpha} \frac{\partial P}{\partial t} + T_0 , \tag{9.14}$$

where T_0 is a real single-valued function, independent of t, which satisfies the equation

$$\frac{\partial^2 T_0}{\partial z \, \partial \bar{z}} = 0 \tag{9.15}$$

and P is a real function which satisfies the equation

$$\frac{\partial^2 P}{\partial z \, \partial \bar{z}} = 0. \tag{9.16}$$

Since T_0 satisfies (9.15) and is real and single-valued, we see, as we did in discussing the function T_0 given by the first of Eqs. (9.1), that there exists* a function $R_0(z, \bar{z})$ which is real and single-valued together with its derivatives up to the fourth order and satisfies the equation

$$T_0 = 4 \frac{\partial^2 R_0}{\partial z \, \partial \bar{z}}. \tag{9.17}$$

Defining $R_1(z, \bar{z}, t)$ by

$$R = P + R_0 + R_1 , \tag{9.18}$$

we see, from (9.12), (9.14), (9.16), (9.17) and (9.18), that R_1 must satisfy the equation

$$4 \frac{\partial^2 R_1}{\partial z \, \partial \bar{z}} - \frac{1}{\alpha} \frac{\partial R_1}{\partial t} = 0. \tag{9.19}$$

Substituting from (9.18) in the second of Eqs. (9.10) and employing (9.16) and (9.17), we have

$$T = T_0 + T_1 , \tag{9.20}$$

where

$$T_1 = 4 \frac{\partial^2 R_1}{\partial z \, \partial \bar{z}}. \tag{9.21}$$

From (9.19) and (9.21)

$$T_1 = \frac{1}{\alpha} \frac{\partial R_1}{\partial t}. \tag{9.22}$$

Since T and T_0 are real and single-valued, we see from (9.20) that T_1 must be a real single-valued function of z and $\bar{z}$ for every t. It follows from (9.22) that we may choose R_1 to be a real single-valued function of z and $\bar{z}$ for all t. It must, of course, also be single-valued in t and possess derivatives with respect to z, $\bar{z}$ and t of any required order. We have already seen that R_0 can be chosen to be real and single-valued and P has been chosen to be real and single-valued. It follows from (9.18) that R is real and single-valued.

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*In deriving this result, we note that in this case also T_0 must have the form (9.2) and we assume as in the earlier discussion that T_0 is such that $h''(z)$ is a regular function of z in the open region S.