HEAT FLOW IN A CYLINDER*

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Summary. A formal, analytical solution is given to the problem of finding the temperature in a cylinder which is gaining or losing heat by radiation exchange with a concentric, thin-walled metal tube. The tube in turn is radiating to, or receiving radiation from, its surroundings. A general treatment of systems with this type of boundary condition is presented first, and then the given problem is solved as a particular case.

Introduction. The problem being considered is that of radial heat conduction in an infinite, right circular cylinder surrounded by an air space, and then by a thin-walled, concentric, cylindrical tube. The tube is assumed to be a good thermal conductor, so that its temperature may be considered to be uniform throughout at all times. The rates of heat exchange between cylinder and tube, and between tube and surroundings are taken to be proportional to the respective temperature differences. The temperature of the surroundings is assumed to be some specified function of the time.

The feature that makes this problem of interest is the boundary condition. It is given by Eqs. (31) and (32), although the nature of the boundary condition is more easily seen if w (the temperature of the surrounding metal tube) is eliminated from those equations. The surface temperature of the cylinder, u, is then seen to satisfy the equation

$$\sigma_1 u + \sigma_2 \frac{\partial u}{\partial r} + \sigma_3 \frac{\partial u}{\partial t} + \sigma_4 \frac{\partial^2 u}{\partial t\, \partial r} = \sigma_5 g(t), \qquad (t \neq 0),$$

where the σ's are constants. This same boundary condition was handled earlier [1] for a semi-infinite medium. However, in the present case, the medium is of finite extent in the direction of the heat flow, and so the method of Ref. [1] cannot be used here.

It will be seen that the answer to the present problem is obtained as a series of functions which do not satisfy the usual orthogonality equation, but which satisfy instead the modified orthogonality relation given in Eq. (14). Before tackling the given heat flow problem, it is convenient to derive this modified orthogonality condition and some other preliminary results.

Modified orthogonality relations. Let

$$u = \sum u_n\,, \qquad v = \sum v_n\,, \qquad w = \sum w_n\,, \tag{1}$$

$$\frac{\partial}{\partial r}\left[P(r)\,\frac{\partial u_n}{\partial r}\right] + [Q(r) + \xi_n^2 S(r)]u_n = 0, \tag{2}$$

$$v_n = \left(C_1 + D_1\,\frac{\partial}{\partial r}\right)u_n(L_1\,,\, t), \tag{3}$$

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$$w_n = \left(C_2 + D_2 \frac{\partial}{\partial r}\right) u_n(L_2, t), \tag{4}$$

$$0 = \left[(A_1 - C_1\xi_n^2) + (B_1 - D_1\xi_n^2)\frac{\partial}{\partial r}\right] u_n(L_1, t), \tag{5}$$

$$0 = \left[(A_2 - C_2\xi_n^2) + (B_2 - D_2\xi_n^2)\frac{\partial}{\partial r}\right] u_n(L_2, t). \tag{6}$$

Then

$$\frac{\partial}{\partial r}\left[P(r) u_n \frac{\partial u_k}{\partial r} - P(r) u_k \frac{\partial u_n}{\partial r}\right] = u_n \frac{\partial}{\partial r}\left[P(r)\frac{\partial u_k}{\partial r}\right] - u_k \frac{\partial}{\partial r}\left[P(r)\frac{\partial u_n}{\partial r}\right] \tag{7}$$

$$= (\xi_n^2 - \xi_k^2) S(r) u_n u_k . \tag{8}$$

So

$$(A_1 - C_1\xi_k^2)(A_2 - C_2\xi_k^2)(\xi_n^2 - \xi_k^2) \int_{L_1}^{L_2} S(r) u_n u_k \, dr$$

$$= (A_1 - C_1\xi_k^2)(A_2 - C_2\xi_k^2)\left\{P(L_2)\left[u_n(L_2, t)\frac{\partial u_k(L_2, t)}{\partial r} - u_k(L_2, t)\frac{\partial u_n(L_2, t)}{\partial r}\right]\right.$$

$$\left. - P(L_1)\left[u_n(L_1, t)\frac{\partial u_k(L_1, t)}{\partial r} - u_k(L_1, t)\frac{\partial u_n(L_1, t)}{\partial r}\right]\right\} \tag{9}$$

$$= (A_1 - C_1\xi_k^2) P(L_2) \frac{\partial u_k(L_2, t)}{\partial r}\left[(A_2 - C_2\xi_k^2) + (B_2 - D_2\xi_k^2)\frac{\partial}{\partial r}\right] u_n(L_2, t)$$

$$- (A_2 - C_2\xi_k^2) P(L_1) \frac{\partial u_k(L_1, t)}{\partial r}\left[(A_1 - C_1\xi_k^2) + (B_1 - D_1\xi_k^2)\frac{\partial}{\partial r}\right] u_n(L_1, t) \tag{10}$$

$$= (A_1 - C_1\xi_k^2) P(L_2) \frac{\partial u_k(L_2, t)}{\partial r}\left[\left(A_2 + B_2\frac{\partial}{\partial r}\right) - \xi_k^2\left(C_2 + D_2\frac{\partial}{\partial r}\right)\right] u_n(L_2, t)$$

$$- (A_2 - C_2\xi_k^2) P(L_1) \frac{\partial u_k(L_1, t)}{\partial r}\left[\left(A_1 + B_1\frac{\partial}{\partial r}\right) - \xi_k^2\left(C_1 + D_1\frac{\partial}{\partial r}\right)\right] u_n(L_1, t) \tag{11}$$

$$= (A_1 - C_1\xi_k^2) P(L_2) \frac{\partial u_k(L_2, t)}{\partial r} (\xi_n^2 - \xi_k^2) w_n$$

$$- (A_2 - C_2\xi_k^2) P(L_1) \frac{\partial u_k(L_1, t)}{\partial r} (\xi_n^2 - \xi_k^2) v_n . \tag{12}$$

Hence

$$(\xi_n^2 - \xi_k^2)\left[(A_1 - C_1\xi_k^2)(A_2 - C_2\xi_k^2)\int_{L_1}^{L_2} S(r) u_n u_k \, dr\right.$$

$$+ (A_2 - C_2\xi_k^2) P(L_1) v_n \frac{\partial u_k(L_1, t)}{\partial r}$$

$$\left. - (A_1 - C_1\xi_k^2) P(L_2) w_n \frac{\partial u_k(L_2, t)}{\partial r}\right] = 0. \tag{13}$$

The expression in brackets is thus equal to zero for $n \neq k$. This gives a modified orthogonality relation. In the special case when $C_1 = C_2 = D_1 = D_2 = 0$, the development above is seen to reduce to the usual orthogonality treatment, as given, for example, in Ref. [2]. Equation (13) is not useful if A_1 and C_1 are both zero, or if A_2 and C_2 are both zero. However, other equations may be derived in a similar manner for these cases. For example, by multiplying Eq. (8) through by $(B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2)$ and then proceeding as in Eqs. (9)–(13), one obtains:

$$(\xi_n^2 - \xi_k^2)\left[(B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2)\int_{L_1}^{L_2} S(r)u_n u_k\,dr - (B_1 - D_1\xi_k^2)P(L_2)w_n\frac{\partial u_k(L_2\,,\,t)}{\partial r} - (A_2 - C_2\xi_k^2)P(L_1)v_n u_k(L_1\,,\,t)\right] = 0. \tag{14}$$

This equation will be used in solving the present problem.

An expression for the kth coefficient. If

$$u_n = R_n(r)T_n(t), \qquad v_n = \phi_n T_n(t), \qquad w_n = \psi_n T_n(t), \tag{15}$$

then from Eqs. (14) and (15) one finds

$$T_k(t)\left[(B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2)\int_{L_1}^{L_2} S(r)R_k^2\,dr - P(L_2)\psi_k(B_1 - D_1\xi_k^2)\frac{\partial R_k(L_2)}{\partial r} - P(L_1)\phi_k(A_2 - C_2\xi_k^2)R_k(L_1)\right] = (B_1 - D_1\xi_k^2)\int_{L_1}^{L_2} S(r)uR_k\,dr - P(L_2)w(B_1 - D_1\xi_k^2)\frac{\partial R_k(L_2)}{\partial r} - P(L_1)v(A_2 - C_2\xi_k^2)R_k(L_1). \tag{16}$$

Both sides of this equation were obtained by summing the brackets of Eq. (14) over n. On the left side of Eq. (16), however, use has been made of Eq. (15), and the fact that the brackets of Eq. (14) is zero for $n \neq k$, while the right side of Eq. (16) was determined by interchanging operations of integration and summation, and using Eqs. (1) and (15).

If one sets $t = 0$ in Eq. (16), he obtains $T_k(0)$, and hence the kth coefficient of the Fourier series for u. For the case when the surroundings are at constant temperatures, Eq. (16) therefore enables one to complete the solution to the problem. Equation (16) with $t = 0$ is usually obtained by getting the Fourier series for the initial value of u, multiplying by the kth eigenfunction, and integrating. The orthogonality relation then causes all terms of the series but the kth to drop out, and thus gives the kth coefficient. However, that procedure is not convenient to use in the present case because of the nature of the orthogonality relation—see Eq. (14)—hence the need for Eq. (16).

The fact that the temperature of the surroundings in the present problem of heat conduction in a cylinder is a prescribed function of the time can be taken care of by means of Duhamel's theorem. However, this turns out to be a bit lengthy in the present case. For this reason, the following development will be used instead.

A substitute for Duhamel's theorem. Let it be assumed that, in addition to Eqs. (1)–(6) and (15), the following equations are also satisfied

$$\frac{\partial}{\partial r}\left[P(r)\frac{\partial u}{\partial r}\right] + uQ(r) = \frac{S(r)}{\kappa}\frac{\partial u}{\partial t}, \tag{17}$$

$$\left[(A_1 - C_1\xi_n^2) + (B_1 - D_1\xi_n^2)\frac{\partial}{\partial r}\right]u(L_1, t) = E_1G_1(t) - \left(\frac{d}{dt} + \kappa\xi_n^2\right)\frac{v(t)}{\kappa}, \tag{18}$$

$$\left[(A_2 - C_2\xi_n^2) + (B_2 - D_2\xi_n^2)\frac{\partial}{\partial r}\right]u(L_2, t) = E_2G_2(t) - \left(\frac{d}{dt} + \kappa\xi_n^2\right)\frac{w(t)}{\kappa}. \tag{19}$$

Then, from Eqs. (2), (15) and (17)

$$\begin{aligned}\kappa\frac{\partial}{\partial r}\left[P(r)R_k\frac{\partial u}{\partial r} - P(r)u\frac{\partial R_k}{\partial r}\right] &= \kappa R_k\frac{\partial}{\partial r}\left[P(r)\frac{\partial u}{\partial r}\right] - \kappa u\frac{\partial}{\partial r}\left[P(r)\frac{\partial R_k}{\partial r}\right]\\ &= S(r)R_k\left(\frac{\partial}{\partial t} + \kappa\xi_k^2\right)u.\end{aligned} \tag{20}$$

So, from Eqs. (5), (6), (15), (18), (19) and (20), one finds

$$(B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2)\left(\frac{\partial}{\partial t} + \kappa\xi_k^2\right)\int_{L_1}^{L_2} S(r)uR_k\,dr$$

$$\begin{aligned}&= \kappa(B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2)P(L_2)\left[R_k(L_2)\frac{\partial u(L_2, t)}{\partial r} - u(L_2, t)\frac{\partial R_k(L_2)}{\partial r}\right]\\ &\quad - \kappa(B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2)P(L_1)\left[R_k(L_1)\frac{\partial u(L_1, t)}{\partial r} - u(L_1, t)\frac{\partial R_k(L_1)}{\partial r}\right],\end{aligned} \tag{21}$$

$$\begin{aligned}&= -\kappa(B_1 - D_1\xi_k^2)P(L_2)\frac{\partial R_k(L_2)}{\partial r}\left[(A_2 - C_2\xi_k^2) + (B_2 - D_2\xi_k^2)\frac{\partial}{\partial r}\right]u(L_2, t)\\ &\quad - \kappa(A_2 - C_2\xi_k^2)P(L_1)R_k(L_1)\left[(A_1 - C_1\xi_k^2) + (B_1 - D_1\xi_k^2)\frac{\partial}{\partial r}\right]u(L_1, t),\end{aligned} \tag{22}$$

$$\begin{aligned}&= (B_1 - D_1\xi_k^2)P(L_2)\frac{\partial R_k(L_2)}{\partial r}\left[\left(\frac{d}{dt} + \kappa\xi_k^2\right)w(t) - E_2\kappa G_2(t)\right]\\ &\quad + (A_2 - C_2\xi_k^2)P(L_1)R_k(L_1)\left[\left(\frac{d}{dt} + \kappa\xi_k^2\right)v(t) - E_1\kappa G_1(t)\right].\end{aligned} \tag{23}$$

Hence

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \kappa\xi_k^2\right)&\left[(B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2)\int_{L_1}^{L_2} S(r)uR_k\,dr\right.\\ &\left.- (B_1 - D_1\xi_k^2)P(L_2)\frac{\partial R_k(L_2)}{\partial r}w(t) - (A_2 - C_2\xi_k^2)P(L_1)R_k(L_1)v(t)\right]\\ &= (D_1\xi_k^2 - B_1)P(L_2)\frac{\partial R_k(L_2)}{\partial r}E_2\kappa G_2(t) + (C_2\xi_k^2 - A_2)P(L_1)R_k(L_1)E_1\kappa G_1(t).\end{aligned} \tag{24}$$

This is first order linear, and so may be integrated to give

$$\begin{aligned}&\left[(B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2)\int_{L_1}^{L_2} S(r)uR_k\,dr - (B_1 - D_1\xi_k^2)P(L_2)\frac{\partial R_k(L_2)}{\partial r}w(t)\right.\\ &\qquad\left. - (A_2 - C_2\xi_k^2)P(L_1)R_k(L_1)v(t)\right]\exp(\kappa\xi_k^2 t)\\ &\qquad = (B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2)\int_{L_1}^{L_2} S(r)u(r, 0)R_k\,dr\end{aligned}$$

$$- (B_1 - D_1\xi_k^2)P(L_2) \frac{\partial R_k(L_2)}{\partial r} w(0) - (A_2 - C_2\xi_k^2)P(L_1)R_k(L_1)v(0) \tag{25}$$

$$+ (D_1\xi_k^2 - B_1)P(L_2) \frac{\partial R_k(L_2)}{\partial r} E_2\kappa \int_0^t G_2(\tau) \exp(\kappa\xi_k^2\tau)\, d\tau$$

$$+ (C_2\xi_k^2 - A_2)P(L_1)R_k(L_1)E_1\kappa \int_0^t G_1(\tau) \exp(\kappa\xi_k^2\tau)\, d\tau.$$

If this equation is divided through by $\exp(k\xi_k^2 t)$, the left side will be the same as the right side of Eq. (16). Hence

$$T_k(t)\Big[(B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2) \int_{L_1}^{L_2} S(r)R_k^2\, dr - P(L_2)\psi_k(B_1 - D_1\xi_k^2) \frac{\partial R_k(L_2)}{\partial r}$$

$$- P(L_1)\phi_k(A_2 - C_2\xi_k^2)R_k(L_1)\Big] = \Big[(B_1 - D_1\xi_k^2)(A_2 - C_2\xi_k^2) \int_{L_1}^{L_2} S(r)u(r, 0)R_k\, dr$$

$$- (B_1 - D_1\xi_k^2)P(L_2) \frac{\partial R_k(L_2)}{\partial r} w(0) - (A_2 - C_2\xi_k^2)P(L_1)R_k(L_1)v(0)\Big] \exp(-\kappa\xi_k^2 t) \tag{26}$$

$$+ (D_1\xi_k^2 - B_1)P(L_2) \frac{\partial R_k(L_2)}{\partial r} E_2\kappa \int_0^t G_2(\tau) \exp[\kappa\xi_k^2(\tau - t)]\, d\tau$$

$$+ (C_2\xi_k^2 - A_2)P(L_1)R_k(L_1)E_1\kappa \int_0^t G_1(\tau) \exp[\kappa\xi_k^2(\tau - t)]\, d\tau.$$

This equation may be used to obtain $T_k(t)$ when $R_k(r)$ is known. Its use will be illustrated by solving the problem stated in the introduction.

A formal solution to the problem of heat flow in a cylinder. With the results developed above, it will now be relatively easy to solve the heat flow problem stated in the introduction, and, incidently, a number of related problems.

Let $u(r, t)$ be the temperature in the cylinder at a distance r from its axis, and $w(t)$ the temperature of the surrounding tube. Then the problem may be formulated mathematically as follows:

$$\frac{\kappa}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t}, \tag{27}$$

$$u(r, 0) = f(r), \tag{28}$$

$$w(0) = W, \tag{29}$$

$$\frac{\partial u(0, t)}{\partial r} = 0, \tag{30}$$

$$\frac{\partial u(L, t)}{\partial r} = a[w(t) - u(L, t)], \tag{31}$$

$$h \frac{dw}{dt} = a[u(L, t) - w(t)] + b[g(t) - w(t)]. \tag{32}$$

In these equations k, a, h and g are positive constants, $f(r)$ is the initial temperature of the cylinder, W is the initial temperature of the tube, and $g(t)$ is the temperature of the surroundings. Equation (32) arises from physical considerations. However, for mathematical purposes it is preferable to replace it by the following equation, which may be obtained algebraically from Eqs. (31) and (32):

$$a(b - h\kappa\xi_k^2)u(L, t) + (a + b - h\kappa\xi_k^2)\frac{\partial u(L, t)}{\partial r} = abg(t) - ah\left(\frac{d}{dt} + \kappa\xi_k^2\right)w(t). \tag{33}$$

In order to solve for u it is merely necessary to find the radial eigenfunctions, R_n, and the equation satisfied by the eigenvalues, ξ_n. The solution can then be completed by means of the results developed above. The R_n and ξ_n will be obtained by the method of separation of variables. To apply this method, one must first set $g(t) = 0$. The R_n and ξ_n obtained in this way will still apply for $g(t) \neq 0$.

Let u and w of Eqs. (27)–(33) be represented by the series given in Eq. (1), where u_n and w_n are assumed to be of the form given in Eq. (15). Then, for $g(t) = 0$, the u_n and w_n will each satisfy Eqs. (27)–(33), and so by the method of separation of variables, one finds from Eqs. (27) and (30) that

$$u_n = E_n \exp(-\kappa\xi_n^2 t)J_0(\xi_n r) \quad \text{when} \quad g(t) = 0. \tag{34}$$

From Eq. (31), then

$$aw_n = E_n \exp(-\kappa\xi_n^2 t)[aJ_0(\xi_n L) - \xi_n J_1(\xi_n L)] \quad \text{when} \quad g(t) = 0. \tag{35}$$

Hence, from Eq. (33) with $g(t)$ set equal to zero, one obtains

$$a(b - h\kappa\xi_n^2)J_0(\xi_n L) = \xi_n(a + b - h\kappa\xi_n^2)J_1(\xi_n L). \tag{36}$$

This is the equation which determines the eigenvalues, ξ_n.

Now that the eigenvalues have been determined, the standard separation of variable procedure will be discontinued. The restriction that $g(t) = 0$ will no longer be imposed, and Eqs. (34) and (35) will be generalized to

$$u_n = T_n(t)J_0(\xi_n r), \tag{37}$$

$$aw_n = T_n(t)[aJ_0(\xi_n L) - \xi_n J_1(\xi_n L)]. \tag{38}$$

Equations (27), (30) and (33) are special cases of Eqs. (17), (18) and (19) with

$$P(r) = r, \quad Q(r) = 0, \quad S(r) = r, \quad A_1 = C_1 = D_1 = E_1 = L_1 = v(t) = 0, \tag{39}$$

$$A_2 = b/(h\kappa), \qquad B_2 = (a + b)/(ah\kappa), \qquad C_2 = 1, \qquad D_2 = 1/a \tag{40}$$

$$E_2 = b/(h\kappa), \qquad G_2(t) = g(t), \qquad L_2 = L. \tag{41}$$

With the particular choices for constants and functions given by Eqs. (39)–(41), it will now be seen that the u_n and w_n of Eqs. (37) and (38), with ξ_n given by Eq. (36), satisfy Eqs. (1)–(6). Moreover, Eq. (31) agrees with Eqs. (4) and (1), and Eqs. (27), (30) and (33) with Eqs. (17), (18) and (19). Hence all the conditions used in deriving Eq. (26) are now satisfied, and so $T_n(t)$ may be obtained from Eq. (26). Upon using Eqs. (15), (28), (29), (37) and (38), and changing subscripts from k to n, one finds

$$\frac{T_n(t)}{\exp(-\kappa\xi_n^2 t)} \tag{42}$$

$$= \frac{a(b-h\kappa\xi_n^2)\int_0^L rf(r)J_0(\xi_n r)dr + ah\kappa\xi_n WLJ_1(\xi_n L) + ab\kappa\xi_n LJ_1(\xi_n L)\int_0^t g(\tau)\exp(\kappa\xi_n^2\tau)d\tau}{a(b-h\kappa\xi_n^2)\int_0^L rJ_0^2(\xi_n r)dr + h\kappa L[aJ_0(\xi_n L) - \xi_n J_1(\xi_n L)]\xi_n J_1(\xi_n L)}.$$

Temperatures u and w are now found by combining Eqs. (1), (37), (38) and (42).

References

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