NOTE ON A PAPER BY J.R.M. RADOK*

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It is of interest to note that the complex variable solution of the equations of dynamic plane elasticity recently derived by Radok† can be arrived at by splitting the displacement field by the classical decomposition

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}$$

and noting that the wave equations

$$c_1^2 \nabla_1^2 \phi = \phi^{\cdot\cdot}, \qquad c_2^2 \nabla_1^2 \psi = \psi^{\cdot\cdot}$$

to which they lead have solutions of the form

$$\phi(x - ct \pm i\beta_1 y), \qquad \psi(x - ct \pm i\beta_2 y)$$

with $\beta_1^2 = 1 - c^2/c_1^2$, $\beta_2^2 = 1 - c^2/c_2^2$. The details are given in a paper published a few years ago**. Radok's solution (4.4) is immediately derivable from my equation (8) by replacing my f' and g' by $-\phi/\mu$ and $i(1 + \beta_2^2)\psi/(2\mu\beta_2)$ respectively.

AN ALTERNATE SOLUTION OF STEFAN'S PROBLEM‡

BY

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The author, using a simple transformation, has solved two cases of Stefan's problem. These problems have been solved earlier by Stefan [1] and by Neumann [1] by other methods. There are a number of problems governed by the parabolic differential equation with a moving boundary condition. Such a problem is the melting of a solid in which case a finite heat sink exists at the position of the moving boundary. A similar problem exists in certain crystalline transformations. These problems, known as moving boundary problems or Stefan's problems, also arise in studies of gravity drainage and seepage of oil from sand beds during pumping.

The first problem considered here is the same as that studied by Stefan, being the case of a solid initially at the melting point with one face brought instantaneously to some temperature above the melting temperature and held constant thereafter. The remaining sides are considered as insulated so that the heat flow is one dimensional.

The second problem is the same as that discussed by Neumann, i.e., the moving boundary problem for a semi-infinite bar initially at a constant temperature *below* the

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†J. R. M. Radok, Quart. Appl. Math. XIV, 289 (1956).

**I. N. Sneddon, Rend. Circ. Mat. Palermo, (*ii*) 1, 57 (1952).

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melting temperature with the leading face brought to some higher temperature instantaneously.

Solution of Case I. Consider a rod of ice at 32°F. Bring the face of the rod to some temperature $u(0, t)$, the other faces being insulated. At a given time the equation

$$\alpha \frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial u(x, t)}{\partial t}, \tag{1}$$

where

$$\alpha = k/C_p\rho$$

describes the temperature in the liquid region. The boundary conditions are, say,

$$u(0, t) = 0, \tag{2a}$$

$$u(x(t), 0) = C, \tag{2b}$$

$$-k \frac{\partial u}{\partial x_{x=x(t)}} = \rho \Delta H_f x^{\cdot}(t),$$

where C is the temperature of fusion, $x(t)$ is the distance of the ice-water interface from the front face, ρ is the density, ΔH_f is the heat of fusion and $x^{\cdot}(t)$ is dx/dt or the rate at which the interface moves.

This last equation may be written as

$$-\frac{\partial u}{\partial x}\bigg|_{x=x(t)} = \frac{A}{\alpha} x^{\cdot}(t), \tag{2c}$$

where $A = \Delta H_f/C_p$.

Transforming Eqs. (1) and (2) in terms of

$$y(x, t) = \frac{x}{2(\alpha t)^{1/2}}, \tag{3}$$

we obtain the equation

$$\frac{d^2u}{dy^2} + 2y \frac{du}{dy} = 0 \tag{4}$$

with the boundary conditions

$$u(0) = 0, \tag{5a}$$

$$u(y(t)) = C, \tag{5b}$$

$$-\frac{du}{dy}\bigg|_{y=y(t)} = 2A[y(t) + 2ty^{\cdot}]. \tag{5c}$$

By direct quadrature Eq. (4) yields

$$\frac{du}{dy} = \beta e^{-y^2} \tag{6}$$

and

$$\int du = \beta \int e^{-y^2}\, dy + c. \tag{7}$$

Applying (5a) and (5b) to the latter gives

$$C = \beta \int_0^{y(t)} e^{-y^2}\, dy. \tag{8}$$

Hence $y(t) = \gamma$, a constant, since C is a constant. This immediately gives the time dependence of the position of the moving boundary,

$$x(t) = 2\gamma(\alpha t)^{1/2}. \tag{9}$$

Equation (5c) now becomes

$$-\beta e^{-\gamma^2} = 2A\gamma \tag{10a}$$

and Eq. (8) becomes

$$C = \beta \int_0^{\gamma} e^{-y^2}\, dy. \tag{10b}$$

The equations (10a) and (10b) can be readily solved for γ and β by first eliminating β and solving the resulting equation for γ since the distribution function and the error function are both tabulated.

Finally

$$u = \beta \int_0^{y} e^{-y^2}\, dy \tag{11}$$

subject to $0 \leq y \leq \gamma$, a condition which requires that the solution hold in the liquid region only.

Solution of Case II. Dividing the bar into region I which is the fluid region and region II the solid region, we obtain the following conditions on the rate of heat flow:

$$\alpha_{\mathrm{I}} \frac{\partial^2 u_{\mathrm{I}}}{\partial x^2} = \frac{\partial u_{\mathrm{I}}}{\partial t}, \tag{12}$$

$$u_{\mathrm{I}}(0, t) = 0, \tag{12a}$$

$$u_{\mathrm{I}}[x(t), t] = C, \tag{12b}$$

$$-k_{\mathrm{I}} \frac{\partial u_{\mathrm{I}}}{\partial x} = \rho \Delta H_f x^{\cdot}(t) - k_{\mathrm{II}} \frac{\partial u_{\mathrm{II}}}{\partial x}, \tag{12c}$$

$$\alpha_{\mathrm{II}} \frac{\partial^2 u_{\mathrm{II}}}{\partial x^2} = \frac{\partial u_{\mathrm{II}}}{\partial t}, \tag{13}$$

$$u_{\mathrm{II}}[x(t), t] = C, \tag{13a}$$

$$u_{\mathrm{II}}(\infty, t) = u(x, 0) = T_0, \tag{13b}$$

$$x(0) = 0. \tag{13c}$$

For convenience it is assumed that the density does not change. A further assumption is that equilibrium conditions are satisfied at the interface as regards temperature and state of aggregation; since the process of melting is a rate process it is expected that the above approximation would not be valid for extremely rapid heating rates. The problem of recrystallization of metals is often complicated in such a way. For example, the transformation from the face-centered cubic form in steel to the body-centered form is a rather slow process so that the condition (12c) is further complicated by the addition of some time-dependent factor on the expression $\rho \Delta H_f x^{\cdot}(t)$. There are, however, crystalline transformations which fall into the category described by Eqs. (12 and 13), these being the metals in which pure martensitic-type transformations occur.

In order to solve the given equation let

$$z = x/t^{1/2}. \tag{14}$$

Equation (12) now becomes

$$\alpha_{\mathrm{I}} \frac{d^2 u_{\mathrm{I}}}{dz^2} + \frac{z du_{\mathrm{I}}}{2dz} = 0, \tag{15}$$

while (12a, b and c) become

$$u_{\mathrm{I}}(0) = 0, \tag{15a}$$

$$u_{\mathrm{I}}[z(t)] = C, \tag{15b}$$

$$-k_{\mathrm{I}} \frac{du_{\mathrm{I}}}{dz}\bigg|_{z=z(t)} = \rho \Delta H_f(tz^{\cdot} + z/2) - k_{\mathrm{II}} \frac{du_{\mathrm{II}}}{dz}\bigg|_{z=z(t)} \tag{15c}$$

Similarly Eq. (13) yields

$$\alpha_{\mathrm{II}} \frac{d^2 u_{\mathrm{II}}}{dz^2} + \frac{z}{2}\frac{du_{\mathrm{II}}}{dz} = 0 \tag{16}$$

with the conditions

$$u_{\mathrm{II}}[z(t)] = C, \tag{16a}$$

$$u_{\mathrm{II}}(\infty) = T_0 . \tag{16b}$$

We leave the condition $z(0)$ unspecified for the time being. Treating du_I/dz as the dependent variable Eq. (15) can be directly integrated, yielding

$$\frac{du_{\mathrm{I}}}{dz} = \beta \exp \frac{-z^2}{4\alpha_{\mathrm{I}}}. \tag{17}$$

Repeated integration gives

$$C = \int_0^C du_{\mathrm{I}} = \beta_{\mathrm{I}} \int_0^{z(t)} \exp \frac{-z^2}{4\alpha_{\mathrm{I}}}\, dz. \tag{18}$$

This implies that $z(t) = \gamma$, a constant. This then gives for the temperature in the liquid region

$$u_{\mathrm{I}} = \beta_{\mathrm{I}} \int_0^z \exp \frac{-z^2}{4\alpha_{\mathrm{I}}}\, dz, \tag{19}$$

where β_{I} is yet to be evaluated. Moreover, it gives the position of the moving boundary as a function of time

$$x(t) = \gamma t^{1/2}, \tag{20}$$

where γ is yet to be evaluated.

Integration of Eq. (16) is carried out in the same way, yielding

$$\frac{du_{II}}{dz} = \beta_{II} \exp \frac{-z^2}{4\alpha_{II}} \tag{21}$$

and

$$\int_C^{T_0} du_{II} = \beta_{II} \int_{z(t)}^{\infty} \exp \frac{-z^2}{4\alpha_{II}}\, dz \tag{22}$$

where $z(t) = \gamma$.

Equation (22) may be rewritten in the form

$$T_0 - C = \beta_{II} \int_0^{\infty} \exp \frac{-z^2}{4\alpha_{II}}\, dz - \beta_{II} \int_0^{\gamma} \exp \frac{-z^2}{4\alpha_{II}}\, dz. \tag{23}$$

The heat balance condition (15c) at the moving boundary has not yet been used.

$$-k_I\beta_I \exp \frac{-\gamma^2}{4\alpha_I} = \frac{\rho\Delta H_f}{2} - k_{II}\beta_{II} \exp \frac{-\gamma^2}{4\alpha_{II}}. \tag{24}$$

Equations 18, 23 and 24 are sufficient for obtaining β_1, β_2 and γ. From Eq. 18 we obtain

$$\beta_I = C\left[\int_0^{\gamma} \exp \frac{-z^2}{4\alpha_I}\, dz\right]^{-1} \tag{25}$$

and Eq. (23) yields

$$\beta_{II} = (T_0 - C)\left[\int_0^{\infty} \exp \frac{-z^2}{4\alpha_{II}}\, dz - \int_0^{\gamma} \exp \frac{-z^2}{4\alpha_{II}}\, dz\right]^{-1}. \tag{26}$$

Substitutions of (25) and (26) into (24) gives an equation involving γ only. This can be solved by trial and error for γ. Then β_1 and β_{II} are readily available from Eqs. 25 and 26.

Finally, the position of the moving boundary is given by Eq. 20 and the expression for the temperature is given for regions I and II, respectively, by

$$u_I = \beta_I \int_0^{z} \exp \frac{-z^2}{4\alpha_I}\, dz, \tag{27}$$

where $z \leq z(t) = \gamma$,

and,

$$u_{II} = T_0 - \beta_{II} \int_z^{\infty} \exp \frac{-z^2}{4\alpha_{II}}\, dz \tag{28}$$

with $z \geq z(t) = \gamma$.

Bibliography

(1) L. R. Ingersol, O. J. Zobel and A. C. Ingersol, *Heat conduction with engineering, geological, and other applications*, The University of Wisconsin Press, Madison, 1954, p. 290-296