GENERALIZED RAYLEIGH PROCESSES*

BY

K. S. MILLER

New York University and Electronics Research Laboratories

AND

R. I. BERNSTEIN AND L. E. BLUMENSON

Columbia University Electronics Research Laboratories

1. Introduction. Properties of Gaussian processes must be investigated in many problems involving the analysis of random noise. Almost as common as the normal distribution is the Rayleigh distribution which occurs in work on radar, the detection of signals in the presence of noise, properties of a sine wave plus noise, etc. [1, 2, 3, 4]. It is the purpose of this paper to investigate certain properties of generalized Rayleigh processes.

We shall use the notation $\mathfrak{X}[a, \psi(\tau)]$ to indicate that $\mathfrak{X}$ is a Gaussian process with mean a and autocorrelation function $\psi(\tau)$. [That is, $\psi(\tau) = \langle [X(t) - a]\,[X(t+\tau) - a\rangle]$, where $X(t)$ is a member of $\mathfrak{X}$.] A Rayleigh process, $\mathfrak{R}$, may then be defined as

$$\mathfrak{R}^2 = \mathfrak{X}^2[a, \psi(\tau)] + \mathfrak{Y}^2[b, \psi(\tau)], \tag{1.1}$$

where $\mathfrak{X}$ and $\mathfrak{Y}$ are independent Gaussian processes. Certain classical results that are associated with $\mathfrak{R}$ may be found in the above quoted references. For example,

(a) The first order probability distribution of $\mathfrak{R}$:

$$p(R) = \frac{R}{\psi_0} \exp\,[-(R^2 + A^2)/2\psi_0] I_0\left(\frac{RA}{\psi_0}\right), \tag{1.2}$$

where $\psi_0 \equiv \psi(0)$, $A^2 = a^2 + b^2$ and I_0 is the Bessel function of the first kind and order zero with purely imaginary argument.

(b) The second order distribution of $\mathfrak{R}$:

$$p(R_1\,, R_2) = \frac{R_1R_2}{\psi_0^2(1-\lambda^2)} \exp\left[-\frac{R_1^2 + R_2^2}{2\psi_0(1-\lambda^2)}\right] I_0\left(\frac{\lambda R_1R_2}{\psi_0(1-\lambda^2)}\right), \tag{1.3}$$

where $\lambda = \psi(\tau)/\psi_0$ is the normalized autocorrelation function and we have assumed $a = 0 = b$.

(c) The correlation function of $\mathfrak{R}$:

$$C(\tau) = \psi_0[2E(\lambda) - (1 - \lambda^2)K(\lambda)], \tag{1.4}$$

where $K(\lambda)$ and $E(\lambda)$ are the complete elliptic integrals of the first and second kind respectively.

In the present paper we wish to consider generalized Rayleigh processes which we define as follows: Let $\mathfrak{X}_1[a_1\,, \psi(\tau)]$, $\mathfrak{X}_2[a_2\,, \psi(\tau)]$, $\cdots$, $\mathfrak{X}_N[a_N\,, \psi(\tau)]$ be N independent Gaussian processes. Then we define the generalized Rayleigh process, $\mathfrak{R}$, as

$$\mathfrak{R} = [\mathfrak{X}_1^2 + \mathfrak{X}_2^2 + \cdots + \mathfrak{X}_N^2]^{1/2}. \tag{1.5}$$

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The general problem is to compute the joint M-dimensional distribution of $\Re$. We have not been so fortunate. While it is of course possible to write down the distribution as a multiple integral, our objective has been to obtain compact usable formulas. Towards this end we have calculated:

(i) The first order distribution of $\Re$:

$$p(R) = \frac{A}{\psi_0}\left(\frac{R}{A}\right)^{N/2} \exp\left[-(R^2 + A^2)/2\psi_0\right] I_{(N-2)/2}\left(\frac{RA}{\psi_0}\right), \tag{1.6}$$

where $A^2 = a_1^2 + a_2^2 + \cdots + a_N^2$ and $I_\nu(x)$ is the Bessel function of the first kind and order ν with purely imaginary argument.

(ii) The joint probability density of $\Re$:

$$\begin{aligned} p(R_x, R_y) = {} & \frac{2\eta R_x R_y}{\psi_0} \frac{\Gamma(N/2)}{N-2}\left(\frac{2}{\lambda\omega^2\psi_0}\right)^{(N-2)/2} \exp\left\{-[(R_x^2 + R_y^2)\eta + A\omega]\right\} \\ & \times \sum_{k=(N/2)-1}^{\infty} 2k\binom{(N/2)+k-2}{N-3} I_k(2\lambda\eta R_x R_y) I_k(\omega R_x) I_k(\omega R_y), \end{aligned} \tag{1.7}$$

where

$$\omega = \frac{A}{\psi_0(1+\lambda)}, \qquad \eta = \frac{1}{2\psi_0(1-\lambda^2)} \tag{1.8}$$

and $\binom{\mu}{\nu}$ is the binomial coefficient.

(iii) The correlation function of $\Re$:

$$C(\tau) = \frac{2\psi_0(1-\lambda^2)^{(N+2)/2}\Gamma^2\left(\dfrac{N+1}{2}\right)}{\Gamma^2(N/2)} \cdot {}_2F_1\left(\frac{N+1}{2}, \frac{N+1}{2}, \frac{N}{2}, \lambda^2\right), \tag{1.9}$$

where ${}_2F_1(\alpha, \beta, \gamma, z)$ is the hypergeometric function.

(iv) The three dimensional distribution of $\Re$:

$$\begin{aligned} p(R_1, R_2, R_3) = {} & \frac{2(R_1R_2R_3) \mid M \mid^{N-3}}{(M_{12}M_{23}M_{31})^{(N-2)/2}} \exp\left[-\frac{M_{11}R_1^2 + M_{22}R_2^2 + M_{33}R_3^2}{2\mid M \mid}\right] \\ & \times \sum_{k=(N/2)-1}^{\infty} (-1)^{k+1-(N/2)} \frac{k}{N-2}\binom{(N/2)+k-2}{N-3} I_k(\alpha)I_k(\beta)I_k(\gamma), \end{aligned} \tag{1.10}$$

where

$$\alpha = \frac{R_1M_{12}R_2}{\mid M \mid}, \qquad \beta = \frac{R_2M_{23}R_3}{\mid M \mid}, \qquad \gamma = \frac{R_3M_{31}R_1}{\mid M \mid}, \tag{1.11}$$

$$M = \left\|\begin{matrix} \psi(0) & \psi(\tau) & \psi(\tau+\xi) \\ \psi(\tau) & \psi(0) & \psi(\xi) \\ \psi(\tau+\xi) & \psi(\xi) & \psi(0) \end{matrix}\right\| \tag{1.12}$$

is the correlation matrix, $\mid M \mid$ the determinant of M and M_{ij} the cofactors of M. We have assumed $A = 0$, that is, the unbiased case.

Other auxiliary equations, relations and special results will be pointed out as we develop the above formulas. Note that the results of (a), (b), and (c) are all subsumed under (i), (ii), and (iii) above. There are, of course, many other ways in which one could generalize Rayleigh processes. Some of the methods and results are described in the references [5, 6, 7,11].

The authors wish to thank the referee for pointing out certain techniques which resulted in simplifications in the proofs of some results and led them to generalizations of other theorems.

2. One dimensional biased Rayleigh. Our first problem is to find the fr.f. of R,

$$R = [X_1^2 + X_2^2 + \cdots + X_N^2]^{1/2}, \tag{2.1}$$

where $X_1, X_2, \cdots, X_N$ are independent random variables with the biased Gaussian distribution

$$p(X_i) = \frac{1}{(2\pi\psi_0)^{1/2}} \exp[-(X_i - a_i)^2/2\psi_0], \qquad i = 1, 2, \cdots, N. \tag{2.2}$$

Introduce an orthogonal set of coordinates in N-space with the vector $A = \{a_1, a_2, \cdots, a_N\}$ as the polar axis. If θ is the angle between R and A, then the components of R in a spherical coordinate system in N-space are

$$\begin{aligned} R_1 &= R\cos\theta \\ R_\alpha &= R\sin\theta \prod_{k=1}^{\alpha-2} \sin\phi_k \cos\phi_{\alpha-1}, \qquad \alpha = 2, 3, \cdots, N-1 \\ R_N &= R\sin\theta \prod_{k=1}^{N-2} \sin\phi_k . \end{aligned} \tag{2.3}$$

Unit vectors in the directions $X_1, X_2, \cdots, X_N$ form an orthogonal system in N-space which are related to $R_1, R_2, \cdots, R_N$ by an orthogonal transformation. Thus in particular

$$R^2 = \sum_{\alpha=1}^{N} R_\alpha^2 = \sum_{\alpha=1}^{N} X_\alpha^2 \tag{2.4}$$

and the Jacobian from $R_1, R_2, \cdots, R_N$ to $R, \theta, \phi_1, \cdots, \phi_{N-2}$ is identical with the Jacobian from $X_1, X_2, \cdots, X_N$ to R, θ, ϕ_k. Note that ϕ_{N-2} ranges from 0 to 2π while θ, and $\phi_1, \cdots, \phi_{N-3}$ range from 0 to π.

The fr.f. of R is given by the marginal distribution

$$p(R) = \int_0^{2\pi} d\phi_{N-2} \int_0^{\pi} \cdots \int_0^{\pi} Jp(X_1, \cdots, X_N)\, d\theta\, d\phi_1\, d\phi_2 \cdots d\phi_{N-3}, \tag{2.5}$$

where the Jacobian J is

$$J = R^{N-1} \sin^{N-2}\theta \prod_{\alpha=1}^{N-3} \sin^{N-2-\alpha}\phi_\alpha . \tag{2.6}$$

Since the X_i are independent,

$$p(X_1, \cdots, X_N) = \prod_{i=1}^{N} p(X_i) = \frac{1}{(2\pi\psi_0)^{N/2}} \exp\left[-\sum_{i=1}^{N} (X_i - a_i)^2/2\psi_0\right]. \tag{2.7}$$

But

$$\sum (X_i - a_i)^2 = \sum (X_i^2 - 2a_iX_i + a_i^2) = R^2 - 2A\cdot R + A^2, \tag{2.8}$$

where $A \cdot R$ is the inner product of the vectors A and R and

$$A \cdot R = AR \cos \theta. \tag{2.9}$$

Hence

$$p(R) = \frac{2\pi R^{N-1} \exp[-(R^2 + A^2)/2\psi_0]}{(2\pi\psi_0)^{N/2}} \int_0^\pi \sin^{N-2} \theta \exp(+ AR \cos \theta/\psi_0)\, d\theta \times \prod_{\alpha=1}^{N-3} \int_0^\pi \sin^{N-2-\alpha} \phi_\alpha \, d\phi_\alpha . \tag{2.10}$$

Now the first integral is

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)\left(\frac{2\psi_0}{AR}\right)^{(N-2)/2} I_{(N-2)/2}\left(\frac{AR}{\psi_0}\right)$$

and the product is

$$\prod_{\alpha=1}^{N-3} B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right) = \frac{\pi^{(N-3)/2}}{\Gamma([N-1]/2)}.$$

Using these results in Eq. (2.10) yields Eq. (1.6).

It is easily seen that as A approaches zero, that is, as $a_1, \cdots, a_N$ approach zero, $p(R)$ as given by Eq. (1.6) approaches

$$p(R) = \frac{2R^{N-1}}{(2\psi_0)^{N/2}\Gamma(N/2)} \exp(-R^2/2\psi_0) \tag{2.11}$$

which is the square root of the chi-squared distribution.

Note that if N is an *odd* integer, say $N = 2n + 3$, then Eq. (1.6) may be written in terms of elementary functions by use of the identity

$$I_{n+(1/2)}(x) = \frac{(2x)^{n+(1/2)}}{\pi^{1/2}} \frac{d^n}{d(x^2)^n}\left(\frac{\sinh x}{x}\right). \tag{2.12}$$

3. The two dimensional distribution of $\mathfrak{R}$. We shall establish a formula slightly more general than that given by Eq. (1.7). The proof is no longer nor more difficult than a direct derivation of this equation.

Let

$$R_x = [X_1^2 + X_2^2 + \cdots + X_N^2]^{1/2}, \qquad R_y = [Y_1^2 + Y_2^2 + \cdots + Y_N^2]^{1/2},$$

where the joint $2N$-dimensional distribution of $X_1, \cdots, X_N, Y_1, \cdots, Y_N$ is of the form $\prod_{i=1}^N p(X_i, Y_i)$ and

$$p(X_i, Y_i) = [2\pi(\det M)^{1/2}]^{-1} \exp\{-\tfrac{1}{2}[C_{11}(X_i - a_i)^2 + 2C_{12}(X_i - a_i)(Y_i - b_i) + C_{22}(Y_i - b_i)^2]\}.$$

The covariance matrix is

$$M = \begin{Vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{Vmatrix} = \begin{Vmatrix} \operatorname{Var} X_i & \operatorname{Cov}(X_i, Y_i) \\ \operatorname{Cov}(Y_i, X_i) & \operatorname{Var} Y_i \end{Vmatrix}$$

and $C = \| C_{ij} \| = M^{-1}$ is the inverse matrix.

We shall show that

$$p(R_x, R_y) = \Omega R_x R_y e^{-S/2} \sum_{k=(N/2)-1}^{\infty} k(-1)^{k-(N/2)+1} I_k(\omega_1 R_x) I_k(\omega_2 R_y) I_k(R_x C_{12} R_y)\, \delta_k, \tag{3.1}$$

where

$$\Omega = \frac{2^{(N-2)/2}\Gamma([N-2]/2)}{(\det M)^{N/2}(\omega_1 C_{12}\omega_2)^{(N-2)/2}},$$

$$S = C_{11}(R_x^2 + |A|^2) + C_{22}(R_y^2 + |B|^2) + 2C_{12}A\cdot B,$$

$$\omega_1 = |C_{11}A + C_{12}B|, \qquad \omega_2 = |C_{22}B + C_{12}A|,$$

$$\delta_k = C_{k-(N/2)+1}^{(N/2)-1}(\cos\alpha) = \text{Gegenbauer function.}$$

The quantities A and B are vectors with components $\{a_1, a_2, \cdots, a_N\}$ and $\{b_1, b_2, \cdots, b_N\}$ respectively while bars denote the magnitude of a vector; for example,

$$|C_{11}A + C_{12}B|^2 = \sum_{i=1}^{N} (C_{11}a_i + C_{12}b_i)^2.$$

We have let α be the angle between the vectors $C_{11}A + C_{12}B$ and $C_{22}B + C_{12}A$. Also, $C_n^\nu(t)$, the Gegenbauer function, is the coefficient of β^n in the expansion of $(1 - 2\beta t + \beta^2)^{-\nu}$ in ascending powers of β.

If we let $A = B$, $M_{11} = M_{22} = \psi_0$, $M_{12} = \lambda\psi_0$, then Eq. (3.1) reduces to Eq. (1.7). Note that in this case $\alpha = 0$ and

$$C_n^\nu(1) = \frac{\Gamma(2\nu + n)}{n!\Gamma(2\nu)}.$$

If, further, we let $A = 0$, then $p(R_x, R_y)$ assumes the elegant closed form

$$p(R_x, R_y) = \frac{(R_x R_y)^{N/2}}{\Gamma\left(\frac{N}{2}\right)\psi_0^{(N+2)/2}(2\lambda)^{(N-2)/2}(1-\lambda^2)} \cdot \exp\left[-\frac{R_x^2 + R_y^2}{2\psi_0(1-\lambda^2)}\right] I_{(N-2)/2}\left(\frac{\lambda R_x R_y}{\psi_0(1-\lambda^2)}\right). \tag{3.2}$$

One application of Eq. (3.2) is to a generalization of the second example in the paper by Barrett and Lampard [8]. This was pointed out to the authors in a private communication from Dr. Lampard.

We now turn to the evaluation of the density function of Eq. (3.1). To obtain $p(R_x, R_y)$ we must evaluate the integral

$$\begin{aligned} p(R_x, R_y) &= \int_{|Y|=R_y} dY \int_{|X|=R_x} dX \prod_{i=1}^{N} p(X_i, Y_i) \\ &= [2\pi(\det M)^{1/2}]^{-N} e^{-S/2} \int_{|Y|=R_y} \exp\,[(C_{12}A + C_{22}B)\cdot Y]\, dY \\ &\quad \cdot \int_{|X|=R_x} \exp\,[(C_{11}A + C_{12}B - C_{12}Y)\cdot X]\, dX, \end{aligned} \tag{3.3}$$

where X and Y are the N-dimensional vectors with components $\{X_1, X_2, \cdots, X_N\}$ and $\{Y_1, Y_2, \cdots, Y_N\}$ respectively. The notation $\int_{|X|=R_x}$ means: integrate over the surface of an N-dimensional sphere of radius R_x.

To evaluate the integral with respect to X we choose the direction of $Z = C_{11}A + C_{12}B - C_{12}Y$ as the polar axis, make an orthogonal transformation and then a general spherical coordinate transformation as in Sect. 2. We find that

$$\int_{|X|=R_x} e^{Z\cdot X}\, dX = (2\pi R_x)^{N/2} \mid Z \mid^{-(N/2)+1} I_{(N-2)/2}(\mid Z \mid R_x), \tag{3.4}$$

where $\mid Z \mid^2 = \omega_1^2 + C_{12}^2 R_y^2 - 2C_{12}(C_{11}A + C_{12}B)\cdot Y$. Thus $p(R_x, R_y)$ becomes the multiple integral

$$\begin{aligned} p(R_x, R_y) &= [2\pi R_x^{-1} \det M]^{-N/2} e^{-S/2} \\ &\quad \cdot \int_{|Y|=R_y} \exp\,[(C_{12}A + C_{22}B)\cdot Y] \mid Z \mid^{-(N/2)+1} I_{(N-2)/2}(\mid Z \mid R_x)\, dY. \end{aligned} \tag{3.5}$$

Now the vectors $C_{12}A + C_{22}B$, $C_{11}A + C_{12}B$ and Y span a three dimensional subspace of N-dimensional space. Choose the direction of $C_{11}A + C_{12}B$ as the z-axis and let the zx-plane be the plane spanned by $C_{11}A + C_{12}B$ and $C_{12}A + C_{22}B$. Let α be the angle between these two vectors, θ the angle between Y and $C_{11}A + C_{12}B$, ψ the angle between Y and $C_{12}A + C_{22}B$ and ϕ the angle between the projection of Y on the xy-plane and the x-axis. Then

$$\cos\psi = \cos\alpha \cos\theta + \sin\alpha \sin\theta \cos\phi. \tag{3.6}$$

Making an orthogonal transformation and then a general spherical coordinate transformation as before we have

$$\begin{aligned} p(R_x, R_y) &= \frac{2R_y^{N-1}}{\pi\Gamma([N-2]/2)} \left(\frac{R_x}{2\det M}\right)^{N/2} e^{-S/2} \int_0^\pi \exp\,(\omega_2 R_y \sin\alpha \sin\theta \cos\phi) \sin^{N-3}\phi\, d\phi \\ &\quad \times \int_0^\pi \exp\,(\omega_2 R_y \cos\alpha \cos\theta) \mid Z \mid^{-(N/2)+1} I_{(N-2)/2}(\mid Z \mid R_x) \sin^{N-2}\theta\, d\theta \\ &= e^{-S/2}\left(\frac{R_y}{2\pi}\right)^{1/2}\left(\frac{R_x R_y}{\det M}\right)^{N/2}\left(\frac{1}{\omega_2 \sin\alpha}\right)^{(N-3)/2} \\ &\quad \times \int_0^\pi \exp\,(\omega_2 R_y \cos\alpha\cos\theta) I_{(N-3)/2}(\omega_2 R_y \sin\alpha \sin\theta) \sin^{(N-1)/2}\theta \\ &\qquad\qquad \cdot \mid Z \mid^{-(N/2)+1} I_{(N-2)/2}(\mid Z \mid R_x)\, d\theta, \end{aligned} \tag{3.7}$$

where $\mid Z \mid^2 = \omega_1^2 + C_{12}^2 R_y^2 - 2C_{12}\omega_1 R_y \cos\theta$. Using the generalized Neumann addition theorem [10]

$$\begin{aligned} (R_x \mid Z \mid)^{-(N/2)+1} I_{(N-2)/2}(\mid Z \mid R_x) &= 2^{(N-2)/2}\Gamma\left(\frac{N-2}{2}\right) \sum_{k=(N/2)-1}^{\infty} k(-1)^{k-(N/2)+1} \\ &\quad \cdot (R_x\omega_1)^{-(N/2)+1} I_k(\omega_1 R_x)(R_x C_{12} R_y)^{-(N/2)+1} I_k(R_x C_{12} R_y) C_{k-(N/2)+1}^{(N/2)-1}(\cos\theta) \end{aligned} \tag{3.8}$$

and the formula [9]

$$\int_0^\pi \exp(y\cos\alpha\cos\theta) I_{\nu-1/2}(y\sin\alpha\sin\theta) C_n^\nu(\cos\theta)\sin^{\nu+1/2}\theta\, d\theta = \left(\frac{2\pi}{y}\right)^{1/2}\sin^{\nu-1/2}\alpha C_n^\nu(\cos\alpha) I_{\nu+n}(y) \tag{3.9}$$

the formula of Eq. (3.7) reduces to Eq. (3.1).

4. The correlation function of $\mathfrak{R}$. By definition, the correlation function $C(\tau)$ of $\mathfrak{R}$ is given by

$$C(\tau) = \int_0^\infty \int_0^\infty R_x R_y p(R_x, R_y)\, dR_x\, dR_y, \tag{4.1}$$

where the density function $p(R_x, R_y)$ is given by Eq. (3.2) and we recall that $p(R_x, R_y) = 0$ for $R_x, R_y < 0$. If we perform the integration,

$$\begin{aligned} C(\tau) &= \frac{2\psi_0(1-\lambda^2)^{(N+2)/2}}{\Gamma\left(\frac{N}{2}\right)} \sum_{p=0}^\infty \frac{\lambda^{2p}}{p!\Gamma\left(p+\frac{N}{2}\right)} \Gamma^2\left(p+\frac{N}{2}+\frac{1}{2}\right) \\ &= 2\psi_0(1-\lambda^2)^{(N+2)/2} \frac{\Gamma^2\left(\frac{N+1}{2}\right)}{\Gamma^2\left(\frac{N}{2}\right)}\, {}_2F_1\left(\frac{N+1}{2}, \frac{N+1}{2}, \frac{N}{2}, \lambda^2\right), \end{aligned} \tag{4.2}$$

where ${}_2F_1$ is the hypergeometric function.

If N is even, say $N = 2\nu$, then by use of well known identities [9] involving the hypergeometric function

$$C(\tau) = \frac{2\psi_0(1-\lambda^2)^{\nu+1}}{\Gamma(\nu)} \frac{d^{\nu-1}}{d(\lambda^2)^{\nu-1}} \left[\frac{E(\lambda)}{(1-\lambda^2)^2} - \frac{K(\lambda)}{2(1-\lambda^2)}\right]. \tag{4.3}$$

In particular, if $N = 2$, $\nu = 1$ and $C(\tau)$ reduces to the formula of Eq. (1.4). Since the derivatives of the elliptic integrals $K(\lambda)$ and $E(\lambda)$ are linear combinations of K and E with rational functions of λ as coefficients, we note that $C(\tau)$, for N even, can also be expressed as a linear combination of $K(\lambda)$ and $E(\lambda)$ with rational functions of λ as coefficients.

If N is odd, say $N = 2\nu + 1$, then

$$C(\tau) = \frac{2\psi_0(1-\lambda^2)^{\nu+(3/2)}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)} \frac{d^\nu}{d(\lambda^2)^\nu}\, {}_2F_1\left(1, 1, \frac{1}{2}, \lambda^2\right).$$

The identity

$${}_2F_1\left(1, 1, \frac{1}{2}, \lambda^2\right) = \frac{d}{d\lambda}\left[(1-\lambda^2)^{-1/2} \arctan \lambda(1-\lambda^2)^{-1/2}\right]$$

implies that in this case $C(\tau)$ is an elementary function of λ.

5. Three dimensional unbiased Rayleigh. We shall derive Eq. (1.10) for a more general covariance matrix than that given by Eq. (1.12).

Let

$$R_j = (X_{j1}^2 + X_{j2}^2 + \cdots + X_{jN}^2)^{1/2}, \qquad j = 1, 2, 3$$

where the joint $3N$-dimensional distribution of the X_{ij} is of the form $\prod_{i=1}^{N} p(X_{1i}, X_{2i}, X_{3i})$ and

$$p(X_{1i}, \quad X_{2i}, \quad X_{3i}) = [(2\pi)^3 (\det M)]^{-1/2} \exp(-\tfrac{1}{2}X_i'CX_i). \tag{5.1}$$

The covariance matrix is $M = \| M_{ij} \|$ and $C = \| C_{ij} \| = M^{-1}$ is its inverse. The column vector X_i has components $\{X_{1i}, X_{2i}, X_{3i}\}$ and X_i' is its transpose.

We shall prove by induction on N that

$$p_N(R_1, R_2, R_3) = \Omega \mid M \mid^{-N/2}(R_1R_2R_3) \exp\left[-\tfrac{1}{2} \sum_{j=1}^{3} C_{jj}R_j^2\right]$$
$$\times \sum_{k=(N/2)-1}^{\infty} v_k I_k(\alpha) I_k(\beta) I_k(\gamma), \tag{5.2}$$

where

$$\Omega = \frac{\Gamma([N-2]/2)}{(C_{12}C_{23}C_{31})^{(N-2)/2}\Gamma(N/2)}$$
$$v_k = k(-1)^{k-(N/2)+1}\binom{(N/2)+k-2}{N-3}$$

and α, β, γ are defined in Eq. (1.11).

The proof of Eq. (5.2) for $N = 2$ follows the pattern used in Sec. 3 and will be omitted. Of course one could also use the methods of that section to obtain Eq. (5.2) directly and thus entirely avoid the induction proof. However, the induction proof, in some respects, is neater and will perhaps suggest additional generalizations.

Let us therefore assume the validity of Eq. (5.2) for N. We shall prove it true for $N + 1$. Perhaps the easiest way of doing this is by the method of moment generating functions. Let $u_j = R_j^2$, $j = 1, 2, 3$ in $p_N(R_1, R_2, R_3)$. Then

$$p_N(u_1, u_2, u_3) = \frac{\Omega \mid M \mid^{-N/2}}{2^3} \exp\left[-\tfrac{1}{2} \sum_{j=1}^{3} C_{jj}u_j\right] \sum_{k=(N/2)-1}^{\infty} v_k I_k(\alpha') I_k(\beta') I_k(\gamma'), \tag{5.3}$$

where $\alpha' = u_1^{1/2}C_{12}u_2^{1/2}$, etc. Using the fact that the integral of $p_N(u_1, u_2, u_3)$ over the whole range is equal to one, that the C_{jj} appear only in the exponent in Eq. (5.3) in a linear fashion, and setting $C_{jj} = C'_{jj} - 2t_j$ we find, after integrating and dropping the primes that the moment generating function of $p_N(u_1, u_2, u_3)$ is

$$m_N(t_1, t_2, t_3) = \mid C \mid^{N/2} \mid C - 2T \mid^{-N/2}, \tag{5.4}$$

where

$$T = \begin{Vmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{Vmatrix}.$$

Equation (5.4) is valid if $t_j < \tfrac{1}{2}C_{jj}$.

Now let

$$S_j = R_j^2 + X_{j,N+1}^2 = u_j + X_{j,N+1}^2, \qquad j = 1, 2, 3$$

where $p_N(u_1, u_2, u_3)$ is given by Eq. (5.3) and $p(X_{1,N+1}, X_{2,N+1}, X_{3,N+1})$ is given by Eq. (5.1). In order to prove the induction we must show that the moment generating function of S_1, S_2, S_3 is equal to

$$m_{N+1}(t_1, t_2, t_3) = m_N(t_1, t_2, t_3) \mid C \mid^{1/2} \mid C - 2T \mid^{-1/2}.$$

Let $\phi(t_1, t_2, t_3)$ be the moment generating function of S_1, S_2, S_3. Then

$$\phi(t_1, t_2, t_3) = m_N(t_1, t_2, t_3) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[\sum_{i=1}^{3} t_i X_{i,N+1}^2\right]$$

$$\cdot p(X_{1,N+1}, X_{2,N+1}, X_{3,N+1})\, dX_{1,N+1}\, dX_{2,N+1}\, dX_{3,N+1}$$

$$= m_N(t_1, t_2, t_3) \mid C \mid^{1/2} \mid C - 2T \mid^{-1/2},$$

which proves the induction hypothesis. An identification of M with the matrix of Eq. (1.12) establishes Eq. (1.10).

Note the similarity in form of the first order biased distribution, Eq. (1.6), and the second order unbiased, Eq. (3.2), as well as the similarity of the second order biased, Eq. (1.7) and the third order unbiased, Eq. (1.10).

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