DETERMINATION OF UPPER AND LOWER BOUNDS FOR SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS*

By R. M. DURSTINE and D. H. SHAFFER (*Westinghouse Research Laboratories, Pittsburgh, Penn.*)

Analytic upper and lower bounds may be constructed on the solutions to linear differential systems of a certain kind, as described in this paper. The principal requirement of the method presented here is that two functions satisfying the boundary conditions placed on the system—and certain other conditions—must first be constructed. The bounds are then generated as linear combinations of these two functions. The development proceeds as follows.

Consider a linear differential equation

$$L(u) + \phi = 0$$

defined in a suitable region D with boundary B, and subject to the linear conditions

$$M_j(u) = \lambda_j \quad \text{on} \quad B, \qquad j = 1, 2, \cdots, q,$$

where L and M_j are homogeneous linear differential operators, and q is appropriate to the particular problem. Here ϕ is taken as a known piecewise continuous function in D. The functions λ_j are known on B.

Now select a function w which has the following properties:

1) w is piecewise continuous in the highest derivatives appearing in L;

2) $M_j(w) = \lambda_j$ on B, $j = 1, 2, \cdots, q$.

Then define ϵ as

$$L(w) + \phi = \epsilon.$$

If $v = u - w$, then

$$L(v) = -\epsilon \quad \text{in} \quad D$$

$$M_j(v) = 0 \quad \text{on} \quad B, \qquad j = 1, 2, \cdots, q.$$

It is assumed that the original system was such that the above problem has a unique solution, and furthermore possesses a Green's function which does not change sign at any point of D. We have then

$$v = \int_D G(x, \xi)\epsilon(\xi)\, d\xi,$$

where x and ξ represent points of the space in which we are working, regardless of dimension.

Let w_1 and w_2 be two functions, each satisfying the conditions previously imposed on w. Corresponding to these are two functions ϵ_1 and ϵ_2. It is required in addition that

1) ϵ_2/ϵ_1 be continuous in D;

2) ϵ_1 not change sign in D;

3) one of the following inequalities holds

$$1 > M \geq \epsilon_2/\epsilon_1 \geq m$$

or

$$M \geq \epsilon_2/\epsilon_1 \geq m > 1.$$

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We have then

$$\frac{u - w_2}{u - w_1} = \frac{\int_D G(x, \xi)\epsilon_2(\xi)\, d\xi}{\int_D G(x, \xi)\epsilon_1(\xi)\, d\xi}.$$

Application of a mean value theorem to the right hand side of this expression gives

$$\frac{\int_D G(x, \xi)\epsilon_2(\xi)\, d\xi}{\int_D G(x, \xi)\epsilon_1(\xi)\, d\xi} = \frac{\epsilon_2[\xi_1(x)]}{\epsilon_1[\xi_1(x)]},$$

where ξ_1 is an interior point of D, and clearly a function of x. For brevity we denote $\epsilon[\xi_1(x)]$ by $\delta(x)$. Thus we have

$$\frac{u - w_2}{u - w_1} = \frac{\delta_2}{\delta_1}.$$

Rearrangement of this formula gives

$$u = w_1 + (w_1 - w_2) \Big/ \left(\frac{\delta_2}{\delta_1} - 1\right).$$

If the exact forms of δ_1 and δ_2 were known, this would give the solution to the original problem. However, even though these forms are not known, the ratio is, by hypothesis, bounded.

Since ϵ_2/ϵ_1 is bounded away from unity in D it follows that $w_1 - w_2$ is always of one sign in D, as will be determined by the sign of G. So the second term in the above expression for u may be bounded in D by replacing the denominator by its minimum and maximum values. Define u_1 and u_2 by

$$u_1 = w_1 + (w_1 - w_2)/(M - 1)$$

$$u_2 = w_1 + (w_1 - w_2)/(m - 1).$$

The true value of u must lie between u_1 and u_2.

The "efficiency" of these bounds may be defined as the spread between them, given by

$$S = | w_1 - w_2 | \left[\frac{M - m}{(m - 1)(M - 1)}\right].$$

This formula shows the curious fact that very good bounds may be obtained from functions w_1 and w_2 which are themselves poor approximations to u.

As one example of this method consider the Bessel equation

$$u'' + u'/x + 4u = 0$$

in $0 < x < 1$, subject to $u(0) = 1$. We use the $0 < x < 1$ range to satisfy the conditions imposed on the Green's function. The solution to this problem is $u = J_0(2x)$. As approximating functions we take

$$w_1 = 1$$

$$w_2 = 1 - 1.1167x^2 + .3570x^3.$$

Associated with these are

$$\epsilon_1 = 4$$
$$\epsilon_2 = -.467 + 3.213x - 4.467x^2 + 1.428^3.$$

From these we compute

$$m = -.11675$$
$$M = .05125$$

and hence

$$1.054w_2 - .0540 \leq J_0(2x) \leq .8955w_2 + .1045$$

which produces a rather accurate pair of bounds, particularly near the origin. The function w_2 was of course itself a good approximation to $J_0(2x)$, chosen by another approximation scheme.

As another example consider

$$\Delta\psi = -2$$

in the region interior to $x^2/a^2 + y^2/b^2 = 1$ subject to $\psi = 0$ on the boundary. Choose

$$w_1 = 1 - x^2/a^2 - y^2/b^2$$
$$w_2 = 0.$$

The conditions requisite for the application of this method are fulfilled and we have

$$\epsilon_1 = 2 - 2/a^2 - 2/b^2$$
$$\epsilon_2 = 2.$$

Since both errors are constant, it follows that the precise form of δ_2/δ_1 is known. Hence instead of bounds we obtain the exact solution

$$\psi = (a^2b^2 - b^2x^2 - a^2y^2)/(a^2 + b^2).$$

This is the classical problem of the torsion of an elliptic cylinder.

It should be noted in passing that the results presented here are obtainable under the weaker assumption of only piecewise continuity for ϵ_2/ϵ_1.