Hence, if $P_\alpha^{(i)} = 0$ on each boundary, J vanishes for all admissible choices of $\tau_{\alpha\beta}$. Hence, by the converse theorem mentioned above, $\epsilon'_{\alpha\beta}$ is derivable from a single valued displacement. The same is then necessarily true of $\epsilon^*_{\alpha\beta} = \epsilon_{\alpha\beta} + \epsilon'_{\alpha\beta}$, and hence $\sigma_{\alpha\beta}$ remains a solution for the stress when Poisson's ratio is changed. On the other hand, if $P_\alpha^{(i)}$ does not vanish on some boundaries, a suitable choice of $\tau_{\alpha\beta}$ can always be made to render J non-zero. But this would necessarily imply that the strains $\epsilon'_{\alpha\beta}$ (and hence $\epsilon^*_{\alpha\beta}$) are *not* derivable from a single valued displacement, whence $\sigma_{\alpha\beta}$ would certainly not constitute a solution for the new material.

The present theorem can be useful in simplifying the initial formulation of some problems. For example, the choice $\nu = \frac{1}{2}$ in conjunction with a total stress-strain law of plasticity permits the use of a single formula for the sum of the elastic and plastic components of strain; in other problems, the choice $\nu = 0$ might be more appropriate. In addition, the present theorem may conceivably have significance in connection with photoplasticity.

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ON ISOPERIMETRIC INEQUALITIES IN PLASTICITY*

By WALTER SCHUMANN (*Brown University*)

Abstract. The purpose of this paper is the proof of the inequality $P \geq 6\pi M_0$, where P is the total limit load, M_0 the yield moment of a thin, perfectly plastic, simply supported, uniformly loaded plate of arbitrary shape and connection.

Introduction. The theory of thin, rigid-perfectly plastic plates, given by Hopkins and Prager [1]** has been applied to circular plates with various load and edge conditions. However, if one tries to extend this theory to non-symmetrical cases, serious difficulties arise in seeking examples of exact solutions, although some cases have been solved (see for instance [2]). As a contribution to the estimation of the limit load in an arbitrary plate we shall use here the isoperimetric inequality, which relates a circular domain to an arbitrary domain in a convenient manner. One of the principal theorems of limit analysis [3] and the methods for isoperimetric problems given in Polya's and Szegö's book [4] will be used. Similar problems have been proposed and solved for other physical quantities, as for example the torsional rigidity, the principal frequency, etc.

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**Numbers in square brackets refer to the bibliography at the end of the paper.

2. A lower bound for the limit load of a simply supported plate of arbitrary shape under uniform load. Let us consider a simply supported plate of a rigid-perfectly plastic material obeying Tresca's yield condition (Fig. 1), and let p be the limit load

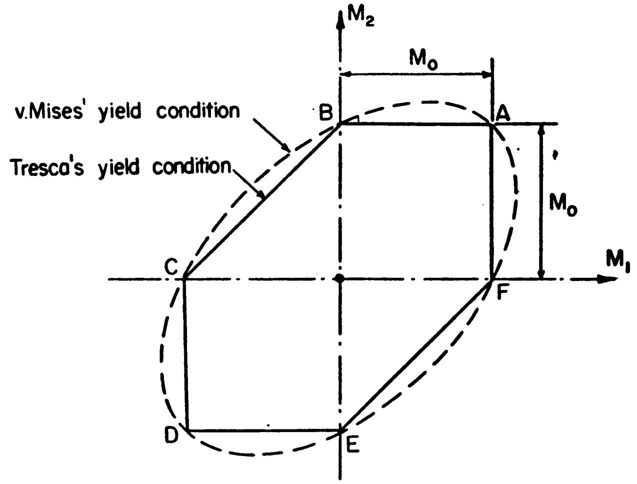

FIG. 1.

per unit area, which is assumed to be constant, A the area of the domain G (Fig. 2), $P = pA$ the total limit load and M_0 the yield moment. Then the following inequality is true:

$$P \geq 6\pi M_0 , \tag{1}$$

where the equality sign holds *only* in the case of the circular plate.

To prove this statement, we consider, in addition to the domain G, a circular domain G' of equal area A (Fig. 2), and we map the actual velocity field v of G into a new field v' over G' in a certain way, that will be defined later. All further quantities for the new

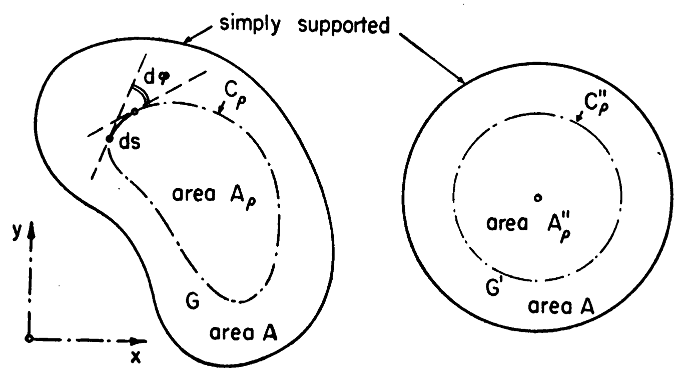

FIG. 2. Schwarz' symmetrization: $A_p = A_p''$.

field will be distinguished by primes from the corresponding quantities of the original field. Denoting by D the dissipation function per unit area, we have

$$\iint_G pv \, dA = \iint_G D \, dA. \tag{2}$$

Using the second theorem of limit analysis [3] and assuming that v' is kinematically admissible for G', we have

$$\iint_{G'} p'v' \, dA' \le \iint_{G'} D' \, dA'. \tag{3}$$

Suppose for the moment $v \ge 0$, and let D_{tot} be the total rate of dissipation, V the volume between the plane of the plate and the surface into which the plate deforms; then (2) and (3) give

$$P = \frac{D_{\text{tot}}}{V} A, \qquad P' \le \frac{D'_{\text{tot}}}{V'} A. \tag{4}$$

To get the inequality (1), we look whether there exists a mapping $v \to v'$ such that

$$\frac{D_{\text{tot}}}{V} \ge \frac{D'_{\text{tot}}}{V'}. \tag{5}$$

For this purpose let us first investigate the dissipation function D, which is (see [6], p. 50)

$$D = \frac{M_0}{2} (|\kappa_1| + |\kappa_2| + |\kappa_1 + \kappa_2|), \tag{6}$$

where κ_1 and κ_2 are the principal rates of curvature associated with the velocity field v. For domains of (positive) elliptic and parabolic curvature corresponding to the regimes A, AB and AF of the yield hexagon we have

$$D = M_0(\kappa_1 + \kappa_2) = 2M_0 H = -M_0 \nabla^2 v, \tag{7}$$

where H is the rate of the mean curvature, and v is counted positive when directed downwards. On the other hand we may write for all regimes

$$D \ge M_0 \max |\kappa_i|, \qquad D \ge 2M_0 H. \tag{8}$$

We shall later use the fact that (8) is valid at every point of the field v, even if hinges occur. A hinge line may be considered as a narrow strip, where one of the rates of curvature is very large and the other finite.

Finally, we note from (8) and Green's formula, that $v \ge 0$ everywhere, since a domain with $v < 0$ can be removed by $v^* = 0$, thus diminishing $D_{\text{tot}}/\iint v dA$, so that $v < 0$ cannot be the actual field.

Denote now by C_ρ the contour line $v = \rho$ of the surface $v = v(x, y)$ (Fig. 2). We note that C_ρ may consist of several branches. Let κ_t be the rate of curvature tangential to the contour line C_ρ, κ the curvature of the contour line in its plane, $d\varphi = \kappa \, ds$ the increment of the angle of the tangent at C_ρ, when a point of v moves an increment ds on C_ρ, and finally let $\partial/\partial n$ denote differentiation normal to C_ρ into its "interior", i.e. the direction of increasing ρ. We integrate the dissipation function D over an infinitestimal strip between C_ρ and $C_{\rho+d\rho}$, which gives, by using (8),

$$\begin{aligned} dD_{\text{tot}} &\ge M_0 \oint_{C_\rho} \max |\kappa_i| \, dn \, ds \ge M_0 \oint_{C_\rho} |\kappa_t| \, dn \, ds \\ &= M_0 \oint_{C_\rho} |\kappa| \frac{\partial v}{\partial n} dn \, ds \ge M_0 \int_0^{2\pi} d\rho \, d\varphi = 2\pi M_0 \, d\rho. \end{aligned} \tag{9}$$

The equality sign in the third inequality of (9) holds, when C_ρ is convex and consists only of one branch.

After these preparations we now specify the mapping of v in two steps as follows: $v(G) \rightarrow v''(G')$, $v''(G') \rightarrow v'(G')$. The first transformation is the so-called *Schwarz' symmetrization* (see [4], p. 190 or [5]). Let C_ρ'' be the contour line of v'', which corresponds to C_ρ (same height ρ), and let A_ρ and A_ρ'' be the areas, which C_ρ and C_ρ'' "surround" (increasing ρ). Schwarz' symmetrization is then defined as follows

I. C_ρ'' is a circle, concentric in G'.

II. The areas A_ρ and A_ρ'' are equal.

It is easy to see, that Schwarz' symmetrization does not change the volume V. However D_{tot} might be, at least in certain cases, increased. We introduce therefore a further transformation. The velocity field v'' consists, because of the rotational symmetry, of several ring-shaped circular zones of elliptic, parabolic and hyperbolic rate of curvature (Fig. 3). We replace the body between v'' and the plane of G' by its *convex hull* (surface v' indicated by the dotted lines in Fig. 3), which has only elliptic and parabolic curvature.

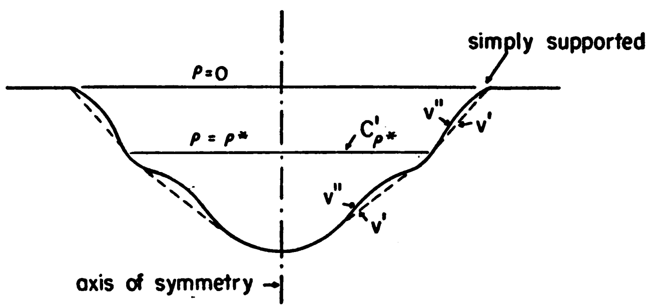

FIG. 3.

Let C'_{ρ^*} be the smallest contour circle, such that outside C'_{ρ^*} no elliptic curvature exists. The transformation has then the following properties

a) The volume is increased: $V' \geq V''$.

b) C'_{ρ^*} is on v' *and* on v'';

$$A'_{\rho^*} = A''_{\rho^*} = A_{\rho^*} .$$

c) $$\frac{\partial v'}{\partial n'} = \frac{\partial v''}{\partial n''}, \quad \text{for} \quad \rho = \rho^*;$$

$$\frac{dA'_\rho}{d\rho} = \frac{dA''_\rho}{d\rho} = \frac{dA_\rho}{d\rho}, \quad \text{for} \quad \rho = \rho^*.$$

As outside C'_{ρ^*}, $D' = M_0\kappa'_t$, we shall have the equality signs in (9) there. As on the other hand the inequalities (9) are valid everywhere in G, we obtain

$$D_{tot} \geq D'_{tot}, \quad \text{for} \quad \rho < \rho^*. \tag{10}$$

"Inside" C_{ρ^*}, the second inequality (8) can be applied, which gives

$$D_{tot/\rho \geq \rho^*} \geq -M_0 \iint_{\rho \geq \rho^*} \nabla^2 v \, dA = M_0 \oint_{C_{\rho^*}} \frac{\partial v}{\partial n} ds, \tag{11}$$

and inside C'_{ρ^*} we may apply Eq. (7) (regimes A, AB)

$$D'_{tot/\rho\geq\rho^*} = -M_0 \iint_{\rho\geq\rho^*} \nabla^2 v'\, dA' = M_0 \oint_{C'_{\rho^*}} \frac{\partial v'}{\partial n'}\, ds'. \tag{12}$$

As $\partial v/\partial n \geq 0$, $\partial v'/\partial n' \geq 0$, per definition of the contour lines, there exists an important inequality between the last terms in (11) and (12), which is actually the key point of the proof. It follows namely from *Schwarz' inequality* for $\partial v/\partial n$ and $(\partial v/\partial n)^{-1}$ and from the *isoperimetric inequality* $4\pi A_{\rho^*} \leq l^2_{\rho^*}$, where l_{ρ^*} is the length of C_{ρ^*} (see [4], p. 234)

$$\oint_{C_{\rho^*}} \frac{\partial v}{\partial n}\, ds \geq \frac{4\pi A_{\rho^*}}{\left|\dfrac{dA_\rho}{d\rho}\right|_{\rho=\rho^*}} = \oint_{C'_{\rho^*}} \frac{\partial v'}{\partial n'}\, ds'. \tag{13}$$

Therefore we obtain

$$D_{tot} \geq D'_{tot}, \quad \text{for} \quad \rho \geq \rho^*. \tag{14}$$

From (14) and (10) we conclude, that D_{tot}/V, and also P, are not increased by the mapping $G \rightarrow G'$, and as $6\pi M_0$ is actually the limit load of the circular plate (see [6], p. 55), inequality (1) is proved.

It remains to show that the equality sign in (1) is valid only in the case of the circular plate. The equality sign in (13) holds only when C_{ρ^*} is a circle, and when $\partial v/\partial n$ is constant. The equality signs in the two last inequalities of (9) hold, when every contour line outside C_{ρ^*} consists of one convex branch and is a line of principal curvature, which means that $\partial^2 v/\partial n \partial s = 0$. As one of them, namely C_{ρ^*}, is circular, they must all be circular; therefore, the edge is a circle.

3. v. Mises' yield condition. If one takes v. Mises' yield criterion instead of Tresca's condition, the limit load is not diminished, because the ellipse surrounds the hexagon [7] (Fig. 1). Therefore the inequality (1) remains true

$$P \text{ (v. Mises)} \geq 6\pi M_0 . \tag{15}$$

However the inequality is not isoperimetric.

4. Minimum weight design of a sandwich plate. As there is a certain duality between analysis and design problems [8], we expect also an isoperimetric inequality in the latter case. However the result is less useful, because a bound for the minimum volume, for example, does not help in finding the actual design. Nevertheless let us look at a sandwich plate of variable thickness h of the sheets, but constant thickness H_0 of the core, with a homogeneous material obeying Tresca's yield criterion. The yield moment is given by

$$M_0 = \sigma_0 h H_0 , \tag{16}$$

where σ_0 is the yield stress. Looking for a statically admissible stress field with regime A of the hexagon (Fig. 1), Prager [9] has shown that h must satisfy the equation

$$\nabla^2 h = -\frac{p}{\sigma_0 H_0}, \tag{17}$$

and he mentioned the analogy between this problem and that of the membrane. The volume of the sheets

$$V_{\text{statically admissible}} = 2 \iint_G h\, dA \tag{18}$$

corresponds to the torsional rigidity, if we take the torsion analogy instead of the membrane analogy. For the torsional rigidity the isoperimetric inequality has been proved long ago; thus we may write, by using the first theorem of limit analysis,

$$V \le V_{\text{statically admissible}} \le V_{\text{circle}} = \frac{pA^2}{4\pi\sigma_0 H_0}. \tag{19}$$

5. Steiner's symmetrization. In the case of a very long but narrow domain G, the isoperimetric inequality gives a very bad bound. Steiner's symmetrization (Fig. 4, see also [10]) does not change G so much as Schwarz' symmetrization, if the axis is chosen conveniently. Steiner's symmetrization, which increases the torsional rigidity, therefore

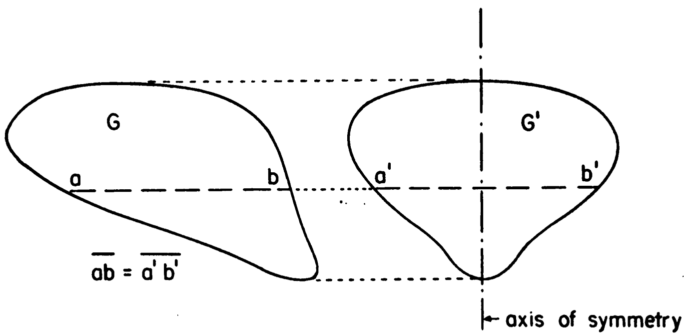

FIG. 4. Steiner's symmetrization.

increases also the volume of a design given in (18), and probably diminishes (or leaves constant) the limit load of a simply supported, uniformly loaded plate. Unfortunately, we are not able to prove the last of these two statements, which would be useful.

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