# THE THICKNESS OF CYLINDRICAL SHOCKS AND THE PLK METHOD*

BY

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**Summary.** Cylindrical shocks occur when a viscous heat-conducting gas flows radially in a plane. This is a singular perturbation problem in which the perturbing parameter is the reciprocal of $R_e$, the Reynolds number of the flow. It is shown that for both inward facing shocks (source flow) and outward facing shocks (sink flow) the shock thickness is of order $R_e^{-1}S^{-1} \log (R_e S^3)$ where $S$ is the shock strength. This is contrary to results for sink flow which have been obtained by the use of Lighthill's technique for rendering approximations uniformly valid—the *PLK* method. It is shown that this method fails when applied to singular perturbation problems of the type discussed here in which the small parameter multiplies the highest derivatives.

**Introduction.** The steady radial two dimensional flow of a viscous heat conducting gas has attracted some attention in recent years [1, 2, 3]. Perhaps the main interest has followed from the fact that if an inviscid gas with zero thermal conductivity existed, it could only flow in such a manner exterior to the sonic circle which is a limiting line of the flow. Outside the sonic circle the speed could either be supersonic everywhere to infinity where the vacuum speed is attained or subsonic everywhere to infinity where the speed is zero. It is apparent that this is a singular perturbation problem because the flow of a real gas in the limit of vanishing viscosity and thermal conductivity should not have this character.

Since there is only one independent variable for this simple configuration a treatment of the problem is feasible. A Reynolds number $R_e$ may be defined in terms of the mass flow and the viscosity, and the solutions sought when $R_e$ is large and the Prandtl number, $\sigma$, is finite.

It has been shown by the writer [2] that when the gas flows outwards, source flow, the limit of the solutions as $R_e \to \infty$ exhibit no ambiguities. Although there is a singular stagnation circle, the "source", which falls outside the scope of the Navier-Stokes equations, the important feature is that a typical solution curve follows the supersonic branch of the inviscid solution outwards from the sonic circle for some distance (dependent on external boundary conditions), then crosses via an inward facing cylindrical shock to the subsonic branch of the inviscid solution to complete its journey to infinity. If $R_e$ is large but finite the shock is not a discontinuity and has a thickness of order $R_e^{-1}S^{-1} \log (R_e S^3)$, where $S$ is the shock strength.

More recently Wu [3] has investigated the case of inward flow, sink flow, and finds an analogous situation. In the limit $R_e \to \infty$, a typical solution curve follows the supersonic branch of the inviscid solution from infinity in for some distance, then crosses via an outward facing cylindrical shock to the subsonic branch of the inviscid solution. The continuation beyond the sonic circle exhibits a singular vacuum circle, the "sink."

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However he finds that for finite $R_e$ an outward facing shock can only be of strength of order $R_e^{-1/3}$ and have a thickness of order $R_e^{-2/3}$.

Although it would be expected that the results should agree, a proof seems to demand a discussion of the whole flow field. It is for this reason that the difference is important, for instead of the topological methods used by the writer, Wu has employed an analytical method, the PLK method, to investigate the flow field, and this approach has been endorsed by Tsien [4]. Hence the procedure which will be adopted is to show that these results are wrong in general, Sec. 2, and to indicate the reasons why invalid results were obtained, Sec. 3.

The governing equations for the flow are found in suitable form in Sec. 1 and after a brief consideration of the singularities for the general case, the discussion centres on the iso-energetic flows, Sec. 2, for a particular value of $\sigma$ close to $\frac{3}{4}$. For this case a topological discussion of the governing equations is possible, and as has been pointed out by the writer [2] variations in $\sigma$ will not affect the nature of the results. After a brief review of source flow, entirely analogous results are derived for sink flow. In particular it is shown that, generally, the outward facing cylindrical shocks which occur have a thickness, $\Delta$, of order $R_e^{-1}S^{-1} \log (R_e S^3)$, while, in their interior, the maximum non-dimensional velocity gradient, $V$, is of order $R_e S^2$. It is only for the very weak shocks for which $S$ is of order $R_e^{-1/3}$ that Wu's results, $\Delta = 0(R_e^{-2/3})$, $V = 0(R_e^{-/3})$, hold, and these are still covered by the general result. An interesting sidelight here is that the *weakest* cylindrical shocks which can occur are of strength of order $R_e^{-1/3}$.

Now Wu's results are obtained from similarity solutions valid for speeds near sonic, but the essential feature is that the PLK method is used to show that *all* solution curves join on to these similarity solutions. Hence it is *his application* of this method, the Lighthill "technique for rendering approximations uniformly valid" [5], which comes under fire in Sec. 3.

In Lighthill's expository paper this technique was applied to singular perturbation problems which arise because singularities are wrongly placed by the zeroth perturbation. There is no reason *a priori* to expect that it will be of any use for the different class of perturbation problems which are singular essentially because the highest order derivatives are dropped in the equations governing the zeroth perturbation, as in our case. The effect of the highest derivative is felt almost everywhere in this problem, and neither a conventional perturbation expansion nor a Lighthill perturbation expansion, which still loses the highest derivative in each perturbation, leads to valid solutions. This is brought out clearly in Sec. 3. A critical examination of the application of the PLK method to the sink flow problem is followed by an example of its application to a simpler equation of the same broad type, for which the explicit solution is known. The manner in which it fails to provide any improvement to a conventional perturbation expansion is then easily seen.

The moral, of course, is that it is useless to expect the PLK method to be a panacea for all singular perturbation troubles. Their nature must be assessed as suitable for the treatment first, and it is possible that some other applications of the method should be critically reexamined.

**1. The fundamental equations.** For purely radial two dimensional flow of a fluid the speed, $u$, is only a function of $r$, the radial distance from the origin. Let $p$, $\rho$, $T$, $\mu$, $\sigma$, $R$, $c_p$ and $a$ denote respectively the pressure, density, temperature, viscosity, Prandtl number, gas constant, specific heat at constant pressure and local speed of sound. Then the equations for the conservation of mass, momentum and energy take the form [2]

$$\rho u r = \kappa, \quad \text{(a constant)}, \tag{1.1}$$

$$\rho u \frac{du}{dr} + \frac{dp}{dr} = \frac{4}{3}\frac{d}{dr}\left(\mu \frac{du}{dr}\right) + \frac{2\mu}{r}\frac{du}{dr} - \frac{2\mu u}{r^2} - \frac{2}{3}\frac{d}{dr}\left(\frac{\mu u}{r}\right), \tag{1.2}$$

and

$$\kappa\left(c_p T + \frac{1}{2}u^2\right) + \frac{2}{3}\mu u\left(r\frac{du}{dr} + u\right) - 2\mu u r\frac{du}{dr} - \frac{c_p \mu r}{\sigma}\frac{dT}{dr} = \kappa I_0 . \tag{1.3}$$

In addition, if the fluid is an ideal gas the thermodynamic variables are related by the equation of state

$$p = R\rho T. \tag{1.4}$$

By analogy with inviscid flow the constant $I_0$ is essentially positive while $\kappa$ is positive (negative) for source (sink) flow.

In terms of the non-dimensional variables

$$w = |u|(2\beta I_0)^{-1/2}, \qquad \theta = c_p T/I_0, \qquad \alpha = 4\mu(1+\beta)/(3\kappa), \tag{1.5}$$

with

$$\beta = (\gamma - 1)/(\gamma + 1) = R/(2c_p - R), \tag{1.6}$$

elimination of $p$ and $\rho$ with the aid of (1.1) and (1.4) yields the two equations

$$\frac{1+\beta}{r}\frac{dw}{dr} + \frac{d}{dr}\left(\frac{\theta}{wr}\right) = \alpha\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} - \frac{w}{r^2}\right) + \left(\frac{dw}{dr} - \frac{w}{2r}\right)\frac{d\alpha}{dr}, \tag{1.7}$$

and

$$\theta + \beta\left(1 + \frac{\alpha}{\beta + 1}\right)w^2 - \frac{3\alpha r}{4(1+\beta)\sigma}\frac{d\theta}{dr} - \frac{2\alpha\beta w r}{1+\beta}\frac{dw}{dr} = 1. \tag{1.8}$$

We may regard $|\alpha|$ as the reciprocal of the Reynolds number of the flow.

If $\alpha$ is placed equal to zero, these equations together yield the inviscid approximation

$$-r(dw/dr) = w(1 - \beta w^2)(1 - w^2)^{-1} \tag{1.9}$$

in which $w = 1$ is the sonic speed and $w = \beta^{-1/2}$ is the vacuum speed. The solution of this equation is

$$(r/r_s) = w^{-1}[(1 - \beta w^2)/(1 - \beta)]^{-(1-\beta)/(2\beta)} \tag{1.10}$$

where $r_s$ is the radius of the sonic circle, and shows that external to the sonic circle the field is doubly covered, so that at each point the speed may be either subsonic or supersonic, while inside the sonic circle there is no solution. When $r \to \infty$ either $w \to 0$ or $w \to \beta^{-1/2}$ (see Fig. 4). Hereafter we shall suppose $r$ to be rendered non-dimensional in such a manner that the radius of the sonic circle is unity in this corresponding inviscid flow—this does not alter the form of the equations (1.7) and (1.8).

For simplicity, we shall only consider the flows for which $\alpha$ is constant. This implies in particular that the viscosity is independent of the temperature but previous results [2] suggest that this will not affect the general nature of the conclusions to be drawn. In this case, if $\sigma$ takes the particular value $\sigma_0$ where

$$4\sigma_0 = 3(1 + \alpha/(\beta + 1)], \tag{1.11}$$

it is possible to obtain an energy integral. For if we define

$$E = \theta - 1 + \beta[1 + \alpha/(\beta + 1)]w^2, \tag{1.12}$$

so that $1 + E$ is essentially the total energy of the flow, Eq. (1.8) yields

$$E = cr^{4(1+\beta)\sigma_0/(3\alpha)}, \tag{1.13}$$

where $C$ is an arbitrary constant.

Now, consider the equations obtained in terms of the new variable

$$V = -r\, dw/dr, \tag{1.14}$$

so that $V$ is essentially a velocity gradient. They are

$$\alpha w^2 V(dV/dw) = \alpha w^3 - (1 + \beta)w^2 V - wV(d\theta/dw) + V\theta - w\theta \tag{1.15}$$

and

$$(1 + \beta)(\theta - 1) + \beta(1 + \beta + \alpha)w^2 + 3\alpha(4\sigma)^{-1} V(d\theta/dw) + 2\alpha\beta wV = 0. \tag{1.16}$$

It may be shown that the singular points of these equations are at $w^2 = w_c^2 = [\beta + \alpha(1 + 2\beta)/(1 + \beta)]^{-1}$ where $\theta = \theta_c = \alpha w_c^2$ and $V = 0$, and at $w = 0$ where $\theta = 1$ and $V = 0$.

In the $w$-$V$ plane the point $w = 0$, $V = 0$ is a double point*, and the possible gradients there are $(dV/dw) = \infty$ and $(dV/dw) = 1$. For both $\alpha > 0$ (source flow) and $\alpha < 0$ (sink flow), there is only one solution curve of infinite slope through the origin, namely $w \equiv 0$. For source flow there is an infinite number of solution curves with $(dV/dw) = 1$ through the origin, but there is only one solution curve through the origin with this slope for $\alpha < 0$.

The singular point at $w = w_c \doteqdot \beta^{-1/2}$ (the vacuum speed), $V = 0$, is a triple point in general in the sense used above. For the case of source flow, there is one curve through this point with slope $(dV/dw) = \lambda_1$, on which $(d\theta/dw) = \mu_1$, corresponding to $r \to \infty$ (this is the physically interesting one and corresponds to one branch of the inviscid solution) while corresponding to $r \to 0$ there are curves described by

$$\left.\begin{aligned} V &\sim \lambda_3(w - w_c) + C\,|\,w - w_c\,|^{\lambda_3/\lambda_2}, \\ \theta - \theta_c &\sim \mu_3(w - w_c) + C[(\mu_2 - \mu_3)/(\lambda_2 - \lambda_3)]\,|\,w - w_c\,|^{\lambda_3/\lambda_2}, \end{aligned}\right\} \tag{1.17}$$

and if $C \neq 0$,

$$V \sim \lambda_2(w - w_c), \qquad \theta - \theta_c \sim \mu_2(w - w_c), \tag{1.18}$$

where

$$\left.\begin{aligned} \lambda_1 &= (2\beta)/(1 - \beta) + O(\alpha), & \mu_1 &= -2\beta w_c + O(\alpha), \\ \lambda_2 &= -(1 + \beta)\alpha^{-1} + O(1), & \mu_2 &= 16\sigma\beta\alpha[3(1 + \beta) - 4\sigma(1 - \beta)]^{-1} + O(\alpha^2), \\ 3\lambda_3 &= -4\sigma(1 - \beta)\alpha^{-1} + O(1), & 3\mu_3 &= [4\sigma(1 - \beta) - 3(1 + \beta)]w_c + O(\alpha). \end{aligned}\right\} \tag{1.19}$$

*This is used in the sense that there are two possible solution curve gradients through $V = w = 0$.

If $\sigma = \sigma_0$, then it may be shown that on these curves

$$E \sim cr^{(1+\beta)/\alpha},$$

but from (1.13), in this case

$$E \sim cr^{(1+\beta)/\alpha}$$

for all $r$, which is unacceptable if the inviscid solution is to be a limiting case when $\alpha \to 0$ for large $r$. Hence $C = 0$, $E \equiv 0$, and the only curves left are

$$V \sim \lambda_3(w - w_c), \qquad (r \to 0),$$

$$V \sim \lambda_1(w - w_c), \qquad (r \to \infty).$$

On the other hand, for sink flow, this point is again a triple point, but all the curves through it correspond to $r \to \infty$. If

$$4\sigma < 3(1 + \beta)/(1 - \beta)$$

they are

$$\left.\begin{aligned} V &\sim \lambda_1(w - w_c) + (\lambda_3 - \lambda_1)k_1 \mid w - w_c \mid^{\lambda_3/\lambda_1} + (\lambda_2 - \lambda_1)k_2 \mid w - w_c \mid^{\lambda_2/\lambda_1}, \\ &\text{with} \\ \theta - \theta_c &\sim \mu_1(w - w_c) + (\mu_3 - \mu_1)k_1 \mid w - w_c \mid^{\lambda_3/\lambda_1} + (\mu_2 - \mu_1)k_2 \mid w - w_c \mid^{\lambda_2/\lambda_1}, \end{aligned}\right\} \quad (1.20)$$

(which corresponds to the supersonic branch of the inviscid curve),

$$\left.\begin{aligned} V &\sim \lambda_3(w - w_c) + (\lambda_2 - \lambda_3)Ck_2 \mid w - w_c \mid^{\lambda_2/\lambda_3}, \\ &\text{with} \\ \theta - \theta_c &\sim \mu_3(w - w_c) + (\mu_2 - \mu_3)Ck_2 \mid w - w_c \mid^{\lambda_2/\lambda_3}, \end{aligned}\right\} \quad (1.21)$$

and, if $k_2 \neq 0$,

$$\left.\begin{aligned} V &\sim \lambda_2(w - w_c) \\ &\text{with} \\ \theta - \theta_c &\sim \mu_2(w - w_c). \end{aligned}\right. \quad (1.22)$$

When $\sigma = \sigma_0$, $\mu_3 = \mu_1$, and we find that

$$E \sim C'k_2 r^{-(1+\beta)/|\alpha|} \quad \text{for} \quad r \to \infty.$$

Again from (1.13) we have

$$E \sim Cr^{-(1+\beta)/|\alpha|}$$

for all $r$ for which the solution exists and it is tempting to argue that in order to keep $E$ bounded when $r$ is small and $|\alpha| \to 0$ we should put $C = 0$ and hence $k_2 = 0$. But in the absence of a general solution it is by no means certain that $r$ is ever less than one on a solution curve yielding a shock, in fact in the isoenergetic case it is not true anywhere where the solution has physical meaning.

Thus, while for source flow the special case $\sigma = \sigma_0$ yields the result that all flows of bounded total energy have constant total energy it is possible that for sink flow *all* flows with $\sigma = \sigma_0$ have bounded total energy. In order to achieve any simplification then we will consider only the sink flows with constant total energy and it seems plausible that the results obtained for this case will be indicative, at least, of those holding for neighbouring flows—and anyway, they will be sufficient for one of the main purposes of this paper.

For this particular case, the singular point at $w = w_c$ becomes a double point, through which the solution curves are

$$V \sim \lambda_1(w - w_c) + (\lambda_3 - \lambda_1)k_1 \mid w - w_c \mid^{\lambda_3/\lambda_1} \tag{1.23}$$

and

$$V \sim \lambda_3(w - w_c) \tag{1.24}$$

on both of which

$$\theta - \theta_c \sim \mu_1(w - w_c) \tag{1.25}$$

since now $k_2 = 0$ and $\mu_3 = \mu_1$ when $\sigma = \sigma_0$ .

Note that if the isoenergetic assumption is not made, then from (1.20) there is a double infinity of curves (involving two arbitrary constants $k_1$ and $k_2$) with slope $\lambda_1$ , which therefore are asymptotic to the inviscid curve.

**2. The isoenergetic flows with $\sigma = \sigma_0$ .** We have now the relation

$$\theta = 1 - \beta[1 + \alpha/(\beta + 1)]w^2 \tag{2.1}$$

and this may be used to eliminate $\theta$ from (1.15) to yield the first order equation

$$\begin{aligned} \alpha w^2 V(dV/dw) = V\{1 - [1 - \alpha\beta/(\beta + 1)]w^2\} \\ - w\{1 - [\beta + \alpha(1 + 2\beta)/(1 + \beta)]w^2\}. \end{aligned} \tag{2.2}$$

There is only a trivial loss of accuracy in replacing the coefficients of $w^2$ in the brackets by 1 and $\beta$ respectively [2], and the equation which will be considered subsequently becomes

$$\alpha w^2 V(dV/dw) = V(1 - w^2) - w(1 - \beta w^2). \tag{2.3}$$

If $\alpha$ is placed equal to zero here we obtain the inviscid equation

$$V = w(1 - \beta w^2)(1 - w^2)^{-1}, \tag{2.4}$$

another form of (1.9).

The equation (2.3), although not soluble explicitly, lends itself to a topological discussion of the solution curves which has been carried out by the writer [2] for the case of source flow. The solution curves of the equation run as in Fig. 1 and exhibit the following features. Those of physical interest 'start' from the point $w = V = 0$ (a stagnation point at $r = \infty$). A typical curve lies close to the subsonic branch of the inviscid curve up to some subsonic speed $w_1$ say, then rises steeply to a large positive value of $V$ and returns to the neighbourhood of the supersonic branch of the inviscid curve near $w = w_1^{-1}$ which it then follows nearly back to $w = 1$. Ultimately the solution curve becomes asymptotic to $w = 0$ with $V \to -\infty$. In the physical plane the steeply humped portion of the curve corresponds to a shock of small, but finite, thickness while the part for which $w \to 0$ with $V \to -\infty$ corresponds to a singularity lying close to $r = 1$ from

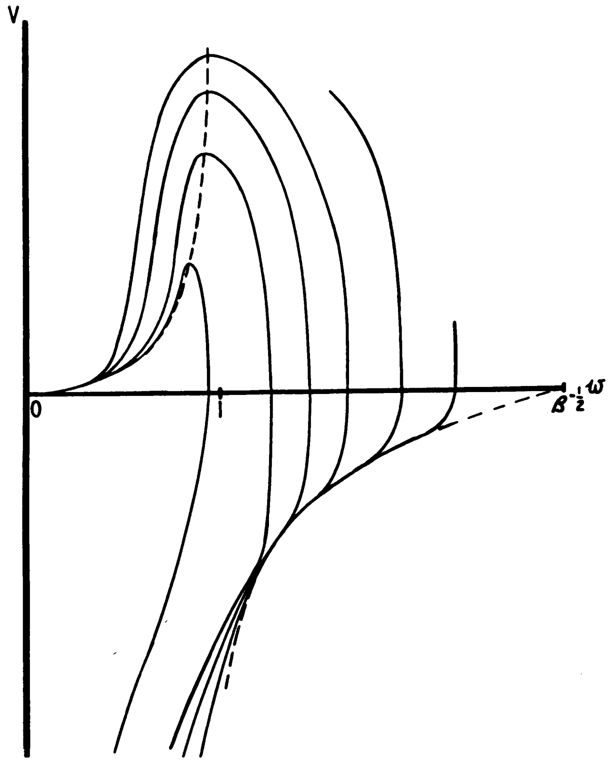


FIG. 1. Source flow: Solution curves in the $w$-$V$ plane

which the fluid issues with infinite density. Thus in the region of interest we have effectively parts of the inviscid solution branches joined by a shock.

It was shown [2] that for such a typical solution curve, if

$$(1 - w_1)^{-1} = o(\alpha^{-1/3}) \tag{2.5}$$

that is, the shock strength, $1 - w_1$, is of larger order than $\alpha^{1/3}$ then

$$\Delta = O[\alpha(1 - w_1)^{-1} \log [(1 - w_1)^3 \alpha^{-1}]] \tag{2.6}$$

where $\Delta$ is the shock thickness. Furthermore, in the interior of the shock

$$V_{max} = O[(1 - w_1)^2 \alpha^{-1}]. \tag{2.7}$$

The case

$$1 - w_1 = O(\alpha^{1/3}) \tag{2.8}$$

was not discussed, but a similarity argument may then be applied and yields the results

$$\Delta = O(\alpha^{2/3}), \tag{2.9}$$

$$V_{max} = O(\alpha^{-1/3}), \tag{2.10}$$

consistent with (2.6) and (2.7).

For *sink flow*, the governing equation is written

$$|\alpha| w^2 V(dV/dw) = -V(1 - w^2) + w(1 - \beta w^2) \tag{2.11}$$

and this has been discussed in some detail by Wu [3]. The curve of zero slope is the inviscid solution curve, $\mathfrak{C}_1$ , given by

$$V = w(1 - \beta w^2)(1 - w^2)^{-1} \tag{2.12}$$

and the curve of inflexions, $\mathfrak{C}_2$ , has the equation

$$2V^3 - w(1 + \beta w^2)V^2 + |\alpha|^{-1}(1 - \beta w^2)[V(1 - w^2) - w(1 - \beta w^2)] = 0. \tag{2.13}$$

Provided that

$$|1 - w|^{-1} = o(\alpha^{-1/3}) \tag{2.14}$$

$\mathfrak{C}_2$ has branches given by

$$V = w(1 - \beta w^2)(1 - w^2)^{-1}[1 + o(1)] \tag{2.15}$$

and

$$V = \pm(1 - \beta w^2)^{1/2}(w^2 - 1)^{1/2}(2|\alpha|)^{-1/2}[1 + o(1)], \qquad w > 1; \tag{2.16}$$

while it has infinite slope at

$$w \sim 1 + 3.2^{-4/3}(1 - \beta)^{1/3}|\alpha|^{1/3}, \qquad V \sim -(1 - \beta)^{-2/3}(4|\alpha|)^{-1/3}, \tag{2.17}$$

and crosses the line $w = 1$ at

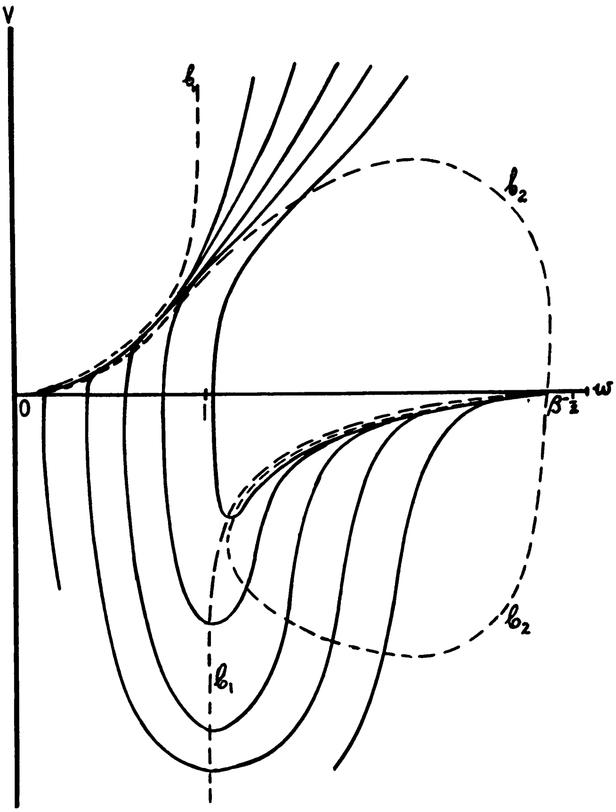


Fig. 2. Sink flow: Solution curves in the $w$-$V$ plane

$$V \sim (1 - \beta)^{2/3}(2 \mid \alpha \mid)^{-1/3}. \tag{2.18}$$

Hence the curve runs as shown in Fig. 2. It is easily verified from power series expansions that although the infinite family of solution curves passing through $w = \beta^{-1/2}$ have the same slope $2\beta/(1 - \beta)$ [see (1.20)] as $\mathfrak{C}_1$ and the branch (2.15) of $\mathfrak{C}_2$, they lie initially below both those curves. Thus the solution curves may be sketched into the figure.

Now consider a typical solution curve from the point $w = \beta^{-1/2}$, $V = 0$, which passes through the point $P(b, c)$ on the lowest supersonic branch of the curve of inflexions, as in Fig. 3, where

$$\mid b - 1 \mid^{-1} = o(\mid \alpha \mid^{-1/3}), \tag{2.19}$$

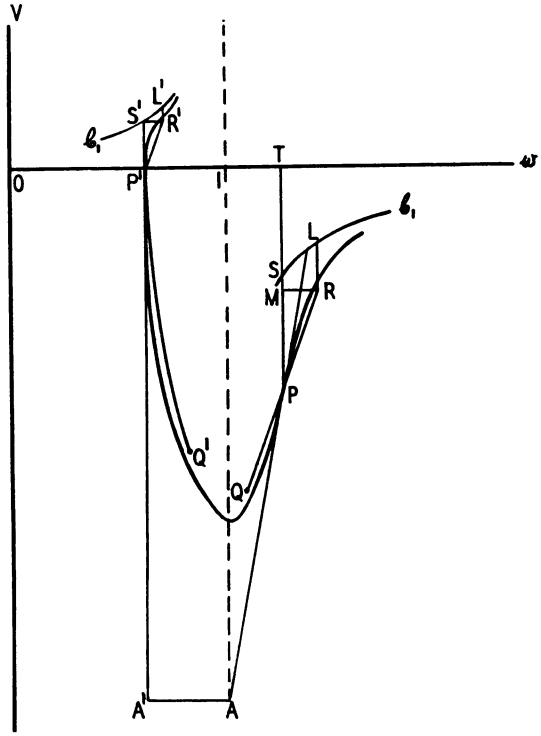


Fig. 3. Sink flow: A typical shock curve in the $w$-$V$ plane

so that from (2.16)

$$c = -(1 - \beta b^2)^{1/2}(b^2 - 1)^{1/2}(2 \mid \alpha \mid)^{-1/2}[1 + o(1)]. \tag{2.20}$$

Since $P(b, c)$ is a point of inflexion then the solution curve certainly lies to the right of its tangent there,

$$V = c + (b^2 - 1)b^{-2} \mid \alpha \mid^{-1} [1 + b(1 - \beta b^2)c^{-1}(b^2 - 1)^{-1}](w - b), \tag{2.21}$$

for $w > b$, and to the left of this tangent for $w < b$. On the other hand, on the line $PQ$,

$$V = c + (b^2 - 1)b^{-2} \mid \alpha \mid^{-1} [1 + b(1 - \beta b^2)c^{-1}(b^2 - 1)^{-1}](1 - \epsilon)(w - b), \quad 0 < \epsilon < 1, \tag{2.22}$$

it may be shown that the slope of the solution curves is greater than the slope of the line for

$$w_Q = b - 3^{-1}\beta b\epsilon(b^2 - 1) \leq w \leq b, \tag{2.23}$$

hence the solution curve through $P$ cannot cross it in this range and must lie to the right of $PQ$.

A repeated application of this kind of reasoning shows that (see Fig. 3)

(i) the curve lies to the left of the line $PR$, where

$$MR = 2^{-1/4}b^{3/2}(1 - \beta b^2)^{3/4}(b^2 - 1)^{1/4}\varphi(b)^{-1/2} \mid \alpha \mid^{1/4} \tag{2.24}$$

and

$$(RL/ST) = 2^{3/4}b^{1/2}(1 - \beta b^2)^{-1/4}(b^2 - 1)^{-3/4}\varphi(b)^{1/2} \mid \alpha \mid^{1/4}, \tag{2.25}$$

where

$$\varphi(b) = 1 + (1 - 3\beta)b^2 + \beta b^4 > 0, \tag{2.26}$$

and

$$ST = b(1 - \beta b^2)(b^2 - 1)^{-1}; \tag{2.27}$$

(ii) the solution curve crosses the $w$ axis at $w = d$, where

$$1 - d = O(b - 1), \tag{2.28}$$

and lies to the left of the curve $P'Q'$ which has the equation

$$\begin{aligned} V = -2^{1/2}(1 - \beta d^2)^{1/2}d^{-1/2} \mid \alpha \mid^{-1/2} (w - d)^{1/2} \\ - 2.3^{-1}(1 - d^2)d^{-2} \mid \alpha \mid^{-1} (w - d), \end{aligned} \tag{2.29}$$

with

$$w_{Q'} - d = b^{-1}d(1 - d^2); \tag{2.30}$$

(iii) the solution curve lies to the left of the line $P'R'$ where

$$SR' = d(1 - \beta d^2)(1 - d^2)^{-1/2}\varphi(d)^{-1/2} \mid \alpha \mid^{1/2}, \tag{2.31}$$

$$(L'R'/P'S') = \varphi(d)^{1/2}(1 - d^2)^{-3/2} \mid \alpha \mid^{1/2}, \tag{2.32}$$

and

$$P'S' = d(1 - \beta d^2)(1 - d^2)^{-1}. \tag{2.33}$$

Hence the solution curve through $P$ lies close to the inviscid curve for $w$ slightly greater than $b$, from (2.24) and (2.25), and for $w$ slightly greater than $d(V > 0)$, from (2.31) and (2.32), and between these points, on the whole, the velocity gradient is large. In fact, the curve in this range represents a shock, and the solution in the $r$-$w$ plane is shown in Fig. 4.

We may estimate the shock thickness by means of the bounding curves. For convenience, we define the shock thickness as the distance between $P$ and $P'$. Since

$$V = -rdw/dr$$

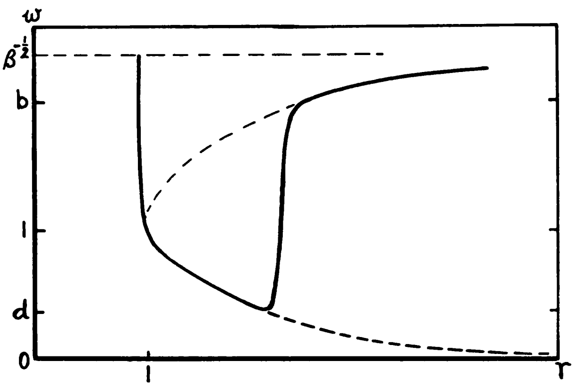


FIG. 4. Sink flow: A typical solution curve in the $r$-$w$ plane

the preceding results show that $\log (r_P/r_{P'})$ is bounded above by

$$\int_{P'}^{Q'} | V |^{-1} dw + \int_{Q}^{P} | V |^{-1} dw + (d - b)[\min. (| V_{Q'} |, | V_Q |)]^{-1},$$

and below by

$$(1 - d) | V_A |^{-1} + \int_{A}^{P} | V |^{-1} dw.$$

It is easily verified that the dominant term in each of these bounds, and hence $\log (r_P/r_{P'})$*, is of order

$$(b - 1)^{-1} | \alpha | \log [(b - 1)^3 (1 - \beta b^2)^{-1} | \alpha |^{-1}] \tag{2.34}$$

since $(1 - d)$ is at least of order $(b - 1)$.

Now $r_P$ will differ little from the value of $r$ at $w = b$ on the supersonic branch of the inviscid curve, namely

$$b^{-1}[(1 - \beta b^2)/(1 - \beta)]^{-(1-\beta)/(2\beta)}, \tag{2.35}$$

so that finally the shock thickness

$$\Delta = r_{P'} - r_P = O\{(b - 1)^{-1} | \alpha | (1 - \beta b^2)^{-(1-\beta)/(2\beta)} \cdot \log [(b - 1)^3 | \alpha |^{-1} (1 - \beta b^2)^{-1}]\}. \tag{2.36}$$

This result may be used to show that Eq. (1.7) integrated between $r_P$ and $r_{P'}$ yields

$$bd \sim 1 + | \alpha |^{1/2} 2^{-1/2} (1 - \beta b^2)^{1/2} (b^2 - 1)^{-1/2} \tag{2.37}$$

which is the Prandtl relation to order $| \alpha |^{1/2}$.

From (2.21), (2.23) and (2.30) the maximum value of $| V |$ attained within the shock lies between $| V_A |$ and the minimum of $| V_Q |$ and $| V_{Q'} |$, which are all of order

$$(b - 1)^2 | \alpha |^{-1}. \tag{2.38}$$

*To extend this result to the neighbourhood of the inviscid curve near the points $L$ and $L'$, more precise bounds than those given in (i) and (iii) are needed, but the result (2.34) still stands.

But we note that at $P$,

$$| V | = O[(b - 1)^{1/2} | \alpha |^{-1/2}] \tag{2.39}$$

from (2.20), and hence the shock curve cannot be obtained by a similarity argument.

However, if

$$(b - 1) = O(| \alpha |^{1/3}) \tag{2.40}$$

the above results suggest that

$$| V | = O(| \alpha |^{-1/3}) \tag{2.41}$$

throughout, and indeed, with the change of variables

$$V = | \alpha |^{-1/3} v, \tag{2.42}$$

$$w - 1 = | \alpha |^{1/3} \omega, \tag{2.43}$$

similarity solutions of (2.3) may be obtained [3] to yield shock type curves for which

$$\Delta = O(| \alpha |^{2/3}). \tag{2.44}$$

But this is obviously a very restricted case, and it is important to investigate the reason why these were the only solutions obtained by the PLK method.

**3. The PLK method.** When the governing equations (including boundary conditions) of a problem contain a small parameter, say $\alpha$, the standard perturbation technique is to expand each dependent variable as a series in $\alpha$. The equations for the $\nu$th perturbation are obtained from the coefficients of $\alpha^{\nu}$ in the governing equations—in particular, the zeroth perturbation satisfies the unperturbed equations.

These perturbations may exhibit, at a point (or on a curve), a singularity which becomes worse as the order $\nu$, increases. If this occurs when there is only a 'slight' difference between the perturbed and unperturbed equations and in particular, they are of the same order, Lighthill [5] has shown that in many cases a uniformly valid solution* may be obtained as follows. Expand also appropriate independent variables as series in $\alpha$ in terms of new independent variables. The coefficients of $\alpha^{\nu}$, except for $\alpha^{0}$, are unknown functions and are used to cancel the worst effect of the singularity, so that in terms of the new independent variables each perturbation is no more singular than the preceding one. A transformation back to the original independent variables yields the uniformly valid solution.

The singular perturbation problems just described are of this nature essentially because a singularity is incorrectly placed by the unperturbed governing equations (it may even lie on another sheet of a Riemann surface), and the Lighthill technique (the PLK method) places the singularity in its correct position. The problem which concerns us is of the singular perturbation type because the unperturbed equations are of lower order than the perturbed equations, and are singular at $w = 1$ whereas the perturbed equations are singular only at $w = 0$ and near $w = \beta^{-1/2}$, see, for example, Eqs. (2.3) and (2.4).

It is this spurious singularity at $w = 1$ which gives rise to a superficial resemblance

*Though not necessarily to all orders, see Tsien [4], p. 334.

to the problems treated by Lighthill. An attempt to solve Eqs. (1.7) and (1.8)* by means of conventional perturbation expansions shows that successive perturbations are more and more singular at $w = 1$. Hence, a new variable [3], $\xi$, is introduced and the expansions

$$w(\xi) = \xi + |\alpha| w_1(\xi) + |\alpha|^2 w_2(\xi) + \cdots, \tag{3.1}$$

$$\log r = \eta_0(\xi) + |\alpha| \eta_1(\xi) + |\alpha|^2 \eta_2(\xi) + \cdots, \tag{3.2}$$

$$\theta(\xi) = \theta_0(\xi) + |\alpha| \theta_1(\xi) + |\alpha|^2 \theta_2(\xi) + \cdots, \tag{3.3}$$

are assumed.

Upon substitution into the differential equations (1.7) and (1.8) and equating coefficients of powers of $|\alpha|$, two differential equations which must be satisfied by $w_n$, $\eta_n$ and $\theta_n$ are obtained, for each $r$. In particular the zero order equations are

$$(1 + \beta)\xi^2 + \xi\theta_0' - \theta_0 - \xi\eta_0'\theta_0 = 0, \tag{3.4}$$

$$\theta_0 - 1 + \beta\xi^2 = 0, \tag{3.5}$$

provided that $\eta_0'(\xi) \not\equiv 0$. The solution for $\eta_0$ is

$$\eta_0(\xi) = -\log \xi - (2\beta)^{-1}(1 - \beta) \log [(1 - \beta\xi^2)/(1 - \beta)], \tag{3.6}$$

where the constant of integration is chosen so that $\eta_0(w)$ is the inviscid solution for $\log r$. The $\eta_n(\xi)$ are now determined in turn so that the $w_n(\xi)$ have the lowest possible order of singularity at $\xi = 1$**. It is found that near $\xi = 1$,

$$\eta_1(\xi) = O[(1 - \xi)^{-1}], \qquad \eta_2(\xi) = O[(1 - \xi)^{-4}], \qquad \eta_n(\xi) = O[(1 - \xi)^{-3r+2}]. \tag{3.7}$$

Since it is noted that the singularity near $w = \beta^{-1/2}$ cannot be accounted for by the expansion procedure adopted, the series (3.2) is finally written

$$\log r = \eta_0(\xi) + |\alpha| \eta_1(\xi) + \cdots + C |1 - \beta\xi^2|^{(1-\beta)^2/(2|\alpha|\beta)} \tag{3.8}$$

where the last term is obtained from a discussion of the singularities of the isoenergetic equation.

It is suggested that this expansion is convergent when $|1 - \xi| \gg |\alpha|^{1/3}$, since successive terms are of smaller order in $|\alpha|$, and moreover, that it remains convergent even when $|1 - \xi| = k|\alpha|$, where $k$ is indeed of order unity, although only the first few terms are known explicitly.

Now, it may be verified, at least for the isoenergetic case, that the series obtained are divergent, even when $|1 - \xi| \gg |\alpha|^{1/3}$, but the important points can be brought out much more clearly by the examination of a simpler equation which, while it exhibits the main features of (2.11), possesses an explicit solution. Consider the equation

$$\alpha(dy/dx) = -xy - 1 \tag{3.9}$$

in which $\alpha > 0$ is a small parameter. The equation has no singularities in the finite plane, but the equation governing the zeroth perturbation,

$$xy + 1 = 0, \tag{3.10}$$

*The formal development which follows is essentially the same for the general case and the isoenergetic case, except for the term with arbitrary constant in (3.8).

**It would seem equally plausible to determine the $w_n$ so that the $\eta_n$ are no more singular than $\eta_0$ but no essential difference results.

obtained by placing $\alpha$ equal to zero, is an equation of lower order than (3.9) and is singular at $x = 0$.

The solution curves of (3.9) are sketched in Fig. 5 and it is seen that for $y < 0$ their general character resembles those of (2.11). The point $x = \infty$ corresponds to $w = \beta^{-1/2}$ and the curve

$$y = -x^{-1} \tag{3.11}$$

corresponds to the inviscid curve (2.12).

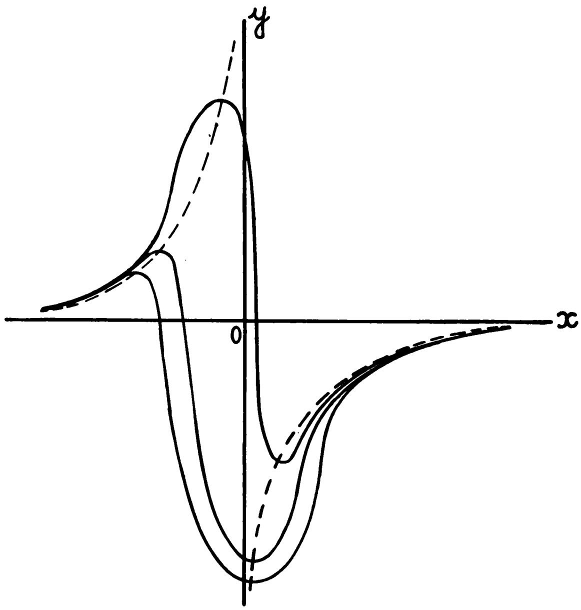


FIG. 5. The solution curves of $\alpha y' = -xy - 1$.

An attempt to solve (3.9) formally by means of an expansion of the form

$$y = y_0(x) + \alpha y_1(x) + \cdots \tag{3.12}$$

leads to

$$y = -x^{-1} - \alpha x^{-3} - \cdots - 1\cdot 3\cdot 5 \cdots (2n - 1)\alpha^n x^{-2n-1} \cdots, \tag{3.13}$$

which is divergent, successive terms being more and more singular at $x = 0$. The exact solution of the equation for the curve which passes through the point $x_0$, $y_0$ is

$$y = y_0 \exp[(x_0^2 - x^2)/(2\alpha)] - \alpha^{-1} \exp[-x^2/(2\alpha)] \int_{x_0}^{x} \exp[t^2/(2\alpha)]\, dt, \tag{3.14}$$

and it may be verified that (3.13) is the asymptotic expansion of (3.14) for $x$ *large enough*, for all $x_0$, $y_0$, but it is not a valid expansion for *any* solution for all $x > 0$. In particular when $x = O(\alpha^{1/2})$ (note that $| 1 - w | = O(| \alpha |^{1/3})$ before), all of the terms in (3.13) are $O(\alpha^{-1/2})$ [note that all terms in (3.2) are of order $| \alpha |^{-1/3}$], but, it can be verified from (3.14), the remainder term is also $O(\alpha^{-1/2})$ and becomes infinite with $r$ for any $x_0$, $y_0$. Thus the series (3.13) is never an adequate approximation to a solution in the critical region when $x = O(\alpha^{1/2})$. (The addition of a term $[\exp(-x^2/(2\alpha)]$ does not improve matters).

Now apply the PLK method to this equation. A new independent variable is chosen so that

$$x = \xi + \alpha x_1(\xi) + \alpha^2 x_2(\xi) + \cdots \tag{3.15}$$

and it is assumed that

$$y = y_0(\xi) + \alpha y_1(\xi) + \alpha^2 y_2(\xi) + \cdots . \tag{3.16}$$

The $x_n(\xi)$ are to be determined successively so that the $y_n(\xi)$ are no more singular than $y_0(\xi)$ at $\xi = 0$. The substitution of these expansions into (3.9) yields the equations

$$\xi y_0 + 1 = 0, \tag{3.17}$$

$$(\xi y_0 + 1)x_1' + x_1 y_0 + \xi y_1 = -y_0' , \tag{3.18}$$

$$(\xi y_0 + 1)x_2' + (x_1 y_0 + \xi y_1)x_1' + \xi y_2 + x_1 y_1 + x_2 y_0 = -y_1' , \cdots . \tag{3.19}$$

It is possible to annihilate all of the $y_n$ for $n > 1$ by suitable choice of the $x_n$ to obtain the formal solution

$$y = -\xi^{-1}, \tag{3.20}$$

$$x = \xi + \alpha\xi^{-1} + \alpha^2\xi^{-3} + \cdots + a_n\alpha^n\xi^{-2n+1} + \cdots , \tag{3.21}$$

where

$$\begin{aligned} a_{2m} &= (4m - 2)\left[\sum_{s=0}^{m-2} a_{2m-1-s}a_{s+1} + \tfrac{1}{2}a_m^2\right], \qquad m \geq 2, \\ a_{2m+1} &= 4m \sum_{s=0}^{m-1} a_{2m-s}a_{s+1} , \qquad m \geq 1. \end{aligned} \tag{3.22}$$

It follows from the last relations that

$$a_n > 2^{n-2}(n - 1)!, \qquad n > 2, \tag{3.23}$$

so that the series (3.21) is certainly divergent, in fact, "more" divergent than (3.13), and the successive terms are more and more singular at $\xi = 0$. The series does not enable the neighbourhood $x = 0(\alpha^{1/2})$ to be approached, and all that has been achieved by the PLK method is the exchange of an invalid expansion of one variable for an invalid expansion of the other. The crucial point is that the small parameter multiplies the highest derivatives in the differential equation. Therefore the highest derivatives are dropped in the sequences of equations set up both by the conventional perturbation analysis and by the PLK method; this is illustrated in (3.17) to (3.19). But the explicit solution (3.14) shows that the highest derivative in (3.9), despite its small coefficient, is at least as important as any other term over a large part of the field.

For the cylindrical shock, the situation is similar. In Wu's application [3] of the PLK method, the highest derivatives are dropped from the sequences of equations, but their importance is indicated by the topological results, at least for the isoenergetic case, and it is not surprising, therefore, that the expansions diverge before a neighbourhood of $\xi = 1$ is reached.

The term in (3.8) with the arbitrary constant serves to represent the effect of the highest derivatives in a neighbourhood of $\xi = \beta^{-1/2}$ for the isoenergetic case [the discussion of the real singularity near $w = \beta^{-1/2}$ in Sec. 1 shows that near $\xi = \beta^{-1/2}$, in general, $\log r \sim \eta_0(\xi) + C \mid 1 - \beta\xi^2 \mid^{\lambda_2/\lambda_1} + D \mid 1 - \beta\xi^2 \mid^{\lambda_2/\lambda_1}$], but even with this term, the expansions are only asymptotic to the solution in that neighbourhood, since the series diverge just the same.

Seeing that all the terms $\eta_r(\xi)$ in the (invalid) expansion (3.8) are $0(\mid \alpha \mid^{2/3})$ when $\mid 1 - \xi \mid = 0(\mid \alpha \mid^{1/3})$, Wu [3] concludes that for *all* cylindrical shocks, $\log r = 0(\mid \alpha \mid^{2/3})$ when $\mid 1 - \xi \mid = 0(\mid \alpha \mid^{1/3})$, and he constructs similarity solutions

$$\log r = \mid \alpha \mid^{2/3} \xi^*, \tag{3.24}$$

$$w = 1 + \mid \alpha \mid^{1/3} w_1^*(\xi^*) + \cdots , \tag{3.25}$$

$$\theta = 1 + \mid \alpha \mid^{1/3} \theta_1^*(\xi^*) + \cdots , \tag{3.26}$$

the validity of which is not contested, in the range

$$1 - k \mid \alpha \mid^{1/3} \leq \xi \leq 1 + k \mid \alpha \mid^{1/3}.$$

The similarity solutions exhibit a shock-like character and imply that across such shocks

$$\Delta w = A \mid \alpha \mid^{1/3}$$

and

$$\Delta \log r = B \mid \alpha \mid^{2/3}.$$

To determine typical values of the constants $A$ and $B$, Wu [3] takes the divergent expansion (3.2) to represent the particular solution corresponding to $C = 0$ in (3.8), and uses the few terms, which are available, of it and of the corresponding expansions for $w$ and $\theta$ to compute values of $\log r$, $w$ and $\theta$ at $\xi = 1 \pm k \mid \alpha \mid^{1/3}$ to serve as boundary values of the similarity solution (3.24) to (3.26).

It is understandable that such a procedure can only lead to the selection of a restricted solution class defined by the similarity assumption. What follows from the existence of the transonic similarity solutions is that shocks of strength of order $\mid \alpha \mid^{1/3}$ have a thickness of order $\mid \alpha \mid^{2/3}$. This restricted class corresponds to the solution curves in Fig. 2 which pass through the curve of inflexions within a distance of order $\mid \alpha \mid^{1/3}$ of its tip. For the stronger shocks the results (2.36) and (2.38) hold.

It is interesting to note that it is impossible for a cylindrical shock of zero strength to exist—at least for the isoenergetic case—for there is a continuous transition from weak shocks with strength $O(\mid \alpha \mid^{1/3})$ to 'incomplete' supersonic (subsonic) 'shocks' of strength $O(\mid \alpha \mid^{1/3})$ for sink (source) flow as the point $P$, Fig. 3, moves along the curve of inflexions to the tip and back again on the other branch. Thus the weakest shocks obtained in cylindrical flow have a strength of order $\mid \alpha \mid^{1/3}$.

**4. Conclusion.** It has been shown, in agreement with the result for inward-facing cylindrical shocks (source flow), that outward facing cylindrical shocks, (sink flow), at least for the isoenergetic case, have a thickness given by

$$\Delta = O[S^{-1} \mid \alpha \mid \log (S^3 \mid \alpha \mid^{-1})], \tag{4.1}$$

where $S$ is the shock strength and $| \alpha |$ may be regarded as the reciprocal of the Reynolds number of the flow. The maximum nondimensional velocity gradient within the shock is

$$(dw/dr)_{\max} = O(S^2 | \alpha |^{-1}). \tag{4.2}$$

The weakest cylindrical shocks which can occur are of strength

$$S = O(| \alpha |^{1/3}) \tag{4.3}$$

and then

$$\Delta = O(| \alpha |^{2/3}), \tag{4.4}$$

$$(dw/dr)_{\max} = O(| \alpha |^{-1/3}), \tag{4.5}$$

in agreement with the results for arbitrary strength. It is *only* in this case that a similarity solution can describe the shock interior.

Singular perturbation problems of this type cannot be treated by a straightforward application of the PLK method. This approach leads to the invalid result that all cylindrical shocks have a strength of order $| \alpha |^{1/3}$.

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