IMPROVING THE CONVERGENCE IN AN EXPANSION OF SPHEROIDAL WAVE FUNCTIONS*

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1. Introduction. Spheroidal configurations have proved to be of considerable value in various types of diffraction and radiation problems. Of these we mention only one example, the prolate spheroidal radiating antenna. This has been discussed recently by Myers [1] who studied the radiation patterns and by Wells [2] who studied the near field of the antenna. Other examples and references to the literature can be found in the books of Meixner and Schafke [3] and of Flammer [4].

The advantages of spheroidal models both in a mathematical and in a physical sense are well known and will not be discussed here. The disadvantages are the complexity of the spheroidal functions and the lack of sufficient numerical values of the functions. Further, the expansions in spheroidal functions which represent the near field, e.g. the current on the antenna, converge slowly, a fact that is also true of expansions for other geometric models. It is this slow convergence which will concern us in this note and we shall show that the convergence can be improved in the sense that fewer terms of the expansion are needed to obtain the desired accuracy. The method is not new; it has been used, e.g. by Meixner and Kloepfer [5] in the problem of the ring shaped antenna. However in the case of the spheroidal functions some device such as this is particularly useful and we present it here with the idea that it may be of use to others.

By improving convergence we mean the following: Let $\sum a_n\psi_n$ be an expansion of eigen functions ψ_n and consider the asymptotic behaviour of $a_n\psi_n$ for large n. Suppose that to orders of $1/n$, $a_n\psi_n \approx b_n\varphi_n$. Then in the expansion

$$\sum (a_n\psi_n - b_n\varphi_n) \tag{1}$$

the terms of order $1/n$ are absent and this sum can be expected to converge faster than the original expansion. If in addition it is possible to write the sum $\sum b_n\varphi_n$ in closed form, then

$$\sum a_n\psi_n = \sum (a_n\psi_n - b_n\varphi_n) + f, \tag{2}$$

where f is the sum $\sum b_n\varphi_n$. The results of this procedure, based on the prolate spheroidal antenna, will be given in what follows. Obviously the method can be applied to other types of eigen function expansions.

2. Notation and expansions. A variety of notations have been used in spheroidal functions most of which have been summarized in [4]. For theoretical work the notation used in [3] is preferable while for numerical work other notations have some advantage. We shall use the notation of [3] where also the relevant information concerning spheroidal functions can be found.

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The spheroidal coordinates are ξ, η, φ where ξ is the radial variable and η and φ are angular variables. The ranges of the variables are $\xi > 1$, $|\eta| \leq 1$, $0 \leq \varphi \leq 2\pi$. The angular functions for the symmetrically driven antenna are $ps_n^1(\eta; \gamma^2)$, where $\gamma = 2\pi a/\lambda$, a is the semi-focal length of the spheroid $\xi =$ const. and λ is the wave length. The radial functions are $S_n^{1(4)}(\xi; \gamma)$ and are functions of the third kind in that they provide the proper wave function behaviour for large ξ. The φ-component of the magnetic field is given by

$$H_\varphi = \sum_{n=1}^{\infty} b_n S_n^{1(4)}(\xi; \gamma) ps_n^1(\eta; \gamma^2), \tag{3}$$

where b_n is given by

$$2[(\xi_0^2 - 1)^{1/2} S_n^{1(4)}(\xi_0; \gamma)]' b_n/(2n + 1) = i\omega\epsilon a \int_{\eta_1}^{\eta_2} E_\eta(\xi_0; \eta)(\xi_0^2 - \eta^2)^{1/2} ps_n^{-1}(\eta; \gamma^2)\, d\eta$$

and where $\xi_0 =$ const. represents the antenna, E_η is the tangential component of the electric field and the prime indicates differentiation with respect to ξ_0.

Equation (3) is our basic expansion. An indication of how slowly it converges for $\xi = \xi_0$ is given in Table 4. Since the spheroidal functions are difficult at best to compute and existing tables so far are limited in range, some method which cuts down this computing and speeds up convergence is desirable.

3. Comparison series. Ignoring the physical constants and the integral over the gap in (3), the problem is the convergence of

$$V = \sum_{n=1}^{\infty} ps_n^{-1}(\eta'; \gamma^2) ps_n^1(\eta; \gamma^2)(n + 1/2) V_n(\xi), \tag{4}$$

where

$$V_n(\xi) = S_n^{1(4)}(\xi; \gamma)/[(\xi^2 - 1)^{1/2} S_n^{1(4)}(\xi; \gamma)]'.$$

We now consider some comparison series formed by replacing the terms of (4) by their asymptotic expressions for large n. The details of obtaining these expressions are outlined in the Appendix. According to Lemma I in the Appendix we can choose as a comparison series

$$W = \sum_{n=1}^{\infty} P_n^{-1}(\eta') P_n^1(\eta)(n + 1/2) W_n(\xi) \tag{5}$$

with

$$W_n(\xi) = (\xi^2 - 1)^{1/2} Q_n'(\xi)/n(n + 1) Q_n(\xi),$$

and where P_n^1, P_n^{-1}, Q_n are Legendre functions. Numerically the comparison of V_n with W_n depends on γ and the smaller the γ the better the comparison. Table 1 shows this comparison for $\gamma = 1$ and $\gamma = 2$ and for two different values of ξ. In this table only the the real part of V_n is given; the imaginary part falls off rapidly as n increases.

One concludes from Table 1 that (5) is, in fact, a very good comparison series if only γ is not too large. The difficulty in using it lies in the fact that W cannot be summed in closed form. But even so there is some practical value in replacing the terms of (4) with those of (5) with some accuracy after, say, $n = 6$, especially in view of the limited tables of spheroidal functions. The functions $Q_n(\xi)$ at least for values of ξ near 1 as well as the functions $P_n^1(\eta)$ can easily be computed.

TABLE 1.

	$\xi = 1.00001$			
	$\gamma = 1$		$\gamma = 2$	
n	$\mathrm{Re}(V_n)$	W_n	$\mathrm{Re}(V_n)$	W_n
1	−37.16	−21.923	29.396	−21.923
2	−8.922	−8.098	−13.199	−8.098
3	−4.548	−4.362	−5.235	−4.362
4	−2.845	−2.774	−3.065	−2.774
5	−1.979	−1.946	−2.058	−1.946
6	−1.491	−1.477	−1.521	−1.477
7		−1.163	−1.141	−1.163

	$\xi = 1.001$			
	$\gamma = 1$		$\gamma = 2$	
n	$\mathrm{Re}(V_n)$	W_n	$\mathrm{Re}(V_n)$	W_n
1	−6.763	−3.962	7.322	−3.962
2	−1.744	−1.586	−2.593	−1.586
3	−.947	−.911	−1.080	−.911
4	−.626	−.613	−.669	−.613
5	−.456	−.450	−.476	−.450
6	−.353	−.351	−.364	−.351
7			−.292	−.286

From the comparison series W, with the help of Lemmas II and III in the appendix, we obtain as an approximation to (5)

$$X = \sum_{n=1}^{\infty} P_n^{-1}(\eta')P_n^1(\eta)(n + 1/2)X_n(\xi)/n(n + 1), \tag{6}$$

where

$$X_n(\xi) = -\tfrac{1}{2} \coth u - (n + 1/2)[K_1[(n + 1/2)u]/K_0[(n + 1/2)u] - 1 - 1/(2n + 1)u - n(n + 1)/(n + 1/2) - 1/(4n + 2)]$$

with $\xi = \cosh u$. Let us put $X = X_1 + X_2 + X_3 + X_4$ and

$$X_i = \sum_{n=1}^{\infty} P_n^{-1}(\eta')P_n^1(\eta)X_{ni}(\xi), \qquad i = 1, 2, 3, 4$$

where

$$X_{n1} = -\tfrac{1}{2}[(n + 1/2)/n(n + 1)] \coth u,$$

$$X_{n2} = -1,$$

$$X_{n3} = -1/4n(n + 1),$$

$$X_{n4} = -[(n + 1/2)^2/n(n + 1)][K_1(u_n)/K_0(u_n) - 1 - 1/(2n + 1)u]$$

and $u_n = (n + 1/2)u$. The coefficients are of the order $X_{n1} = 0\ (n^{-1})$, $X_{n3} = 0\ (n^{-2})$, $X_{n4} = 0\ (n^{-3})$. Thus it would appear that it would be sufficient to take only X_2 as a comparison series in order to improve convergence. However this improvement becomes effective only after the coefficients of (6) with X_{n2} in place of X_n approximate those of (5) closely enough. And this is the case only for rather large values of n if ξ is very close to 1. In order to approximate the terms for smaller n also it is useful to take $X_1 + X_2 + X_3 + X_4$ as the comparison series.

As Table 2 shows X_{n3} is never very essential and can be omitted. For $\xi = 1.081$, the main contribution is given by X_{n2} although the additional consideration of X_{n1} and X_{n4} is useful. For ξ very close to 1, X_{n1} and X_{n4} are of much greater importance than X_{n2} .

TABLE 2.

	$\xi = 1.081$				$\xi = 1.02$			
	$u = 0.4$, $\coth u = 2.632$				$u = 0.2$, $\coth u = 5.067$			
n	X_{n1}	X_{n2}	X_{n3}	X_{n4}	X_{n1}	X_{n2}	X_{n3}	X_{n4}
1	−0.988	−1	−0.125	0.181	−1.900	−1	−0.125	0.495
2	−0.548	−1	−0.042	0.077	−1.056	−1	−0.042	0.220
3	−0.385	−1	−0.021	0.041	− .739	−1	−0.021	0.122
4	−0.296	−1	−0.013	0.025	−.570	−1	−0.013	0.081
5	−0.241	−1	−0.008	0.017	−.465	−1	−0.008	0.061
6	−0.204	−1	−0.006	0.012	−.392	−1	−0.006	0.050

	$\xi = 1.001$				$\xi = 1.00001$			
	$u = 0.0447$, $\coth u = 22.38$				$u = 0.0047$, $\coth u = 223.6$			
n	X_{n1}	X_{n2}	X_{n3}	X_{n4}	X_{n1}	X_{n2}	X_{n3}	X_{n4}
1	−8.40	−1	−0.125	3.64	−84.0	−1	−0.125	46.5
2	−4.67	−1	−0.042	1.76	−46.7	−1	−0.042	26.3
3	−3.27	−1	−0.021	1.11	−32.7	−1	−0.021	18.0
4	−2.52	−1	−0.013	0.76	−25.2	−1	−0.013	13.8
5	−2.05	−1	−0.008	0.58	−20.5	−1	−0.008	10.8
6	−1.73	−1	−0.006	0.47	−17.3	−1	−0.006	9.0

In the next section we shall show that X_1 and X_2 can be written in closed form. The series X_3 can be neglected in comparison with X_2 . The series X_4 is more difficult to handle since it cannot be summed in closed form. However as Lemma IV shows, good approximations exist for both large and small values of ξ. Some values of K_1/K_0 and approximations for large and small argument are given in Table 3.

TABLE 3.

u_n	$K_1(u_n)/K_0(u_n)$	$u_n^{-1}[\gamma + \ln u_n/2]^{-1}$	$1 + 1/2u_n$	$1 + \frac{1}{2u_n} - K_1/K_0$
0.02	12.4	−12.4	26	13.6
0.04	7.5	−7.5	13.5	6.0
0.10	4.05	−4.12	6.0	1.95
0.20	2.73	−3.0	3.50	0.77
0.40	1.97		2.25	0.28
1.00	1.43		1.50	0.07
2.00	1.23		1.25	0.02

Returning to the original expansion (4), it can be seen from the coefficients $V_n(\xi)$ that the convergence improves as ξ approaches 1. However it does not follow that the comparison series give the best results for ξ in the neighborhood of 1. If we study these comparison series for various values of ξ we find that X_4 can be neglected in comparison with X_1 and X_2 if $(n + 1/2)u > 0.5$. This holds for all n if $\xi > 1.54$. For smaller values of ξ, X_4 plays an essential role. Thus for $\xi = 1.00001$, X_1 and X_4 give the main contribution for $n < 0.5/u$ or $n < 100$, approximately. This can be seen from studying the coefficients X_{n1} and X_{n4} in comparison with X_{n2}. Hence if we are interested in values of $\xi < 1.54$, the series X_4 is important and since this series cannot be summed we must compute or approximate the sum numerically. This, however, is not difficult in view of the simple analytic form of the coefficients X_{n4} and their approximations as given in Table 3.

4. Closed form expressions for X_1 and X_2. We consider first X_1 and notice that this series is the Green's function for Legendre's differential equation

$$(1 - \eta^2)y'' - 2\eta y' + [n(n + 1) - m^2/(1 - \eta^2)]y = 0$$

with $n = 0$ and $m = 1$. If we label this Green's function as $G(\eta, \eta')$ then

$$\begin{aligned} G(\eta, \eta') &= \tfrac{1}{2}[(1 + \eta)(1 - \eta')/(1 - \eta)(1 + \eta')]^{1/2}, \qquad \eta \le \eta' \\ &= \tfrac{1}{2}[(1 - \eta)(1 + \eta')/(1 + \eta)(1 - \eta')]^{1/2}, \qquad \eta \ge \eta' \end{aligned}$$

when expressed in terms of the solutions $[(1 + \eta)/(1 - \eta)]^{\pm 1/2}$ of the differential equation.

In order to sum X_2 we make use of a result of Watson [6] on summation of the Gegenbauer functions. This result, specialized for our purposes, states that

$$\sin\theta \sin\varphi \int_0^\pi \frac{\sin^2 w\, dw}{(1 - 2t\cos\Omega + t^2)^{3/2}} = -\pi \sum_{n=0}^{\infty} t^n P_{n+1}^1(\cos\theta) P_{n+1}^{-1}(\cos\varphi) \tag{7}$$

where $\cos\Omega = \cos\theta\cos\varphi + \sin\theta\sin\varphi\cos w$. If we make the substitution $w = \pi - 2\psi$, the integral on the left becomes

$$2\int_0^{\pi/2} \frac{\sin^2 2\psi\, d\psi}{(1 - 2t\cos\Omega + t^2)^{3/2}}.$$

For $t = 1$ this becomes

$$\frac{2}{a^{3/2}} \int_0^{\pi/2} \frac{\sin^2 2\psi\, d\psi}{(1 - k^2\sin^2\psi)^{3/2}} = \frac{8}{a^{3/2}k^4}[(2 - k^2)K - 2E],$$

where $a = 2 - 2 \cos(\theta + \varphi)$, $k^2 = 2 \sin\theta \sin\varphi/[1 - \cos(\theta + \varphi)]$ and K and E are the complete elliptic integrals. The integrals fail to exist when $\theta = \varphi$.

Now the expansion on the right of (7) converges for $|t| < 1$ and since $P_n^1(\cos\theta) P_n^{-1}(\cos\varphi)$ is $0(1/n)$ for large n and for θ and φ between 0 and π, the expansion converges also for $t = 1$. Then by Abel's theorem [7] the equality in (7) holds for $t = 1$ except for $\theta = \varphi$. Hence we have

$$X_2 = -\sum_{n=1}^{\infty} P_n^1(\cos\theta) P_n^{-1}(\cos\varphi) = (1/\pi k^2) \csc[(\theta + \varphi)/2][(2 - k^2)K - 2E]$$

with $\eta = \cos\theta$, $\eta' = \cos\varphi$.

5. Some numerical results on convergence. In Table 4 we give some numerical results which offer some indication of how well the convergence can be improved. The first six terms of the series for V and for X as well as for their difference are given for $\gamma = 2$ and for $\xi = 1.001$ and $\xi = 1.00001$; for $\gamma = 1$ only the first five terms are given with $\xi = 1.02$. Such a table is limited by the amount of information available on numerical values of the spheroidal functions and the above values of ξ were chosen since for these we have sufficient information to make a reasonable comparison of V and X. No attempt has been made here to tabulate X_1 and X_2 from their closed form expressions although such computations can readily be done.

TABLE 4.

$\gamma = 2$						$\gamma = 1$		
$\xi = 1.00001$			$\xi = 1.001$			$\xi = 1.02$		
V	X	V-X	V	X	V-X	V	X	V-X
17.05	−16.6	33.65	4.25	−2.54	6.79	−1.695	−1.09	−.605
−2.15	−1.38	−0.77	−.423	−.255	−0.168	−.136	−.121	−.015
.916	.603	.313	.189	.122	.067	.085	.063	.022
−.538	−.602	.064	−.118	−.137	.019	−.0599	−.074	.014
.995	1.107	−.112	.230	.256	−.026	.126	.146	−.020
.156	.135	.021	.037	.033	.004			

For all considered values of ξ the values of $V - X$ are fairly small after $n = 6$ and can be neglected if two or three-place accuracy is wanted. If the comparison series is not used many more terms would be needed to obtain the same accuracy.

Appendix. *Lemma I. For large n, the coefficients $V_n(\xi)$ are given approximately by*

$$W_n(\xi) = (\xi^2 - 1)^{1/2} Q_n'(\xi)/n(n + 1)Q_n(\xi).$$

From reference [3] we find that for large n

$$ps_n^1(\eta; \gamma^2) \approx P_n^1(\eta),$$

$$S_n^{1(4)}(\xi; \gamma) \approx -iQ_n^1(\xi)/K_n^1(\gamma)n(n + 1).$$

Then

$$V_n(\xi) = S_n^{1(4)}(\xi; \gamma)/[(\xi^2 - 1)^{1/2} S_n^{1(4)}(\xi; \gamma)]' \approx Q_n^1(\xi)/[(\xi^2 - 1)^{1/2} Q_n^1(\xi)]'. \tag{8}$$

Now $Q_n^1(\xi) = (\xi^2 - 1)^{1/2} \, dQ_n/d\xi$ and from the differential equation satisfied by the Q_n we find $[(\xi^2 - 1)^{1/2} Q_n^1(\xi)]' = n(n + 1)Q_n$. If we make these substitutions in (8) we get the desired result.

Lemma II. For large n,

$$Q_n(\xi) \approx (u/\sinh u)^{1/2} K_0((n + 1/2)u).$$

with ξ cosh u *and* $K_0(w)$ *the modified Bessel function of the third kind or the Hankel function of imaginary argument,* $H_0(iw)$. *This result follows immediately from the results of reference* [8].

Lemma III. For large n

$$(\xi^2 - 1)^{1/2} Q_n'(\xi)/Q_n(\xi) \approx -(n + 1/2)K_1(w)/K_0(w) + 1/2u - \tfrac{1}{2} \coth u$$

where $w = (n + 1/2)u$ *and* $K_1(w)$ *is again a Hankel function of imaginary argument.* This result follows directly from Lemma II.

Lemma IV. For small w, $K_1(w)/K_0(w) = - w^{-1} [\gamma + \ln w/2]^{-1}$, *approximately, where* $\gamma = 0.5772$ *is Euler's constant; for large* w, $K_1(w)/K_0(w) = 1 + 1/2w - 1/8w^2$, *approximately.*

The first result follows from the definition of $K_0(w)$ and $K_1(w)$ in terms of the modified Bessel functions $I_0(w)$ and $I_1(w)$ together with the approximations, $I_0(w) \approx 1$, $I_1(w) \approx w/2$. The second result follows from the asymptotic expansion of $K_n(w)$ which can be found, e.g. in reference [9].

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