When $D(r)$ is not constant, β is subject to another restriction besides (8). If we represent φ^*, $\varphi_{\text{calc.}}$, and φ^{**} as φ plus an error term, it is easy to show using the method of Sec. 3 of [1] that if ϵ^* is the error in φ^*, then the error in φ^{**}, ϵ^{**}, is to first order approximately

$$\epsilon^{**} = -\frac{\nabla D \cdot \nabla \epsilon^*}{\beta D(F + \omega)}. \tag{10}$$

From (7) it is evident that the x, and y components of $(\nabla D/D)$ are bounded in magnitude by $(\Delta x)^{-1}$ and $(\Delta y)^{-1}$ respectively. The term $F + \omega$ has a minimum value of unity. At a discontinuity (Δx or Δy) ∇ ϵ^* may well be of the order of ϵ^*, hence we see $-\beta(\Delta x)^2$ and $-\beta(\Delta y)^2$ must be at least unity. We find in practice that a satisfactory choice of β may be obtained from the rule that $-\beta(\Delta x)^2$ and $-\beta(\Delta y)^2$ are greater than 2. If the discontinuities in D are slight then we can reduce the value of β correspondingly. It seems desirable to provide a convergence factor of about $\frac{1}{2}$ between ϵ^* and ϵ^{**} in (10).

In the problems ($N \lesssim 2000$) we have run using this method, most took about 7 to 15 iterations to reduce the error by a factor of 100. Large problems with sharp discontinuities in D can take considerably longer, due to the restriction imposed by (10) on β.

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FUNCTIONAL EQUATIONS AND MAXIMUM RANGE*

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1. Introduction. The current interest in rockets and space travel has aroused a corresponding interest in the determination of maximum range, minimum time, and so on, for various types of trajectories.

A variety of questions of this type have been treated by means of the theory of dynamic programming, see [1, 2, 4]. Here we wish to show how to use functional equations to determine the range, the maximum elevation, and similar quantities, as functions of initial position and velocities.

2. Vertical motion—I. Consider an object, subject only to the force of gravity and the resistance of the air, which is propelled straight up. In order to illustrate the technique we shall employ, let us treat the problem of determining the maximum altitude.

Let the defining equation be

$$u'' = -g - h(u'), \tag{1}$$

with the initial conditions $u(0) = 0$, $u'(0) = v$. Here $v > 0$, and $h(u') \geq 0$ for all u'.

Since the maximum altitude is a function of v, let us introduce the function

$$f(v) = \text{the maximum altitude attained starting with initial velocity } v. \tag{2}$$

*Received October 20, 1958.

From the definition of the function it follows that

$$f(v) = v\,\Delta + f(v - [g + h(v)]\,\Delta) + o\,(\Delta), \tag{3}$$

for Δ an infinitesimal. Verbally, this states that the maximum altitude is the altitude gained over an initial time Δ, plus the maximum altitude attained starting with a velocity $v - [g + h(v)]\Delta$, the velocity of the object at the end of time Δ, to within $o(\Delta)$.

Expanding both sides and letting $\Delta \to 0$, we see that

$$f'(v) = \frac{v}{g + h(v)}. \tag{4}$$

Since $f(0) = 0$, this yields

$$f(v) = \int_0^v \frac{v_1\,dv_1}{g + h(v_1)}. \tag{5}$$

In the particular case where $h(v) = 0$, we obtain the standard result $v^2/2g$.

3. Vertical motion—II. Consider the more general case where motion is through an inhomogeneous medium. Let the defining equation be

$$u'' = h(u, u'), \qquad u(0) = c_1\,, \qquad u'(0) = c_2\,. \tag{1}$$

Assume that $h(u, u') \leq 0$ for all u and u', so that $c_2 = 0$ implies no motion.

The maximum altitude is now a function of both c_1 and c_2. Introduce

$f(c_1\,, c_2) =$ the maximum altitude attained starting with the initial position c_1 and initial velocity c_2. (2)

Then, as above,

$$f(c_1\,, c_2) = c_2\,\Delta + f[c_1 + c_2\,\Delta, c_2 + h(c_1\,, c_2)\,\Delta] + o\,(\Delta), \tag{3}$$

which yields in the limit the partial differential equation

$$c_2 + c_2\,\frac{\partial f}{\partial c_1} + h(c_1\,, c_2)\,\frac{\partial f}{\partial c_2} = 0. \tag{4}$$

By virtue of our assumptions, $f(c_1\,, 0) \equiv 0$, for $c_1 \geq 0$.

4. Computational aspects. One can, of course, use the method of characteristics, or standard difference methods, to solve (3.4). Let us present another method which reduces the solution to the tabulation of a sequence of functions of one variable.

In place of (3.4), let us use the discrete approximation of (3.3),

$$f(c_1\,, c_2) = c_2\,\Delta + f[c_1 + c_2\,\Delta, c_2 + h(c_1\,, c_2)\,\Delta]. \tag{1}$$

Since c_2 is monotone decreasing, it can be used to play the role of time. Let us write $c_2 = N\delta$, where δ is a positive quantity, and $f(c_1\,, c_2) \equiv f_N(c_1)$. We consider then only values of c_2 which are multiples of δ. To overcome the fact that $c_2 + h(c_1\,, c_2)\Delta$ in general will not be a multiple of δ, we can either replace it by $[(c_2 + h(c_1\,, c_2)\Delta)/\delta]$, or use interpolation. Although use of an interpolation formula slows up the computation, it greatly improves the accuracy. For an application of the foregoing techniques to a more complicated partial differential equation, see [3].

5. Maximum altitude. Consider now the case where motion takes place in a plane. Let the equations be

$$x'' = g(x', y'), \qquad x(0) = 0, \qquad x'(0) = c_1 , \tag{1}$$
$$y'' = h(x', y'), \qquad y(0) = 0, \qquad y'(0) = c_2 .$$

Introducing, as before, the function $f(c_1 , c_2)$ equal to the maximum altitude, we see that

$$f(c_1 , c_2) = (c_1^2 + c_2^2)^{1/2} \Delta + f[c_1 + g(c_1 , c_2) \Delta, c_2 + h(c_1 , c_2) \Delta] + o (\Delta). \tag{2}$$

Hence,

$$(c_1^2 + c_2^2)^{1/2} + g(c_1 , c_2) \frac{\partial f}{\partial c_1} + h(c_1 , c_2) \frac{\partial f}{\partial c_2} = 0. \tag{3}$$

Once again, let us assume that $c_2 = 0$ implies no vertical motion. Then $f(c_1 , 0) = 0$ for $c_1 \geq 0$. It follows that we can again compute the solution by means of a sequence of functions of one variable.

6. Maximum range. To tackle the problem of maximum range directly requires the introduction of another state variable, the initial altitude. It can also be broken up into two problems, corresponding to the ascent to maximum altitude, and the descent.

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ON THE DETERMINATION OF CERTAIN THERMODYNAMIC AND PHYSICAL QUANTITIES*

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We consider any physical phenomenon where a quantity z is a continuous differentiable function of two independent quantities x and y. Thus:

$$z = z(x, y).$$

Hence

$$dz = F\,dx + G\,dy,$$

where

$$F = F(x, y) = \frac{\partial z}{\partial x},$$

$$G = G(x, y) = \frac{\partial z}{\partial y}.$$

*Received October 27, 1958.