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RESTRICTIONS IMPOSED UPON THE UNIT STEP RESPONSE OF LINEAR PHASE SHIFT NETWORKS*

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Summary. Bounds have been placed upon the 10–90% rise time and overshoot, of the unit step response of phase distortion free networks. Two bounds on the rise time are given, one, in terms of the area under the amplitude function curve, and the other in terms of the area under the square of this curve. Two bounds on the overshoot are given in terms of these areas and the phase slope or its square root.

Best possible bounds on the rise time and overshoot are obtained when the amplitude function is itself bounded and approaches zero at least as fast as ω^{-n}, $n > 0$ in the high frequency region. These bounds may be evaluated readily from tabulated data.

Introduction. The transient response of a network is often substantially improved by the introduction of phase equalization. However, it should be determined whether such phase correction will actually provide sufficient improvement to warrant its use. Computations of the transient response are often tedious. If certain key quantities such as the 10–90% rise time and the overshoot, of the unit step response, of the phase corrected network could be readily estimated, then the advisability of phase correction could be determined. Such readily evaluated bounds are presented here.

Since phase distortion free networks have a linear phase characteristic, it will be assumed throughout this paper that the transfer function of the network in question is of the form $T(\omega)e^{-ik\omega}$ where $T(\omega) \geq 0$ and k is the constant phase slope. Low pass structures will be considered and for convenience $T(\omega)$ will be normalized so that $T(0) = 1$. The unit step response will be written as $A(t)$.

The 10–90% rise time of the unit step responses. Bounds on the 10–90% rise time can be obtained by manipulation of the expression for the unit step response

$$A(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{T(\omega)}{\omega} \sin \omega(t - k)\, d\omega. \tag{1}$$

Utilizing the fact that the unit step response of a linear phase network possesses odd symmetry about the 50% point, which occurs at $t = k$, the 10–90% rise time r_t can be obtained by solution of the equation

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$$\frac{4}{5} = \frac{2r_t}{\pi} \int_0^\infty \frac{T(\omega) \sin [\omega r_t/2]}{\omega r_t} d\omega. \tag{2}$$

It is assumed here that the 10 and 90% points fall on opposite sides of the 50% point. If this is not the case, then the response will fall below 10% after $t = k$. Such a highly oscillatory response is not suitable for most practical applications and will not be considered here.

Equation (2) is difficult to evaluate. In fact, the rise time can, in general be more readily evaluated from a plot of the actual transient response. However, Eq. (2) can be used to obtain some readily evaluated bounds.

Since

$$\left| \frac{\sin x}{x} \right| \le 1,$$

$$r_t \ge \frac{4\pi}{5} \left[\int_0^\infty T(\omega)\, d\omega \right]^{-1}. \tag{3}$$

Often $\int_0^\infty T(\omega)d\omega$ does not exist, which renders inequality (3) useless. Even if the integral does exist, the lower bound may be unduly small.

A somewhat different approach may be used to obtain a bound that exists in all physical cases where parasitic capacitance limits the high frequency response.

Apply Schwartz' inequality to Eq. (2).

$$\frac{4}{5} \le \frac{2}{\pi} r_t \left\{ \int_0^\infty [T(\omega)]^2\, d\omega \int_0^\infty \left[\frac{\sin (\omega r_t/2)}{\omega r_t} \right]^2 d\omega \right\}^{\frac{1}{2}}$$

but,

$$\int_0^\infty \left[\frac{\sin (\omega r_t/2)}{\omega r_t} \right]^2 d\omega = \frac{\pi}{4r_t}.$$

Thus,

$$r_t \ge \frac{16}{25} \pi \left\{ \int_0^\infty [T(\omega)]^2\, d\omega \right\}^{-1}. \tag{4}$$

In many instances the bound presented by relation (4) will be much stronger than that stated in relation (3). Of course, there are also conditions where the converse is true.

If some relatively weak conditions, which are obtained in practical situations, are imposed upon $T(\omega)$, then a very readily evaluated greatest lower bound on the rise time may be obtained.

Consider that $T(\omega) \le \epsilon(\omega_c/\omega)^n$, $n > 0$ for $\omega > \omega_c$. That is $T(\omega_c) \le \epsilon$ and $T(\omega)$ falls off as ω^{-n}, for $\omega > \omega_c$. (For further discussion of such approximations see [1]).

Then the unit step response can be written in the form

$$A(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\omega_c} \frac{T(\omega) \sin \omega(t - k)}{\omega} d\omega + \delta, \tag{5}$$

where

$$| \delta | \le \frac{\epsilon}{n\pi}. \tag{6}$$

Thus, if $|\delta|$ is sufficiently small, the transient response can be assumed to be unaffected by frequency components greater than ω_c.

If the transfer function is such that frequency components greater than ω_c can be neglected, and, in addition, if $T(\omega)$ is bounded so that $T(\omega) \leq M$, then the following procedure leads to a readily evaluated bound.

From Eq. (2)

$$\frac{4}{5} = \frac{2}{\pi} \int_0^{\omega_c r_t/2} T(2x/r_t) \frac{\sin x}{x} \, dx.$$

If $\omega_c r_t/2 \leq \pi$, then the following bound is obtained

$$\frac{4}{5} \leq \frac{2}{\pi} M Si(\omega_c r_t/2)$$

or

$$r_t \geq \frac{2}{\omega_c} \text{Inv. } Si[4\pi/(10M)]. \tag{7}$$

Where Inv. $Si[x]$ is defined by the following relations if $x = \int_0^y [\sin z/z] \, dz = Si(y)$ then $y = \text{Inv. } Si(x)$.

Note that Inv. $Si[4\pi/(10M)]$ is single valued since $M \geq 1$ and $[4\pi/(10M)] \leq Si[2\pi]$ which is the least minimum of $Si[x]$.

In the derivation of relation (7) it was assumed that $\omega_c r_t/2 \leq \pi$. If for some $T(\omega)$, $\omega_c r_t/2 > \pi$, then relation (7) will still be valid, since the result is given in the form of a lower bound and, thus, larger values of r_t are allowed.

It can be shown that values of r_t given by the equality sign of relation (7) exist, and hence, this is the best possible bound. To do this, use the transfer function

$$T(\omega) = \lim_{\epsilon \to 0} \begin{cases} 1, \omega < \epsilon \\ M, \epsilon \leq \omega \leq \omega_c \, . \\ 0, \omega > \omega_c \end{cases}$$

Although such a transfer function cannot be realized, it can be approximated arbitrarily closely.

Note that an increase in the value of M will reduce the bound on the rise time. However, since $T(0) = 1 \leq M$, large values of M can result in large overshoots.

The overshoot of the unit step response. The overshoot of the unit step response will be defined in the usual way

$$o_s = \text{max. } A(t) - 1.$$

If the maximum value of $A(t)$ occurs at time t_m then manipulation of Eq. (1) yields

$$o_s = -\frac{1}{2} + \frac{1}{\pi} \int_0^\infty T\left(\frac{x}{t_m - k}\right) \frac{\sin x}{x} \, dx. \tag{8}$$

In general, this equation cannot be used to obtain an overshoot since t_m the time of its occurrence is not known. However, a bound can be obtained. Since $|\sin x/x| \leq 1$, Eq. (8) leads to

$$o_s \leq -\frac{1}{2} + \frac{t_m - k}{\pi} \int_0^\infty T(\omega)\, d\omega.$$

Of course, this equation still contains the unknown time t_m. The bound attains its maximum value when $t_m - k$ is a maximum. Since the response of a linear phase network is symmetrical about 50% point, and in a physically realizable case is zero for $t < 0$, the maximum value of $t_m - k$ is k. Thus,

$$o_s \leq -\frac{1}{2} + \frac{k}{\pi} \int_0^\infty T(\omega)\, d\omega. \tag{9}$$

The Schwartz' inequality provides a second bound which may yield better results than the above if $\int_0^\infty T(\omega)d\omega$ is unduly large, or does not converge.

This is

$$o_s \leq -\frac{1}{2} + \left\{\frac{k}{2\pi} \int_0^\infty [T(\omega)]^2\, d\omega\right\}^{1/2}. \tag{10}$$

If $T(\omega)$ is such that $|\ \delta\ |$ given by relation (6) is negligible, so that the components of $T(\omega)$ for $\omega > \omega_c$ can be neglected and, in addition, $T(\omega)$ is bounded so that $T(\omega) \leq M$, then a bound on the overshoot which requires no integration can be obtained.

Equation (8) leads to

$$o_s = -\frac{1}{2} + \frac{1}{\pi} \int_0^{\omega_c(t_m - k)} T\left(\frac{x}{t_m - k}\right)[\sin x/x]\, dx.$$

To bound this expression, consider a comb like $T(\omega)$ such that $T[x/(t_m - k)] = 0$ if $\sin x < 0$ and $T[x/(t_m - k)] = M$ if $\sin x \geq 0$. The largest value of 0_s then occurs at $t_m - k\ |_{\max} = k$. Thus,

$$o_s \leq -\frac{1}{2} + \frac{M}{\pi}\{Si(\pi) + [Si(3\pi) - Si(2\pi)] + \cdots\}.$$

The final term in this series depends upon the numerical value of $\omega_c k$ so that the complete form of the bound may be stated in the following way.

If $n\pi \leq \omega_c k < (n + 1)\pi$, $n = 0, 1, 2, 3, \cdots$ and $T(\omega) \leq M$, the following bound on the overshoot is obtained.

$$\begin{aligned} o_s &\leq -\frac{1}{2} + \frac{M}{\pi} \sum_{g=1}^{n} (-1)^{g+1} Si(g\pi), \qquad n = 1, 3, 5, \cdots \\ o_s &\leq -\frac{1}{2} + \frac{M}{\pi}\left\{Si(\omega_c k) + \sum_{g=1}^{n} (-1)^{g+1} Si(g\pi)\right\}, \qquad n = 2, 4, \cdots \\ o_s &\leq -\frac{1}{2} + \frac{M}{\pi} Si(\omega_c k), \qquad n = 0. \end{aligned} \tag{11}$$

It can be shown that these are the best possible bounds by using the comb like transfer function discussed previously.

In order to aid in the computation of these bounds and to provide a means of rapidly estimating the overshoot the following table is included.

TABLE I

$\sum_{g=1}^{n} (-1)^{g+1} Si(g\pi)$ *for odd n*

n	$\sum_{g=1}^{n} (-1)^{g+1} Si(g\pi)$
1	1.8519
3	2.1085
5	2.2504
7	2.3484
9	2.4234
11	2.4840
13	2.5350
15	2.5789

It should be noted that if the term $|\delta|$ defined in relation (6) is not negligible, the bounds given by relation (10) can still be used if they are increased by $|\delta|$.

Conclusion. Two bounds have been placed upon the 10–90% rise time of phase distortion free networks. These are

$$r_t \geq \frac{4}{5}\pi\left[\int_0^\infty T(\omega)\,d\omega\right]^{-1}$$

and

$$r_t \geq \frac{16}{25}\pi\left\{\int_0^\infty [T(\omega)]^2\,d\omega\right\}^{-1}.$$

The following bounds have been placed upon the overshoot.

$$o_s \leq -\frac{1}{2} + \frac{k}{\pi}\int_0^\infty T(\omega)\,d\omega,$$

$$o_s \leq -\frac{1}{2} + \left\{\frac{k}{2\pi}\int_0^\infty [T(\omega)]^2\,d\omega\right\}^{1/2}.$$

The shape of the transfer characteristic will determine which of these bounds is the best and should be used.

If $T(\omega)$ falls off so that its frequency components can be neglected for $\omega > \omega_c$ and in addition, if $T(\omega) \leq M$, then bounds may be obtained which require no integration. These bounds are the best possible and are for the rise time

$$r_t \geq \frac{2}{\omega_c}\,\text{Inv.}\ Si\left[\frac{4\pi}{10M}\right]$$

and for the overshoot

$$o_s \leq -\frac{1}{2} + \frac{M}{\pi}\sum_{g=1}^{n}(-1)^{g+1}Si(g\pi), \qquad n = 1, 3, 5, \cdots$$

$$o_s \leq -\frac{1}{2} + \frac{M}{\pi}\left\{Si(\omega_c k) + \sum_{g=1}^{n}(-1)^{g+1}Si(g\pi)\right\}, \qquad n = 2, 4, 6, \cdots$$

$$o_s \leq -\frac{1}{2} + \frac{M}{\pi}\,Si(\omega_c k), \qquad n = 0,$$

where the constant n is determined from the relation

$$n\pi \leq \omega_c k < (n+1)\pi, \qquad n = 0, 1, 2, 3, \cdots .$$

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Appendix A. Application of the work of Zemanian to linear phase networks. Zemanian [2] has imposed certain restrictions on the unit step response of a physically realizable one port network in terms of the real part of the impedance function $R(\omega)$. These conditions, when modified slightly, apply to the amplitude function of linear phase networks.

Comparison of the expression for the unit step response of a network

$$A(t) = \frac{2}{\pi} \int_0^\infty \frac{R(\omega)}{\omega} \sin \omega t \, d\omega$$

with Eq. (1) indicates the similarity of the roles played by $R(\omega)$ and $T(\omega)$. Some examples of these dual roles will now be given. It has been shown by Zemanian that:

(1) If the unit step response is to be a monotonically increasing function of time, it is necessary that

$$R(0) \geq R(\omega) \quad \text{for all} \quad \omega.$$

(2) If $R(\omega)$ is a monotonically decreasing function of ω which approaches zero as ω approaches infinity, then the overshoot is bounded by

$$o_s \leq -1 + \frac{2}{\pi} Si(\pi) = 0.1790.$$

These may be written for the case of linear phase networks as

(1) If the unit step response is to be a monotonically increasing function of time, it is necessary that $T(0) \geq T(\omega)$ for all ω.

(2) If $T(\omega)$ is a monotonically decreasing function of ω which approaches zero as ω approaches infinity, then the overshoot is bounded by

$$o_s \leq -\frac{1}{2} + \frac{1}{\pi} Si(\pi) = 0.0895.$$

In a similar way other restrictions imposed upon the real part of an impedance function can be carried over to the amplitude function of linear phase networks.

References

1. P. M. Chirlian, *An investigation of linear phase networks*, New York University, Eng. Sc. D. thesis, Department of Electrical Engineering, May, 1956. Also, P. M. Chirlian, *Bounds on the error in the unit step response of a network*, Quart. Appl. Math. **16**, 432-435 (1959)
2. A. H. Zemanian, *Bounds existing on time and frequency responses of various types of networks*, Proc. IRE **42**, 835–839 (May 1954). Also, A. H. Zemanian, *Further bounds existing on the transient responses of various types of networks*, Proc. IRE **43**, 322-326 (Mar. 1955)