5. Maximum altitude. Consider now the case where motion takes place in a plane. Let the equations be

$$x'' = g(x', y'), \quad x(0) = 0, \quad x'(0) = c_1 , \tag{1}$$
$$y'' = h(x', y'), \quad y(0) = 0, \quad y'(0) = c_2 .$$

Introducing, as before, the function $f(c_1, c_2)$ equal to the maximum altitude, we see that

$$f(c_1, c_2) = (c_1^2 + c_2^2)^{1/2} \Delta + f[c_1 + g(c_1, c_2) \Delta, c_2 + h(c_1, c_2) \Delta] + o(\Delta). \tag{2}$$

Hence,

$$(c_1^2 + c_2^2)^{1/2} + g(c_1, c_2) \frac{\partial f}{\partial c_1} + h(c_1, c_2) \frac{\partial f}{\partial c_2} = 0. \tag{3}$$

Once again, let us assume that $c_2 = 0$ implies no vertical motion. Then $f(c_1, 0) = 0$ for $c_1 \geq 0$. It follows that we can again compute the solution by means of a sequence of functions of one variable.

6. Maximum range. To tackle the problem of maximum range directly requires the introduction of another state variable, the initial altitude. It can also be broken up into two problems, corresponding to the ascent to maximum altitude, and the descent.

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ON THE DETERMINATION OF CERTAIN THERMODYNAMIC AND PHYSICAL QUANTITIES*

By A. GLEYZAL (*U. S. Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland*)

We consider any physical phenomenon where a quantity z is a continuous differentiable function of two independent quantities x and y. Thus:

$$z = z(x, y).$$

Hence

$$dz = F\, dx + G\, dy,$$

where

$$F = F(x, y) = \frac{\partial z}{\partial x},$$

$$G = G(x, y) = \frac{\partial z}{\partial y}.$$

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Suppose furthermore that the quantities G and y are readily measured directly but F and x cannot be measured directly except that the family of curves $F = \text{const.}$ may be determined and at least two curves $x = \text{const.}$ may be found. Then the physical quantities F and x themselves may be determined as functions of G and y. We need merely assign a label F to each curve of the family of curves $F = \text{const.}$ in such a manner that the areas enclosed by the two curves $x = \text{const.}$ and two curves $F = \text{const.}$ are proportional to the increment ΔF of F. Then additional curves $x = \text{const.}$ may be constructed and x evaluated so that the area enclosed by the two curves $x = x_1$, $x = x_1 + \Delta x$, and two curves $F = F_1$ and $F = F_1 + \Delta F$ is equal to $\Delta F \Delta x$.

Proof. Given any exact differential

$$dz = F\,dx + G\,dy, \tag{1}$$

where F and G are functions of x and y, then, if the curves $F = \text{const.}$ and $x = \text{const.}$ are plotted in the G, y plane as shown in Fig. 1,
the area ΔA enclosed by the curves:

$$F = F_1\,, \qquad F = F_1 + \Delta F,$$

$$x = x_1\,, \qquad x = x_1 + \Delta x,$$

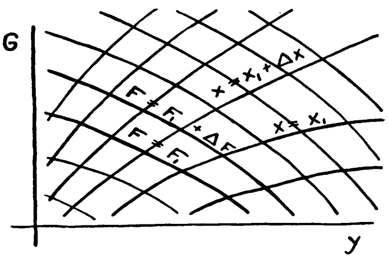

Fig. 1.

is equal to $\Delta F \Delta x$. For, integrating

$$\int_C dz = \int_C F\,dx + \int_C G\,dy = 0,$$

where C is the cycle which proceeds along curves $x = \text{const.}$ or $F = \text{const.}$ from x_1 , F_1 to x_1 , $F_1 + \Delta F$, to $x_1 + \Delta x$, $F_1 + \Delta F$, to $x_1 + \Delta x$, F_1 to x_1 , F_1 , we find:

$$0 = \Delta F\, \Delta x + \int_C G\,dy.$$

Therefore:

$$\int_C G\,dy = \Delta A = -\Delta F\, \Delta x.$$

These statements may be generalized to three or more variables and are themselves generalizations of, and render obvious, certain relationships among thermodynamic variables. Here, in the usual notation, since

$$dE = T\,dS - p\,dV, \tag{2}$$

we have

$$-\int_C p\,dV = \Delta A = -\Delta S\,\Delta T, \tag{3}$$

where C is a Carnot cycle which proceeds first along an adiabatic, then along an isothermal, then an adiabatic, and then along an isothermal to the starting point. We conclude that the family of isothermals $T = T_1 + n\Delta T$ and the family of adiabatics $S = S_1 + n\Delta S$, $n = 0, \pm 1, \pm 2, \pm 3, \cdots$ drawn in the p, V plane map out equal areas ΔA in this plane. Thus, if for any gas the isothermals $p = p_i(V)$ and the adiabatics $p = p_a(V)$ are determined by experiment, the "labels" S and T for the curves may then be determined for it is merely necessary to label as 0 the isothermal along which water freezes and as 100 the isothermal along which water boils at atmospheric pressure. The labels of the curves $T =$ const. are then determined by Eq. (3) uniquely. Any pair of curves may be labeled $S = 1$ and $S = 2$ along any isothermal. The unit of entropy is related to the unit of mechanical energy by resorting again to Eq. (3). The labels for the intervening curves $S =$ const. are also uniquely determined by Eq. (3). Entropy and absolute temperature S and T as thus determined for one gas must be consistent with S and T determined for any other gas due to the principle of conservation of energy. The extent to which the areas ΔA in the p, V plane are equal indicates the extent to which the postulated equation (2) is valid.

Correction to my paper

ON TRANSFER FUNCTIONS AND TRANSIENTS

Quarterly of Applied Mathematics, XVI, 273-294 (1958)

By A. H. ZEMANIAN (*New York University*)

Expression (38) should be replaced by its $(m - n)$th positive root and the second line below this expression should read, "horizontal line whose ordinate is the $(m - n)$th positive root of $(m - n)!$."

The second conclusion of Theorem 8 should be deleted and its proof adjusted such that zeros on the imaginary axis are counted with the zeros in the left half plane (i.e. the symbol q should be discarded and the expression $n - p - q$ should be replaced by $n - p$).