HEAT CONDUCTION FROM A CYLINDRICAL SOURCE WITH INCREASING RADIUS*

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Introduction. The theory of heat flow due to conduction from a moving heat source is of interest in a number of applications; for example, welding [8]**, heat exchangers [2, 3] and progressive freezing of a liquid [2, 3]. In this paper we consider heat conduction in an infinite homogeneous medium from the surface of a cylinder of finite length whose radius is increasing with time. This problem arises in connection with secondary oil recovery by an underground combustion process, e.g., [4; 7]. The restriction of the cylindrical source to a finite length corresponds to considering an oil reservoir of finite thickness and including vertical heat losses to media bounding the reservoir.

The differential equation describing this problem is written and the Greens function method is applied to obtain a solution in the form of an integral. A limiting value of this integral is then obtained for the case of the source moving at a constant velocity with no vertical losses. The problem is to evaluate a limit of the form $\lim_{t\to\infty} \int_0^t f(t, \tau)d\tau$ when $f(t, \tau)$ has an asymptotic representation, for $(t - \tau)/\tau$ sufficiently large, which can be integrated explicitly. For the integrand considered in Sec. 2, it is shown that the integral can be divided into two parts, namely

$$\int_0^t f(t, \tau)\, d\tau = \int_0^{t/N} f(t, \tau)\, d\tau + \int_{t/N}^t f(t, \tau)\, d\tau,$$

where the last integral goes to zero as $t \to \infty$ and the integrand in the range $[0, t/N]$ can be replaced by its asymptotic expression and evaluated in terms of N for $t \to \infty$. Finally, the desired limit is obtained by passing to the limit as $N \to \infty$.

In Sec. 3 an explicit evaluation of the integral solution is obtained for the case of the radius of the cylindrical source increasing at a variable velocity, namely $\dot{r}_F = V/r_F$. This result is obtained by showing that the solution in a transformed coordinate system is independent of time and thus the partial differential equation in the new coordinate system reduces to an ordinary differential equation which is solved explicitly.

1. A heat conduction problem. The partial differential equation in cylindrical coordinates for the temperature in a homogeneous conducting medium is given by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} - a^2 \frac{\partial T}{\partial t} = -\frac{1}{k}\Phi(z, r, t), \tag{1}$$

where r is the radius, t is time, a^2 is reciprocal diffusivity, k is the conductivity, Φ (z, r, t) is the source function and T is the temperature which is independent of the angular position, θ. The source is assumed to be at the surface, $r = r_F(t)$, of a cylinder with a fixed axis at $r = 0$ and height h as shown in figure below.

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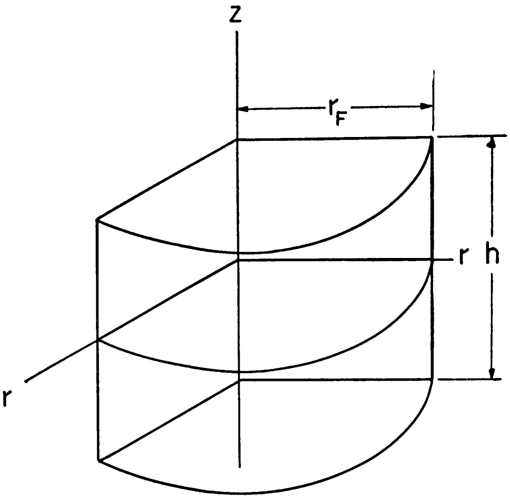

The intensity of the source is given by $L(t)$ in units of heat per unit of surface area per unit time. The source function, $\Phi(z, r, t)$, for this type of source may be expressed in terms of the Dirac delta function, $\delta(r - r_F)$, by the equation

$$\Phi(z, r, t) = L(t)B(z, h/2)\ \delta(r - r_F), \tag{2}$$

where $B(z, h/2) = 1$ for $|z| \leq h/2$, $= 0$ for $|z| > h/2$.

If we assume $T(z, r, 0) = 0$, then Eq. (1) has a solution of the form [see 1,7],

$$T(z, r, t) = (4\pi k)^{-1} \int_0^t dt_0 \int_{r_0=0}^{\infty} \int_{\theta_0=0}^{2\pi} \int_{z_0=-\infty}^{\infty} \Phi(z_0, r_0, t_0) G r_0\, dr_0\, d\theta_0\, dz_0\,, \tag{3}$$

where Φ is given by Eq. (2) and G is the Green's function corresponding to the left side of Eq. (1). G is given by the formula [see 1, 7],

$$G \equiv G(z, r, \theta, t \mid z_0, r_0, \theta_0, t_0)$$
$$= 2^{-1} a\pi^{-1/2}(t - t_0)^{-3/2} \exp\{-a^2[R^2 + (z - z_0)^2]/4(t - t_0)\},$$

where $R^2 = r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)$.

After performing the indicated integrations in Eq. (3) with respect to z_0, θ_0 and r_0 and making the substitution $t - t_0 = \tau$ we obtain

$$T(z, r, t) = (4k)^{-1} \int_0^t d\tau\, \tau^{-1} L r_F \exp\left[\frac{-a^2(r^2 + r_F^2)}{4\tau}\right] I_0\left(\frac{a^2 r r_F}{2\tau}\right)$$
$$\cdot\left[\operatorname{erf} \frac{a(h/2 + z)}{2\sqrt{\tau}} + \operatorname{erf} \frac{a(h/2 - z)}{2\sqrt{\tau}}\right],$$

where L and r_F are evaluated at $t - \tau$, $I_0(z)$ is a modified Bessel function defined in Sec. 2.1 and $\operatorname{erf} x = 2\pi^{-1/2} \int_0^x \exp(-\alpha^2)d\alpha$.

A reasonable assumption in an underground combustion process is that $L = q\dot{r}_F$ where q is a constant. Two cases of this problem are of particular interest in underground combustion, namely: Case I = constant radial velocity, $r_F = Ut$, and Case II = frontal radius given by $r_F^2 = 2Vt$. The solutions for these two cases may be obtained from the above equation; and for no vertical losses, i.e. $h \to \infty$, are given by Eqs. (4) and (5) respectively.

$$T(r, t) = (2k)^{-1}qU^2 \int_0^t d\tau(t - \tau)\tau^{-1} \exp\left\{\frac{-a^2[r^2 + U^2(t - \tau)^2]}{4\tau}\right\} I_0\left[\frac{a^2 r U(t - \tau)}{2\tau}\right] \quad (4)$$

$$T(r, t) = (2k)^{-1}qV \int_0^t d\tau\, \tau^{-1} \exp\left\{\frac{-a^2[r^2 + 2V(t - \tau)]}{4\tau}\right\} I_0\left\{\frac{a^2 r[2V(t - \tau)]^{1/2}}{2\tau}\right\}. \quad (5)$$

In Sec. 2 a steady state solution of (4) is obtained, that is, the $\lim_{t\to\infty} T(r, t)$, assuming that $s = r - Ut$ remains finite as $t \to \infty$. Thus a steady state solution for r positions near the source is determined; this has been called quasi-stationary state [see 6]. In Sec. 3 an explicit evaluation of the solution for Case II is obtained.

2. A steady state solution for the constant velocity case. 2.1. *Preliminary Considerations.* In this section we prove a lemma, state some known properties of $I_0(z)$, and discuss an explicit evaluation of an integral. These results will be needed in Sec. 2.2.

Lemma. Let $f(t)$ be a real function of the real variable t satisfying the conditions (α) $0 < f_2(N) \le \lim_{t\to\infty} f(t) \le f_1(N)$ for all finite N, and

$$(\beta)\ \lim_{N\to\infty} f_1(N) = \lim_{N\to\infty} f_2(N) = P, \quad \text{then} \quad \lim_{t\to\infty} f(t) = P.$$

Proof. By (β) there exists an N_0 such that $| f_1(N) - P | < \epsilon$ and $| f_2(N) - P | < \epsilon$ for all $N > N_0$. Combining this with (α) we have $P - \epsilon < \lim_{t\to\infty} f(t) < P + \epsilon$ and by choosing N_0 sufficiently large we can make ϵ small and the lemma is proved.

The modified Bessel function, $I_0(z)$, is defined by $I_0(z) = (2\pi)^{-1} \int_0^{2\pi} e^{z\cos\theta} d\theta$. $I_0(z)$, for real z, can be represented for large values of z by its asymptotic series $I_0(z) \sim (2\pi z)^{-1/2} e^z [1 + 0(1/z)]$, and thus

$$\lim_{z\to\infty} (2\pi z)^{1/2} e^{-z} I_0(z) = 1. \quad (6)$$

Since $e^{-z}I_0(z)$ is a continuous function of z and $\lim_{z\to\infty} e^{-z}I_0(z) = 0$, then $e^{-z}I_0(z)$ is bounded for $z \ge z_0$ for any z_0.

We define a function $g(z)$ by the equation

$$e^{-z}I_0(z) = (2\pi z)^{-1/2}[1 + g(z)], \qquad z \ne 0. \quad (7)$$

And if we define $g(0) = -1$, then $g(z)$ is a continuous function of z. Finally from (6) and (7) we have that $\lim_{z\to\infty} g(z) = 0$.

An explicit evaluation for the integral,

$$\psi(u, v, t) = \int_0^t x^{-1/2} \exp(-u^2/x) \exp(-v^2 x)\, dx, \qquad u \ne 0,$$

has been given by W. Horenstein [5]. The result is

$$\psi(u, v, t) = (2v)^{-1}\pi^{1/2}[-2\sinh(2\,|u|\,v) + e^{2|u|v} \operatorname{erf}(vt^{1/2} + |u|\,t^{-1/2}) \\ + e^{-2|u|v} \operatorname{erf}(vt^{1/2} - |u|\,t^{-1/2})]. \quad (8)$$

The result of Horenstein does not include the absolute values signs on the u's and these must be added to make the formula correct for $u < 0$, thus we must have $\psi(u, v, t) = \psi(-u, v, t)$. In the application of this formula in Sec. 2.2, the case $u < 0$ corresponds to positions inside the cylinder, i.e. $s < 0$. Passing to the limit as $t \to \infty$ in (8) we obtain

$$\lim_{t\to\infty} \psi(u, v, t) = v^{-1}\pi^{1/2}e^{-2|u|v}. \tag{9}$$

2.2. *Evaluation of a limit.* Equation (4) may be written in the form

$$T = (2k)^{-1}qU^2I, \tag{10}$$

where

$$I = \int_0^t d\tau(t - \tau)\tau^{-1} \exp\{-a^2[r - U(t - \tau)]^2/4\tau\} \exp[-a^2rU(t - \tau)/2\tau] \cdot I_0[a^2rU(t - \tau)/2\tau]$$

and putting $s = r - Ut$, $U > 0$, we obtain

$$I = \int_0^t d\tau(t - \tau)\tau^{-1} \exp[-a^2(s + U\tau)^2/4\tau] \exp[-z(t, \tau)]I_0[z(t, \tau)], \tag{11}$$

where

$$z(t, \tau) = (2\tau)^{-1}a^2U(s + Ut)(t - \tau). \tag{12}$$

We shall temporarily assume $s \neq 0$ and in this case the integrand in (11) is bounded in $0 \leq \tau \leq t$, since for $\tau \neq 0$ the integrand is a product of continuous functions of τ and the integrand approaches zero as $\tau \to 0$.

I may be divided into two parts, $I = I_1 + I_2$, where $I_1 = \int_0^{t/N} f(t, \tau)d\tau$ and $I_2 = \int_{t/N}^t f(t, \tau)d\tau$ with $f(t, \tau)$ written for the integrand in (11), where N is chosen > 1. For $0 < t/N \leq \tau \leq t$, we have $0 \leq (t - \tau)/\tau \leq (t - t/N)/(t/N) = N - 1$. Then it follows from Eq. (12), with $U > 0$ and s finite that $z(t, \tau)$ is bounded below for $0 < t/N \leq \tau < t$ and $N > 1$, say $z(t, \tau) \geq z_0$ and thus, by Sec. 2.1, that $e^{-z(t,\tau)}I_0[z(t, \tau)]$ is bounded. Hence there exists an M such that $0 < e^{-z(t,\tau)}I_0[z(t, \tau)] \leq M$ for $0 < t/N \leq \tau \leq t$, $N > 1$. The left side of the above inequality is true since $I_0(z) \geq 1$ for all z. Using the above evaluations we can now evaluate I_2. I_2 can be written in the form

$$I_2 = \int_{t/N}^t d\tau\, \tau^{-1}(t - \tau) \exp(-a^2s^2/4\tau) \exp(-a^2sU/2) \cdot \exp(-a^2U^2\tau/4) \exp[-z(t, \tau)]I_0[z(t, \tau)]$$

and thus

$$\begin{aligned} 0 \leq I_2 &\leq (N - 1)M \exp(-a^2sU/2) \int_{t/N}^t \exp(-a^2U^2\tau/4)\, d\tau \\ &= (N - 1)M \exp(-a^2sU/2)(-4)(aU)^{-2}[\exp(-a^2U^2t/4) - \exp(-a^2U^2t/4N)] \end{aligned}$$

and thus for $t \to \infty$ and for all finite N we have

$$\lim_{t\to\infty} I_2(t, N) = 0. \tag{13}$$

If we replace $e^{-z(t,\tau)} I_0[z(t, \tau)]$ in I_1 by its equivalent given in Eq. (7) we obtain

$$\begin{aligned} I_1 &= \int_0^{t/N} d\tau(t - \tau)\tau^{-1} \exp[-a^2(s + U\tau)^2/4\tau]\{1 + g[z(t, \tau)]\}[2\pi z(t, \tau)]^{-1/2} \\ &= [a^2U\pi(s + Ut)]^{-1/2} \int_0^{t/N} d\tau(t - \tau)^{1/2}\tau^{-1/2} \\ &\qquad \cdot \exp[-a^2(s + U\tau)^2/4\tau]\{1 + g[z(t, \tau)]\}, \end{aligned} \tag{14}$$

where $z(t, \tau)$ in the last factor of the integrand of the first equation of (14) has been replaced by its expression in Eq. (12). If we define

$$h(t, \tau) = (t - \tau)^{1/2}\tau^{-1/2} \exp[-a^2(s + U\tau)^2/4\tau], \qquad h(t, 0) = 0 \quad \text{and} \quad g[z(t, 0)] = 0,$$

then the integrand of the second equation of (14) can be written as the product of $h(t, \tau)$ and $1 + g[z(t, \tau)]$, where both factors are continuous functions of τ; and $h(t, \tau) \geq 0$ for $0 \leq \tau \leq t/N$ and for all $t \geq 0$. Hence we may apply the first mean value theorem for integrals to the integral in the second equation of (14) and obtain

$$I_1 = [a^2U\pi(s + Ut)]^{-1/2}\{1 + g[z(t, \xi)]\} \int_0^{t/N} d\tau(t - \tau)^{1/2}\tau^{-1/2} \exp[-a^2(s + U\tau)^2/4\tau],$$

where $0 \leq \xi \leq t/N$. For $0 \leq \tau \leq t/N$ we have $(t)^{1/2}(1 - 1/N)^{1/2} \leq (t - \tau)^{1/2} \leq (t)^{1/2}$ and thus

$$\begin{aligned} I_1 &\leq t^{1/2}[a^2U\pi(s + Ut)]^{-1/2}\{1 + g[z(t, \xi)]\}J(t, N) \\ I_1 &\geq t^{1/2}(1 - 1/N)^{1/2}[a^2U\pi(s + Ut)]^{-1/2}\{1 + g[z(t, \xi)]\}J(t, N), \end{aligned} \tag{15}$$

where

$$J(t, N) = \int_0^{t/N} d\tau\, \tau^{-1/2} \exp[-a^2(s + U\tau)^2/4\tau].$$

For $0 \leq \xi \leq t/N$, we have $(t - \xi)/\xi \geq (t - t/N)/(t/N) = N - 1$; then

$$z(t, \xi) = 2^{-1}a^2U(s + Ut)(t - \xi)/\xi \geq 2^{-1}a^2U(s + Ut)(N - 1),$$

and thus $\lim_{t\to\infty} z(t, \xi) = \infty$ and finally we have $\lim_{t\to\infty} g[z(t, \xi)] = \lim_{z\to\infty} g(z) = 0$. If we now pass to the limit as $t \to \infty$ in (15) we obtain

$$(1 - 1/N)^{1/2}(aU)^{-1}\pi^{-1/2} \lim_{t\to\infty} J(t, N) \leq \lim_{t\to\infty} I_1 \leq (aU)^{-1}\pi^{-1/2} \lim_{t\to\infty} J(t, N) \tag{16}$$

and evaluating $\lim_{t\to\infty} J(t, N)$, we have

$$\begin{aligned} \lim_{t\to\infty} J(t, N) &\equiv \lim_{t\to\infty} \int_0^{t/N} d\tau\, \tau^{-1/2} \exp[-a^2(s + U\tau)^2/4\tau] \\ &= \exp(-a^2sU/2) \int_0^{\infty} d\tau\, \tau^{-1/2} \exp(-a^2s^2/4\tau) \exp(-a^2U^2\tau/4) \end{aligned}$$

and from Eq. (9) we have

$$\lim_{t\to\infty} J(t, N) = \exp(-a^2sU/2)2(aU)^{-1}(\pi)^{1/2} \exp(-a^2 \mid s \mid U/2). \tag{17}$$

Combining (13), (16) and (17), we have

$$\begin{aligned} (1 - 1/N)^{1/2}2(aU)^{-2} \exp[-a^2U(s + \mid s \mid)/2] &\leq \lim_{t\to\infty} (I_1 + I_2) \\ &\leq 2(aU)^{-2} \exp[-a^2U(s + \mid s \mid)/2] \end{aligned}$$

and the above inequality is of the form discussed in the lemma of Sec. 2.1 and thus we may pass to the limit as $N \to \infty$ and obtain

$$\lim_{t\to\infty} I = 2(aU)^{-2} \exp[-a^2U(s + \mid s \mid)/2]. \tag{18}$$

In the above calculations we have assumed $s \neq 0$; if $s = 0$, we may replace the lower limit of the integral in Eq. (11) by ϵ, repeat the calculations and then pass to the limit as $\epsilon \to 0$; and we obtain (18) for the case $s = 0$.

If we combine Eqs. (8) and (18) we obtain for $t \to \infty$

$$\lim_{t\to\infty} T \equiv 2k^{-1}qU^2 \lim_{t\to\infty} \int_0^t (t - \tau)\tau^{-1} \cdot \exp\{-a^2[r^2 + U^2(t-\tau)^2]/4\tau\} I_0[a^2rU(t-\tau)/2\tau]\, d\tau \tag{19}$$
$$= qk^{-1}a^{-2} \exp[-a^2U(s + |s|)/2].$$

The above result is the same as the corresponding known result, [see 6], for a plane source moving with a constant velocity. This would be expected since the cylindrical source approaches a plane source for large radii.

3. An explicit solution for Case II. Substituting $x = r/r_F$ and $\tau = \lambda t$ in Eq. (5) gives

$$T(x, t) = (2k)^{-1}qv \int_0^1 \frac{d\lambda}{\lambda} \exp\left[\frac{-a^2V(x^2 + 1 - \lambda)}{2\lambda}\right] I_0\left[\frac{a^2Vx(1-\lambda)^{1/2}}{\lambda}\right]. \tag{20}$$

Thus T, when written as a function of x, is independent of t. The partial differential equation, Eq. (1), for the corresponding case may be written in the form

$$\frac{\partial^2 T}{\partial x^2} + \left(\frac{1}{x} + a^2Vx\right)\frac{\partial T}{\partial x} - 2a^2Vt\frac{\partial T}{\partial t} = 0, \tag{21}$$

where we have made the substitution $x = r/r_F$. The term $\partial^2T/\partial z^2 = 0$, since, for the case $h \to \infty$, T is independent of z. The source term is replaced by its equivalent condition, namely

$$\left.\frac{\partial T}{\partial x}\right|_{x=1+} - \left.\frac{\partial T}{\partial x}\right|_{x=1-} = -(k)^{-1}r_FL(t) = -qV/k, \tag{22}$$

where

$$\left.\frac{\partial T}{\partial x}\right|_{x=1+}, \quad \left.\frac{\partial T}{\partial x}\right|_{x=1-}$$

are the right and left hand derivatives, respectively, at $x = 1$.

Due to symmetry no temperature gradient exists at $r = 0$, also $T \to 0$ as $r \to \infty$. The corresponding boundary conditions on $T(x, t)$ are

$$\frac{\partial T}{\partial x}(0, t) = 0; \qquad \lim_{x\to\infty} T(x, t) = 0. \tag{23}$$

Thus Eq. (20) is a solution of Eq. (21) subject to conditions (22) and (23). Since the solution (20) is independent of t, then $\partial T/\partial t = 0$ and Eq. (21) reduces to the ordinary differential equation

$$x\frac{d^2T}{dx^2} + (1 + a^2vx^2)\frac{dT}{dx} = 0. \tag{24}$$

Equation (24) can be integrated giving the two solutions

$$T = \text{constant}$$

$$T = C_1 \int_{C_2}^{x} \frac{dz}{z} \exp\,[-a^2 V z^2/2].$$

To satisfy the boundary condition, $dT/dx = 0$ for $x = 0$, we must choose the solution $T = \text{constant}$ for $0 \le x \le 1$; thus the boundary condition corresponding to (22) becomes

$$\left.\frac{dT}{dx}\right|_{x=1+} = \frac{-qV}{k}$$

and the solution in the range $x \ge 1$ is given by

$$\begin{aligned} T &= -qV(k)^{-1} \exp\,(a^2V/2) \int_{\infty}^{x} d(z)(z)^{-1} \exp\,(-a^2Vz^2/2) \\ &= -(2k)^{-1} qV \exp\,(a^2V/2)\, Ei(-a^2Vx^2/2), \qquad x \ge 1 \end{aligned} \tag{25}$$

where C_1 and C_2 have been chosen so that (25) will satisfy the boundary conditions at $x = 1$ and $x \to \infty$. Finally, the value of the constant temperature in the range $0 \le x \le 1$ is obtained by requiring that the solution be continuous at $x = 1$ thus

$$T = -(2k)^{-1} qV \exp\,(a^2V/2) Ei(-a^2V/2), \qquad 0 \le x \le 1 \tag{26}$$

where Ei is the exponential integral, $-\,Ei(-x) = \int_x^\infty e^{-z} z^{-1} dz$. Returning to the r coordinate, by putting $r = xr_F(t)$ in the solution (25) and (26), gives the desired explicit solution for Case II.

Since this solution is unique, we may equate the solution (20) to the solution (25), (26) and obtain the following interesting evaluation, with $a^2V/2 = R$,

$$\int_0^1 \frac{d\lambda}{\lambda} \exp\left[\frac{-R(x^2+1)}{\lambda}\right] \cdot I_0\left[\frac{2Rx(1-\lambda)^{\frac{1}{2}}}{\lambda}\right] = \begin{cases} -Ei(-R), & |\,x\,| \le 1 \\ -Ei(-Rx^2), & |\,x\,| \ge 1. \end{cases} \tag{27}$$

References

1. H. S. Carslaw and J. C. Jaeger, *Operational methods in applied mathematics,* Oxford Press, 1941
2. J. Crank, *The mathematics of diffusion,* Oxford Press, 1956
3. P. V. Dankwerts, *Unsteady state diffusion or heat conduction with a moving boundary,* Trans. Faraday Soc. **47**, 701–12 (1950)
4. Don V. Hester and D. E. Menzie, *Development of subsurface combusion drive,* Petroleum Eng. **26**, B-82, B-87-92 (Nov. 1954)
5. W. Horenstein, *On certain integrals in the theory of heat conductions,* Quart. Appl. Math. **3**, 183–84 (1945)
6. Max Jacob, *Heat transfer,* vol. I, Wiley & Sons, New York, 1949, p. 343–52
7. P. M. Morse and H. Feshbach, *Methods of theoretical physics,* McGraw-Hill, 1953, New York, p. 857–95
8. D. Rosenthal, *The theory of moving sources of heat and its application to metal treatments,* Trans. Am. Soc. Mech. Engrs. **68**, 849–65 (1946)