THE EXACT SOLUTION OF BORDA'S MOUTHPIECE IN TWO DIMENSIONS*

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The solution of the semi-infinite two-dimensional Borda mouthpiece is well known. Its coefficient of contraction σ is 0.5. The solution of the finite mouthpiece seems to have been avoided, apparently because it involves elliptic integrals.

The finite Borda mouthpiece, Fig. 1, is formed by two walls DC and $D'C$ of length b

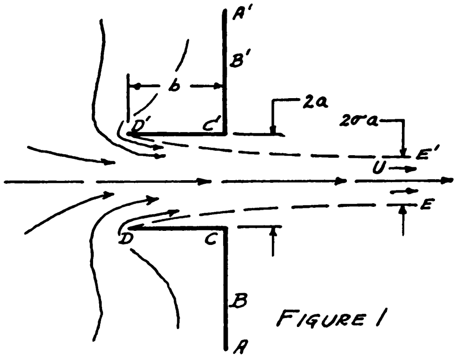

FIGURE 1

projecting into a semi-infinite reservoir which is bounded by the semi-infinite walls $A'B'C'$ and ABC having a gap of width $2a$ between them. Inviscid incompressible fluid flows out of the reservoir through the mouthpiece to form a jet which is bounded by the free stream lines $D'E'$ and DE. The jet contracts to the width $2\sigma a$ at $E'E$ far from the mouthpiece where the speed of the fluid is uniform and of value U. The total efflux from the reservoir is therefore $2\sigma aU$. The speed of the fluid along the free stream lines $D'E'$ and DE is U. Along the wet side of the mouthpiece and reservoir walls, $A'B'C'D'$ and $ABCD$, the surface speed varies. It is zero at the corners C' and C and at the infinite points A' and A. At two places, B' and B, the surface speed has a maximum. This follows because B is between A and C where the surface speed is zero.

The values of the coefficient of contraction σ and the maximum surface speed with its location are of interest.

Because the flow is in the general category of "potential flow", the techniques employing conformal mapping are applicable. This principle of analysis is not new.

Starting with the lower half of the symmetrical geometry of Fig. 1 redrawn on the z-plane of Fig. 2a (the mapping figures are grouped together on a later page) define: the complex potential $P = \phi + \psi i$, where ϕ is the potential function, and ψ is the stream function, so that the z-plane velocity $V_z = V_x + V_y i = -(dP/dz)^*$ where the * means conjugate.

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A pair of mapping sequences, one set involving the potential P, the other the potential derivative dP/dz, is formed to coalesce finally, establishing the desired formulas. This procedure is indicated by the diagram, Figs. 2a through 2e, indicating the sequential transformations.

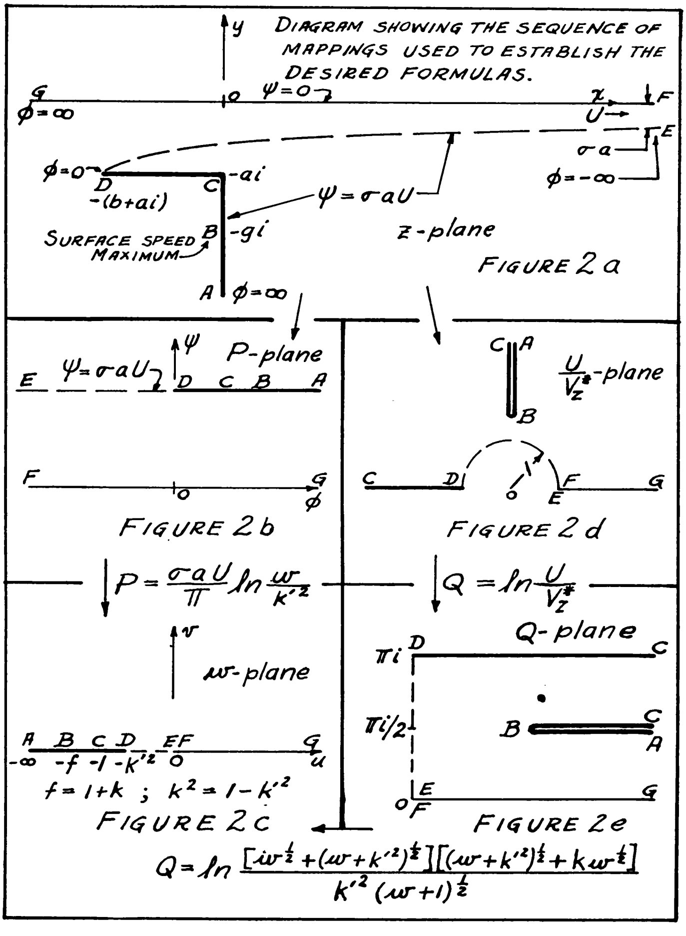
The potential sequence is started by assigning to the potential function ϕ and to the stream function ψ values which are consistent with the existing conditions at specific places in the z-plane:

Along the left infinite semi-circle $\phi = \infty$ (including points A and G), at the point D $\phi = 0$, at points E, F $\phi = -\infty$, along the stream line $ABCDE$ $\psi = \sigma a U$, along the stream line FG $\psi = 0$.

These values are then plotted to form the P-plane map of Fig. 2b. The P-plane strip is then mapped onto the upper half of the w-plane of Fig. 2c by means of the Schwarz-Christoffel transformation

$$dP/dw = \sigma a U/\pi w; \qquad P = (\sigma a U/\pi) \ln (w/k'^2). \tag{1}$$

The choice of -1 and $-k'^2$ as the w-plane locations of C and D respectively is made in the light of subsequent developments.



The potential derivative (velocity) sequence of mappings is started by utilizing certain known facts about the velocity in the z-plane. Rather than mapping the potential derivative, the reciprocal of the conjugate velocity normalized with respect to the terminal velocity U of the jet is mapped. This function maps into a simple geometry. Because

$$V_z^* = -dP/dz = q \exp(-\theta i)$$

where q is the speed and θ is the direction of the velocity,

$$U/V_z^* = (U/q) \exp(\theta i).$$

Along the rigid walls and the stream line of symmetry the direction, θ, is dictated by these boundaries. Along the free stream line the speed q is constant. The values of U/V_z^* along the boundaries can then be tabulated:

Along	ABC	CD	DE	FG
$U/V_Z^* =$	$(U/q)\exp(\pi i/2)$	$(U/q)\exp(\pi i)$	$\exp(\theta i)$	U/q
with	$\infty > U/q > U/q_B$	$\infty > U/q > 1$	$\pi > \theta > 0$	$\infty > U/q > 1$

These values are then plotted to form the U/V_z^*-plane map of Fig. 2d.

The upper half of the U/V_z^*-plane excluding the unit semi-circle is then mapped onto the semi-infinite strip in the Q-plane of Fig. 2e by means of the transformation

$$Q = \ln(U/V_z^*). \tag{2}$$

Finally the two mapping sequences are coalesced by mapping the Q-plane strip onto the upper half of the w-plane of Fig. 2c applying the Schwarz-Christoffel transformation

$$dQ/dw = N(w + f)/[w^{\frac{1}{2}}(w + k'^2)^{\frac{1}{2}}(w + 1)].$$

Both N and f are evaluated by integrating between G to A and C to C in the Q-plane and along the corresponding semi-circles in the w-plane. The results are:

$$N = \tfrac{1}{2} \qquad f = 1 + k \quad \text{with} \quad k^2 = 1 - k'^2.$$

Putting these into the expression for dQ/dw, integrating and making appropriate adjustment for correspondence of points, there results the Q to w-plane transformation

$$Q = \ln \frac{[w^{\frac{1}{2}} + (w + k'^2)^{\frac{1}{2}}][(w + k'^2)^{\frac{1}{2}} + kw^{\frac{1}{2}}]}{k'^2(w + 1)^{\frac{1}{2}}}. \tag{3}$$

The connection between the w- and z-plane is effected by substituting the potential derivative for the velocity in Eq. (2) and using the potential derivative from Eq. (1)

$$Q = \ln \frac{U}{V_z^*} = \ln\left(-U\frac{dz}{dP}\right) = \ln\left(-U\frac{dz}{dw}\frac{dw}{dP}\right) = \ln\left(-\frac{\pi w}{\sigma a}\frac{dz}{dw}\right). \tag{4}$$

Equating the arguments of the logarithms in Eqs. (3) and (4) establishes, after rearrangement, the z to w-plane transformation in derivative form

$$-\frac{\pi}{\sigma a}\,dz = \frac{[w^{\frac{1}{2}} + (w + k'^2)^{\frac{1}{2}}][(w + k'^2)^{\frac{1}{2}} + kw^{\frac{1}{2}}]}{k'^2 w(w + 1)^{\frac{1}{2}}}\,dw. \tag{5}$$

Equation (5) can be integrated to form the z to w-plane transformation. The result involves elliptic integrals of modulus k or k' with arguments that are complex, real or imaginary depending on the locations of the points in the z and w-plane. It is much easier to integrate Eq. (5) between limits which correspond to specific points, arranging the integrand in each case so that the resulting elliptic integrals have real arguments. Three such integrations suffice to establish the relationships among the parameter k, the mouthpiece length (b/a), the coefficient of contraction σ, and the location of the maximum surface speed (g/a).

The relation between the mouthpiece length (b/a) and the parameter k is obtained by integrating Eq. (5) between limits corresponding to the locations of points C and D in the z- and w-planes

$$-\frac{\pi}{\sigma a}\int_{-ai}^{-(b+ai)} dz = \frac{1+k}{k'^2}\int_{-1}^{-k'^2} \frac{w^{\frac{1}{2}}\,dw}{[(w + k'^2)(w + 1)]^{\frac{1}{2}}} + (1 + k)$$

$$\cdot\int_{-1}^{-k'^2} \frac{dw}{[w(w + k'^2)(w + 1)]^{\frac{1}{2}}} + \frac{1+k}{k'^2}\int_{-1}^{-k'^2} \frac{dw}{(w + 1)^{\frac{1}{2}}} + \int_{-1}^{-k'^2} \frac{dw}{w(w + 1)^{\frac{1}{2}}}.$$

After performing the indicated integrations the results are arranged to

$$\frac{\pi b}{2\sigma a} = \frac{k}{1 - k} + \frac{E(k) - k'^2 K(k)}{1 - k} - \tfrac{1}{2}\ln\frac{1 + k}{1 - k}. \tag{6}$$

Integrating between limits corresponding to the locations of points D and E establishes the relation between the coefficient of contraction σ and the parameter k. Here the infinite value of x is avoided by approaching point E in the z-plane:

$$\lim_{x\to\infty} -\frac{\pi}{\sigma a}\int_{-(b+ai)}^{(x-ai)} dz = \frac{1+k}{k'^2}\int_{-k'^2}^{0} \frac{(w + 1)^{\frac{1}{2}}\,dw}{[w(w + k'^2)]^{\frac{1}{2}}} - \frac{k^2(1 + k)}{k'^2}$$

$$\cdot\int_{-k'^2}^{0} \frac{dw}{[w(w + k'^2)(w + 1)]^{\frac{1}{2}}} + \frac{1+k}{k'^2}\int_{-k'^2}^{0} \frac{dw}{(w + 1)^{\frac{1}{2}}} + \int_{-k'^2}^{0} \frac{dw}{w(w + 1)^{\frac{1}{2}}}.$$

The real parts are obviously infinite and so are ignored. The imaginary parts yield the desired formula:

$$\frac{\pi}{2\sigma} = (1 + k)\,\frac{E(k') - k^2 K(k')}{k'^2} + \frac{\pi}{2}. \tag{7}$$

Eliminating σ from Eqs. (6) and (7) establishes the relationship between (b/a) and the parameter k

$$\frac{b}{a} = \frac{k + E(k) - k'^2 K(k) - [(1 - k)/2]\ln[(1 + k)/(1 - k)]}{E(k') - k^2 K(k') + \pi(1 - k)/2}. \tag{8}$$

It is interesting to note that $k = 1$, $k' = 0$ corresponds to $b/a = \infty$, the semi-infinite Borda mouthpiece. Putting these values in Eq. (7) yields $\sigma = 1/2$, the well known coefficient for this mouthpiece. Similarly $k = 0$, $k' = 1$ corresponds to $b/a = 0$, the mouthpiece vanishes and discharge is through the remaining slit in the reservoir wall. Equation (7) now yields $\sigma = 1/(1 + 2/\pi)$, the known value for discharge through a slit.

The location of the maximum surface speed along the reservoir wall is found by integrating (5) between limits corresponding to the locations of points B and C

$$-\frac{\pi}{\sigma a}\int_{-gi}^{-ai} dz = \frac{-(1+k)i}{k'^2}\int_{-(1+k)}^{-1}\frac{(-w)^{\frac{1}{2}}\,dw}{[(-k'^2-w)(-1-w)]^{\frac{1}{2}}} + (1+k)i \cdot\int_{-(1+k)}^{-1}\frac{dw}{[-w(-k'^2-w)(-1-w)]^{\frac{1}{2}}} - \frac{(1+k)i}{k'^2}\cdot\int_{-(1+k)}^{-1}\frac{dw}{(-1-w)^{\frac{1}{2}}} - i\int_{-(1+k)}^{-1}\frac{dw}{w(-1-w)^{\frac{1}{2}}}$$

and

$$\frac{\pi(g-a)}{2\sigma a} = \frac{(1+k)[k^2K(k')-E(k')]}{k'^2} - \tan^{-1}k^{\frac{1}{2}} + \frac{1+k^{\frac{1}{2}}}{2(1-k^{\frac{1}{2}})}. \tag{9}$$

Equation (9) with (7) determines the value of g/a in terms of the parameter k and so the relative location of the maximum surface speed.

The value of the maximum surface speed is found using the potential derivative in terms of the transformation derivatives of Eqs. (1) and (5) evaluated for $w = -(1+k)$, the w-plane location of point B:

$$V_z^* = -\frac{dP}{dz} = -\frac{dP}{dw}\frac{dw}{dz}.$$

Substituting from Eqs. (1) and (5) and making $w = -(1+k)$

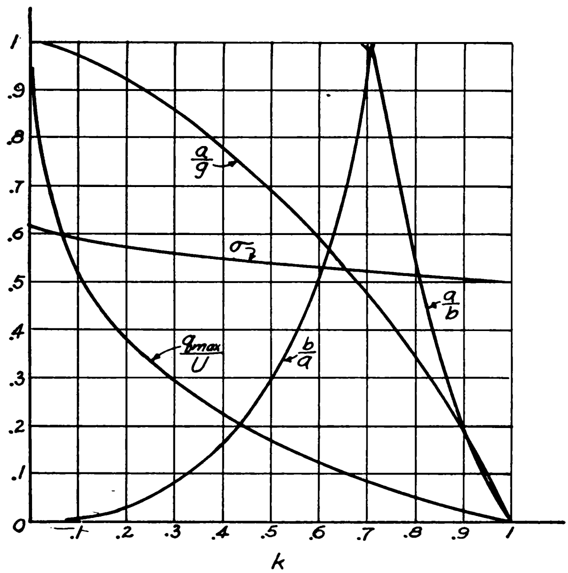


FIG. 3. Graphical representation of Equations (7), (8), (9), and (10) showing the relations of σ, the coefficient of contraction; b/a, the mouthpiece length; g/a, the location of the maximum surface speed; and q_{max}/U, the maximum surface speed, to the parameter k. $2a$ is the width of the mouthpiece. U is the terminal speed of the jet. The values on any ordinate correspond. See Figures 1 and 1a.

$$V_z = \frac{1 - k^{\frac{1}{2}}}{1 + k^{\frac{1}{2}}} Ui, \tag{10}$$

the velocity maximum at point B.

Eqs. (6), (7), (9) and (10) constitute the desired relationships. The results are shown graphically in Fig. 3.

For computation purposes it is convenient to introduce into Eqs. (6), (7) and (9)

$$B = \frac{E(k) - k'^2 K(k)}{k^2}$$

which is tabulated in *Tables of functions* by Jahnke and Emde, Dover Publications, 1945.

The following formulas from *Handbook of elliptic integrals for Engineers and Physicists*, Byrd and Friedman, Springer, 1945, were used for evaluating the various elliptic integrals:

Equation	Formula number
(6)	233.00, 233.01, 110.06, 1110.07
(7)	236.00, 236.01, 110.06, 110.07
(9)	232.06, 321.02, 232.00, 111.03, 122.10

References

1. L. M. Milne-Thomson, *Theoretical hydrodynamics*, Macmillan Co., 3rd ed. 1955
2. E. Weber, *Electro-magnetic fields*, Wiley 1950—Secs. 25-27, vol. 1