ON A FREE BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION

BY

WALTER T. KYNER*

University of Southern California

1. Introduction. W. L. Miranker [1] recently published an existence theorem for a free boundary value problem for the heat equation. Using a method due to I. Kolodner [2], he obtained a functional equation for the free boundary function $R(t)$ and showed that the existence of a solution to the functional equation implied the existence of the solution to the free boundary problem. He then solved the functional equation by an iterative method.

The mathematical problem which Miranker solved represents the heating of a long insulated metal rod which has begun to melt at one end ($x = 0$) and *after* a layer of liquid metal A units thick has formed, heat is applied at $x = 0$. The layer of liquid metal is assumed to have an initial temperature distribution $f(x)$. It is essential for Miranker's formalism that A be positive and that $df(A)/dx$ be negative. Physically, this means that the front separating the liquid and solid metal must be moving before the mathematical model applies. The purpose of this paper is to present a constructive existence and uniqueness theorem which is not subject to this restriction.

The problem is to determine two functions $u(x, t)$ and $R(t)$ satisfying the following:

$$\begin{aligned} &u_{xx} = u_t \,, \qquad 0 < x < R(t), \qquad 0 < t, \\ &u_x(0, t) = -g(t), \qquad 0 < t, \\ &u(R(t), t) = -dR(t)/dt, \qquad 0 < t, \\ &R(0) = A, \\ &u(x, 0) = f(x),\ 0 < x < A, \end{aligned} \tag{1.1}$$

where g is a positive continuous function, and f is a continuous function such that, for some constant b,

$$0 \leq f(x) \leq b(A - x), \qquad 0 \leq x \leq A. \tag{1.2}$$

Miranker required that f be continuously differentiable, non-negative, and that

$$df(0)/dx = -g(0), \qquad df(A)/dx < 0, \tag{1.3}$$

the latter condition being essential for his proof. Although it was not stated explicitly, it follows from his conclusion that u_x is continuous on the boundary** that f must vanish at $x = A$. Hence Miranker's initial value function satisfies (1.2).

In 1951, G. W. Evans [3] published an existence theorem for this problem with $A = 0$ and g constant. His proof consisted of an iterative argument applied to a heat

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**See the proof of lemma 2 in [1].

balance equation. He proved the existence of a solution for t restricted to the interval [0, 1/4]. J. Douglas and T. M. Gallie [4], A. Datzeff [5], G. Sestini [6], A. Friedman [7], and the present author [8], have proved existence theorems for similar problems.

2. The existence theorem: There exists a unique solution to the free boundary problem.

Proof: Following Evans, we derive a heat balance equation by evaluating

$$\int_0^t \int_0^{R(t')} (u_{xx} - u_t)\, dx\, dt' = 0, \tag{2.1}$$

using the boundary conditions (1.1). We obtain

$$R(t) = A + \int_0^t g(s)\, ds - \int_0^{R(t)} u(x, t)\, dx + \int_0^A f(x)\, dx. \tag{2.2}$$

We use this equation to define a transformation $S = F(R)$ by taking R to be a given differentiable monotonic function such that $R(0) = A$, and taking u to be the solution of the *reduced problem*:

$$\begin{aligned} &u_{xx} = u_t\,, \qquad 0 < x < R(t), \qquad 0 < t,\\ &u_x(0, t) = -g(t), \qquad 0 < t,\\ &u(R(t), t) = 0, \qquad 0 < t,\\ &u(x, 0) = f(x), \qquad 0 < x < A. \end{aligned} \tag{2.3}$$

If we can find a differentiable monotonic function which is left invariant by the transformation F, then it, together with the corresponding solution to the reduced problem, satisfies (1.1). In this paper, we show that the boundary function we seek is the limit function of a sequence of iterates, $R_0 = A$, $R_{n+1} = F(R_n)$.

The sequence of iterates is well defined, for if R is differentiable, and if $S = F(R)$, then S is monotonic and differentiable. In fact,

$$0 \le dS(t)/dt = -u_x(R(t), t). \tag{2.4}$$

The equality follows from (2.2) and (2.1). The inequality from the following argument: if u_x were positive on $x = R(t)$, u would be negative nearby. But then, by the maximum principle, u would attain its negative minimum on $x = 0$. This cannot happen since u_x is negative there.

Our goal is to prove the existence of a solution for an arbitrary time interval. The first iterative process which we carry out will converge if the time interval is small. Then, taking as the initial function the solution to the reduced problem corresponding to the limit boundary function, we carry out another iteration with a small time step, etc. We show that a finite number of such processes will give the solution over an arbitrary time interval $[0, T]$. Uniqueness of the solution follows from the contracting character of the transformation F. If we are content with existence alone, we can use a standard fixed point theorem of functional analysis and obtain a (non-constructive) proof which does not require subdividing the time interval.*

We will show that the first iterative process converges to a differentiable function

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*This method was used in [8]. Uniqueness was established by a separate argument.

if $0 < t \le t_1$. Subsequent iterative processes provide solutions to the integral equations

$$R(t) = A_p + \int_{t_p}^{t} g(s)\, ds - \int_0^{R(t)} u(x, t)\, dx + \int_0^{A_p} u_p(x)\, dx, \tag{2.5}$$

where $A_p(= R(t_p))$ and $u_p(x)$ are obtained from the previous process. The function u is the solution to the reduced problem

$$\begin{aligned} &u_{xx} = u_t, \quad 0 < x < R(t), \quad t_p < t < t_{p+1}, \\ &u_x(0, t) = -g(t), \quad t_p < t < t_{p+1}, \\ &u(R(t), t) = 0, \quad t_p < t < t_{p+1}, \\ &u(x, 0) = u_p(x), \quad 0 < x < A_p. \end{aligned} \tag{2.6}$$

In order to prove convergence, we need the following estimates:

Lemma 1. *If u is the solution to the reduced problem, then there exists a number B, independent of $R(t)$, such that for all t in the interval* $[0, T]$,

$$\begin{gathered} A \le R(t) \le Bt, \\ 0 \le u(x, t) \le B(R(t) - x), \quad 0 \le x \le R(t), \\ -B \le u_x(R(t), t) \le 0. \end{gathered} \tag{2.7}$$

Lemma 2. *If u and v are solutions to the reduced problem* (2.6) *with boundary functions R and S respectively, then there exists $q_0 > 0$, independent of the boundary curves and of the subdivision, such that $0 < t - t_p < q_0$ implies that*

$$\int_0^{A_p} | u(x, t) - v(x, t) |\, dx \le (1/2)\, | R - S |_{t_{p+1}}^{*}, \qquad t_{p+1} = t_p + q_0. \tag{2.8}$$

Furthermore, for all $t > t_p$,

$$| u(x, t) - v(x, t)| \le B\, | R - S |_t^{*}, \; 0 \le x \le \min (R(t), S(t)). \tag{2.9}$$

The derivation of these estimates is in the appendix.

Let

$$\begin{aligned} j_n(t) &= \min (R_n(t), R_{n-1}(t)), \\ k_n(t) &= \max (R_n(t), R_{n-1}(t)), \end{aligned} \tag{2.10}$$

then if $R_{n+1} = F(R_n)$ defines the pth iterative process, and if $q < q_0$,

$$\begin{aligned} | R_{n+1} - R_n |_q &\le \int_0^{A_p} | \Delta u_n(x, t) |\, dx + \int_{A_p}^{j_n(t)} | \Delta u_n(x, t) |\, dx \\ &+ \int_{j_n(t)}^{k_n(t)} | \Delta u_n(x, t) |\, dx, \le 1/2\, | R_n - R_{n-1} |_q + B\, | j_n(t) - A |\, | R_n - R_{n-1} |_q \\ &+ B/2\, | R_n - R_{n-1} |_q^2 \le [1/2 + 2B^2 q]\, | R_n - R_{n-1} |_q, \end{aligned} \tag{2.11}$$

where Δu_n is the difference between the solutions to the reduced problems corresponding

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$^*| R - S |_t = \max | R(t') - S(t') |,\ t_p \le t' \le t.$

to R_n and R_{n-1}. We have adopted the convention that the solution to the reduced problem is identically zero outside the original domain of definition, i.e., $u(x, t) = 0$, if $x > R(t)$.

Clearly, if $q < \min(1/4B^2, q_0)$, the sequence converges to a monotonic function $R(t)$. Let $u(x, t; R)$ be the solution to the reduced problem corresponding to the limit function R. Then if $R' = F(R)$,

$$\begin{aligned} &| R' - R | = | F(R) - F(R_n) | + | R_{n+1} - R |, \\ &| R' - R |_t \leq [1/2 + 2B^2 t] \, | R - R_{n-1} |_t + | R_{n+1} - R |_t , \qquad 0 < t < q_1 . \end{aligned} \tag{2.12}$$

Since the right side can be made arbitrarily small, we conclude that R is invariant under F. We repeat this argument for each subinterval.

The functions $R(t)$ and $u(x, t; R)$ are the solution to the free boundary problem if R is differentiable. To prove that R is differentiable, we write

$$\begin{aligned} [R(t + k) - R(t)] + \int_{R(t)}^{R(t+k)} u(x, t) \, dx = \int_{R(t)}^{R(t+k)} g(s) \, ds \\ - \int_0^{R(t)} [u(x, t + k) - u(x, t)] \, dx. \end{aligned} \tag{2.13}$$

Using the law of the mean and the fact that u is the solution to the reduced problem, we get

$$(1/k)[R(t + k) - R(t)] = -u_x(R(t), t) + 0(k). \tag{2.14}$$

This concludes the proof of the theorem.

APPENDIX

Proof of Lemma 1. In proving inequality (2.4), we found that u_x must be non-positive on $x = R(t)$ and that u must be non-negative in the interior of the domain. To obtain the lower bound on u_x, we pick a constant B so that

$$\begin{aligned} g(t) < B, \qquad 0 < t < T, \\ 0 \leq f(x) \leq B(A - x), \qquad 0 \leq x \leq A. \end{aligned} \tag{a1}$$

We extend f as an even function and take v to be the solution of the heat equation taking on the boundary values

$$\begin{aligned} &v(x, 0) = f(x) + B \, | x |, \qquad -A < x < A, \\ &v(\pm R(t), t) = BR(t), \qquad 0 < t < T. \end{aligned} \tag{a2}$$

$R(t)$ is monotonic, so by the maximum principle,

$$\begin{aligned} &0 \leq v(x, t) \leq BR(t), \\ &0 \leq v_x(R(t), t). \end{aligned} \tag{a3}$$

Since v is an even function, $v_x = 0$ on $x = 0$. Hence, if we let

$$w(x, t) = v(x, t) - Bx - u(x, t), \tag{a4}$$

then

$$w_x = v_x - B - u_x < 0 \quad \text{on} \quad x = 0, \qquad 0 < t < T. \tag{a5}$$

By construction, $w = 0$ on $x = R(t)$. It follows from the maximum principle that

$$\begin{gathered} 0 \leq w(x, t), \qquad 0 \leq x \leq R(t), \qquad 0 < t < T, \\ w_x(R(t), t) \leq 0. \end{gathered} \tag{a6}$$

We conclude that

$$\begin{gathered} -B \leq v_x(R(t), t) - B \leq u_x(R(t), t), \qquad 0 < t < T, \\ u(x, t) \leq v(x, t) - Bx \leq B(R(t) - x), \qquad 0 \leq x \leq R(t), \qquad 0 < t < T. \end{gathered} \tag{a7}$$

Proof of Lemma 2. It follows from (2.7) that (2.9) is valid on the boundary, $x = R(t)$. By the maximum principle, it is valid in the interior of the domain.

If $R \neq S$, let $t' = \sup \{t \mid R(t) = S(t)\}$. Then for any $r > 0$,

$$\begin{gathered} | u(x, t) - v(x, t) | \leq w(x, t) \, | R - S |_{t'+r} \, , \; t' < t < t' + r, \\ 0 \leq x \leq C = R(t') = S(t'), \end{gathered} \tag{a8}$$

where w is the solution to

$$\begin{aligned} &w_t = w_{xx} \, , && 0 < x < C, \qquad t' < t, \\ &w(x, t') = 0, && 0 < x < C, \\ &w(C, t) = B, && t' < t, \\ &w_x(0, t) = 0, && t' < t. \end{aligned} \tag{a9}$$

Note that

$$A \leq C \leq BT, \qquad 0 \leq t' < T. \tag{a10}$$

Clearly,

$$\int_0^C | u(x, t) - v(x, t) | \, dx \leq | R - S |_{t'+r} \int_0^C w(x, t) \, dx. \tag{a11}$$

Since

$$\int_0^C w(x, t) \, dx = 4C \sum_{n=0}^{\infty} (1/(2n + 1)^2(1 - \exp -\{(t - t')(2n + 1)^2/4C^2\}) \tag{a12}$$

approaches zero uniformly in C and t' as t approaches t', we can restrict r so that the integral (a12) is less than 1/2.

If $A = 0$, the estimate (2.8) is not needed for the first iterative process. The lower bound on C will then be $R(t_1)$.

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