AN IMPLICIT, NUMERICAL METHOD FOR SOLVING THE n-DIMENSIONAL HEAT EQUATION*

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1. Introduction. The work of this paper represents an extension of the work of Baker and Oliphant [1] to the n-dimensional, linear, heat-flow problem with arbitrary spacing between the mesh points. We give an implicit scheme which we are able to solve exactly, and we prove that it is unconditionally stable. We treat only the linear case because the generalization to the non-linear case may be treated in precisely the same way as was done by Baker and Oliphant and the same conclusions hold in the n-dimensional case concerning this generalization as in the two-dimensional case.

2. The difference approximation. The basic partial differential equation we wish to consider is

$$\nabla^2\theta = S(\mathbf{x}, t) + A^2\theta + B^2 \frac{\partial\theta}{\partial t}, \tag{2.1}$$

where ∇^2 is the n-dimensional Laplacian with respect to $\mathbf{x}$, $\mathbf{x}$ is an n-dimensional position vector, t is time, and S is a source function. Not both A and B may be zero. By setting $B = 0$ $(A \neq 0)$ we may solve at one step for the asymptotic solution of (2.1). We wish to approximate this differential equation by an implicit, difference scheme to allow approximate numerical calculation of the function θ. We shall define our mesh by n sequences

$$0 < x_{1_k} < \cdots < x_{I_k} < x_{I+1_k} < \cdots x_{M_k} < L_k, \qquad k = 1, \cdots, n \tag{2.2}$$

which are the values of the coordinates of the mesh points in an n-dimensional vector space.

Let us define the matrices

$$D_{J_k}^{I_k} = \frac{2\,\delta_{J_k}^{I_k-1}}{(x_{J_k+1} - x_{J_k})(x_{J_k+1} - x_{J_k-1})} - \frac{2\,\delta_{J_k}^{I_k}}{(x_{J_k+1} - x_{J_k})(x_{J_k} - x_{J_k-1})} + \frac{2\,\delta_{J_k}^{I_k+1}}{(x_{J_k} - x_{J_k-1})(x_{J_k+1} - x_{J_k-1})}, \qquad k = 1, n, \tag{2.3}$$

where δ_i^j is the Kronecker delta. Let us approximate

$$\frac{\partial\theta}{\partial t} = [3\theta(t) - 4\theta(t - \Delta t) + \theta(t - 2\,\Delta t)]/(2\,\Delta t) \tag{2.4}$$

and define

$$\beta = -[A^2 + 3B^2/(2\,\Delta t)]. \tag{2.5}$$

We may now consider the nth order tensor (suppressing the various Kronecker δ's)

$$\sum_{I_k} \beta^{1-n} \prod_{k=1}^{n} (\beta + D_{J_k}^{I_k})\theta_{I_1\cdots I_k\cdots I_n} = \sum_{I_k}\left[\beta + \sum_{k=1}^{n} D_{J_k}^{I_k} + \beta^{-1}\sum_{j<k} D_{J_k}^{I_k} D_{J_j}^{I_j} + \cdots\right]\theta_{I_1\cdots I_k\cdots I_n}. \tag{2.6}$$

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It is evident that

$$\sum_{I_k} D_{J_k}^{I_k} \theta_{I_k} \quad \text{represents} \quad \frac{\partial^2 \theta}{\partial x_k^2} \tag{2.7}$$

to within terms of the second order in Δx_k. Hence (2.6), neglecting 4th and higher order derivatives, represents to within second order in Δx_k the left hand side of

$$\beta\theta + \nabla^2\theta = S(\mathbf{x}, t) + A^2\theta + B^2 \frac{\partial \theta}{\partial t} + \beta\theta. \tag{2.8}$$

By definition β has been chosen so that the right-hand side of (2.8) is independent of $\theta(t)$ and depends only on $\theta(t - \Delta t)$, $\theta(t - 2\Delta t)$, and a known function, S. We can solve the implicit equations (2.6) for $\theta_{I_1 \cdots I_k \cdots I_n}$, as $\beta\delta_{J_k}^{I_k} + D_{J_k}^{I_k}$ is a triple diagonal matrix. We can factor it as

$$\beta\, \delta_{J_k}^{I_k} + D_{J_k}^{I_k} = \sum_{L_k} w_{J_k}^{L_k} b_{L_k}^{I_k}, \tag{2.9}$$

where w is a lower triangular matrix. The matrix elements are given by the recursion relations

$$\begin{aligned}
&w_{J_k}^{L_k} = 0 \quad \text{if} \quad J_k - L_k \neq 0, 1 \\
&b_{L_k}^{I_k} = 0 \quad \text{if} \quad I_k - L_k \neq 0, 1 \\
&b_{I_k}^{I_k} = 1 \\
&w_{J_k}^{J_k-1} = D_{J_k}^{J_k-1} \\
&w_{J_k}^{J_k} = \beta + D_{J_k}^{J_k} - w_{J_k}^{J_k-1} b_{J_k-1}^{J_k} \; (b_0^1 = 0) \\
&b_{J_k}^{J_k+1} = D_{J_k}^{J_k+1}/w_{J_k}^{J_k}.
\end{aligned} \tag{2.10}$$

Let us now define

$$g_{L_1 \cdots L_k \cdots L_n} = \sum_{I_k} \left(\prod_{k=1}^{n} b_{L_k}^{I_k} \right) \theta_{I_1 \cdots I_k \cdots I_n}. \tag{2.11}$$

Then, using the difference approximations (2.4) and (2.6), we may represent (2.1) by

$$\sum_{L_k} \beta^{1-n} \left(\prod_{k=1}^{n} w_{J_k}^{L_k} \right) g_{L_1 \cdots L_k \cdots L_n} = a_{J_1 \cdots J_k \cdots J_n}, \tag{2.12}$$

where the a's are known quantities. Due to the triangular nature of the w's we may proceed from low index numbers to high ones and solve (2.12) for the g's by straightforward elimination. Then due to the triangular nature of the b's we may proceed from high index number to low ones, and solve (2.25) for the θ's by straightforward elimination.

3. Stability. We shall proceed with a proof of the unconditional stability of the method described in Sec. 2 by a method closely related to those employed by Baker and Oliphant [1]. We consider only the problem with S independent of t. As Eq. (2.1) is linear it will be sufficient to consider the solutions of the difference equation analogue of

$$\nabla^2\theta = A^2\theta + B^2 \frac{\partial \theta}{\partial t} \tag{3.1}$$

with homogeneous boundary conditions. This equation results from the subtraction of the asymptotic solution from the difference equation analogue of (2.1). Let us seek a solution of the difference equation analogue of (3.1), such that, for some Z_ν ,

$$\theta(\nu, \mathbf{x}, t) = \psi(\nu, \mathbf{x})Z_\nu^{t/\Delta t}. \tag{3.2}$$

Equation (3.1) as expressed by (2.4), (2.5), and (2.6) is for a component (3.2)

$$\begin{aligned}\sum_{I_k}\left[\beta^{1-n}\prod_{i=1}^{n}(\beta\,\delta_{J_k}^{I_k} + D_{J_k}^{I_k}) - \beta\prod_{i=1}^{n}\delta_{J_k}^{I_k}\right]\theta_{I_1\cdots I_k\cdots I_n}(\nu)&\\ = \{A^2 + B^2[3 - 4Z_\nu^{-1} + Z_\nu^{-2}]/(2\,\Delta t)\}\,\theta_{J_1\cdots J_k\cdots J_n}(\nu).&\end{aligned} \tag{3.3}$$

Let us define

$$A^2 + \frac{B^2}{\Delta t}(3/2 - 2Z_\nu^{-1} + 1/2Z^{-2}) = \lambda \tag{3.4}$$

and an operator Δ_k , such that

$$\Delta_k\omega_{I_1\cdots I_k\cdots I_n} \equiv \frac{\omega_{I_1\cdots I_k\cdots I_n} - \omega_{I_1\cdots I_k-1\cdots I_n}}{x_{I_k} - x_{I_k-1}}. \tag{3.5}$$

Further let

$$\begin{aligned}F \equiv \sum_{I_k}\Bigg\{\sum_{k=1}^{n}(\Delta_k\theta_{I_1\cdots I_k\cdots I_n})^2 - \beta^{-1}\sum_{j<k}(\Delta_j\,\Delta_k\theta_{I_1\cdots I_k\cdots I_n})^2&\\ + \beta^{-2}\sum_{i<j<k}(\Delta_i\,\Delta_j\,\Delta_k\theta_{I_1\cdots I_k\cdots I_n})^2 - \cdots\Bigg\}&\end{aligned} \tag{3.6}$$

and

$$G \equiv \sum_{I_k}(\theta_{I_1\cdots I_k\cdots I_n})^2 = 1. \tag{3.7}$$

One can now show that the solution of (3.3) is the same problem as finding the extrema of $F + \lambda G$ subject to (3.7), and homogeneous boundary conditions. As we have

$$\prod_{k=1}^{n} M_k = M \tag{3.8}$$

mesh-points, the theory of quadratic forms [2] assures us that since β is negative, there exists a complete set of m orthogonal vectors $(\theta_{I_1\cdots I_k\cdots I_n}(\nu))$ which satisfy (3.6) and (3.7) and therefore (3.3). As λ can be shown to be the negative of F, we must have

$$(3 - 4Z_\nu^{-1} + Z_\nu^{-2})/3 = \Delta t(\lambda - 2A^2/3)/B^2 < 0. \tag{3.9}$$

It easily follows [1] from (3.9) that for both roots

$$|\,Z_\pm\,| < 1. \tag{3.10}$$

Hence, the amplitude of all the $\theta(\nu)$ solutions decays to zero as time goes to infinity. Since the $\theta(\nu)$ form a complete orthonormal set, any initial value problem can be expanded in terms of them, and hence we may conclude on the basis of (3.10) that the method is unconditionally stable.

REFERENCES

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HELICAL FLUID FLOWS*

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Introduction. Potential helical flows have been completely described by G. Hamel [1]. Nemenyi and Prim, and N. Coburn have obtained some Beltrami helical flows [2, 3]. A simple description will be given here of all steady incompressible helical flows and all steady compressible helical flows with entropy constant along stream lines.

The equations of helical flow. The class of compressible flows to be considered here are those governed by the following differential equations in which v^* is the velocity, p is the pressure, ρ is the density, S is the entropy, and t^* is the unit tangent vector along the stream lines;

$$\nabla \cdot \rho v^* = 0 \text{ (continuity equation)} \tag{1}$$

$$(\rho v^* \cdot \nabla) v^* = -\nabla p \text{ (equation of motion)} \tag{2}$$

$$\rho = \rho(p, S) \text{ (equation of state)} \tag{3}$$

$$t^* \cdot \nabla S = 0 \text{ (entropy is constant along stream lines)} \tag{4}$$

For incompressible flows we must satisfy Eqs. (1) and (2) and the special case $\rho =$ constant of Eq. (3).

Now we assume that the flows are helical; i.e., the stream lines of the flow are parallel helices on coaxial circular cylinders. Such flows have the property $\nabla \cdot t^* = 0$. This may be seen immediately if we introduce cylindrical coordinates r, θ, z and decompose t^* according to

$$t^* = \sin \beta \theta^* + \cos \beta z^*.$$

Note that the angle β of the helices is in general a function of r.

With the condition $\nabla \cdot t^* = 0$ the continuity equation reduces to

$$t^* \cdot \nabla \ln \rho q = 0, \tag{5}$$

where q is the magnitude of the velocity. Also, from Eqs. (3) and (4)

$$t^* \cdot \nabla \rho - \frac{\partial \rho}{\partial p} t^* \cdot \nabla p = 0. \tag{6}$$

Finally, we write Eq. (2) in intrinsic form [4]:

$$t^* \cdot \nabla p = -\rho t^* \cdot \nabla \frac{q^2}{2}, \tag{7}$$

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