ON THE APPLICATION OF ELLIPTIC FUNCTIONS IN NON-LINEAR FORCED OSCILLATIONS*

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1. Introduction. A system having an equation of motion of the Duffing type is known to respond in free vibrations in the form of elliptic functions, and the period is expressible in terms of complete elliptic integrals. It seems appropriate, therefore, to analyze the response of such systems when they are subject to periodic forcing functions which are themselves elliptic functions. It is the purpose of this paper to examine this problem in detail.

We shall show that there exist quite general cases where exact single-term and multiple-term elliptic responses can arise under forcing functions consisting either of a single elliptic function, or of a combination of these. It will be shown that all known results of the perturbation or iteration method are simply special cases of these exact solutions, and the former emerge easily from the present analysis if one expands the results of this paper in terms of the moduli of the elliptic functions and retains only terms which are linear in the second power of the modulus. Higher order approximations may easily be found from the multiple-term exact solutions by comparing orders of magnitude of the various components in an appropriate fashion. Finally, we shall demonstrate exact subharmonic solutions which are elliptic functions and which may, or may not, be simple. The simple elliptic subharmonic solutions reduce to the simple subharmonics defined by Rosenberg [6]** if one sets the modulus equal to zero.

In view of the fact that there is a fair amount of uniformity concerning the notation used in connection with elliptic functions, we have adopted the commonly accepted notation of Whittaker and Watson [9].

2. Non-linear free vibrations. The equation of free vibrations of a Duffing system is

$$m \frac{d^2x}{dt^2} + k_{s1}x + k_{s3}x^3 = 0. \tag{1}$$

Introducing $\tau = t(k_{s1}/m)^{1/2}$ and $\beta = k_{s3}/k_{s1}$, we obtain a new equation of motion

$$\frac{d^2x}{d\tau^2} + x + \beta x^3 = 0. \tag{2}$$

Let us assume that the peak amplitude of the oscillation x is equal to A, and it occurs at $\tau = 0$. Then it is well known [9] that the solution of (2) is given by the Jacobian elliptic function *cn u*. The exact form of the solution is

$$x = A \operatorname{cn}(p\tau, k), \qquad p = (1 + \beta A^2)^{\frac{1}{2}}, \qquad k = [\beta A^2/2(1 + \beta A^2)]^{\frac{1}{2}}, \tag{3}$$

where k is the modulus of the elliptic function and p may be taken as the "circular frequency."

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**Numbers in square brackets refer to the bibliography at the end of the paper.

If we have a soft non-linear spring, i.e., β is negative, the solution may be transformed [1] into the following form in order to have the modulus remaining in the usual range from 0 to 1, [2, 5],

$$x = A \text{ sn } [p_1\tau + K(k_1), k_1], \qquad p_1 = (1 + \beta A^2/2)^{\frac{1}{2}}, \qquad k_1 = [-\beta A^2/(2 + \beta A^2)]^{\frac{1}{2}}. \quad (4)$$

Here $K(k)$ is the complete elliptic integral of the first kind.

Similarly, when $\beta A^2 < -1$, the solution can be put in the following form.

$$x = A \text{ ns } [p_2\tau + K(k_2), k_2], \qquad p_2 = (-\beta A^2/2)^{1/2}, \qquad k_2 = [-(2 + \beta A^2)/\beta A^2]^{1/2}. \quad (5)$$

We note that in this case the solution is no longer bounded. The meaning is quite obvious physically. When β is negative and A is so large that $\beta A^2 < -1$, the total spring force—i.e., the linear and the non-linear part together—becomes negative. Under this circumstance the mass will move away from the origin $x = 0$ indefinitely. As a matter of fact, the quantity A should be interpreted here as the amplitude at $\tau = 0$, not as the peak amplitude.

Both the functions *cnu* and *snu* have a period in u equal to $4K$. The complete elliptic integral of the first kind, K, is then the quarter period of the free oscillation. The period in τ is given by

$$T_\tau = 4K/p. \quad (6)$$

For the case of a hard spring it is

$$T_\tau = 4(1 - 2k^2)^{1/2}K(k). \quad (7)$$

For a system with a soft spring it is

$$T_\tau = 4(1 + k_1^2)^{1/2}K(k_1). \quad (8)$$

Figures 1 and 2 indicate these relationships between βA^2, k^2, k_1^2 and T_τ in graphic form.

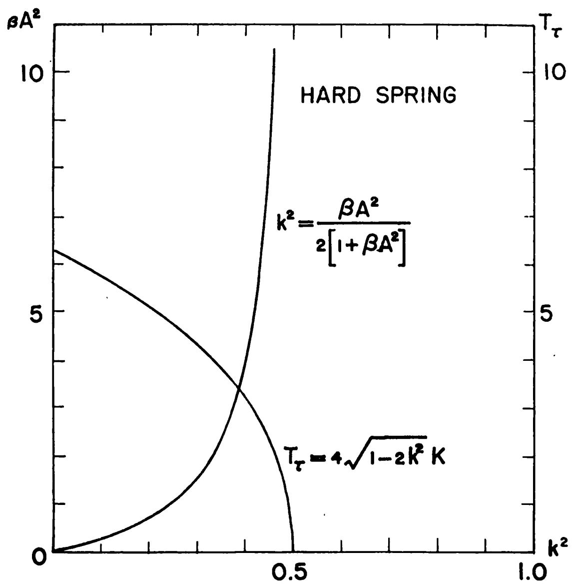

FIG. 1. Free vibration period and amplitude as functions of the modulus for a system with a hard spring.

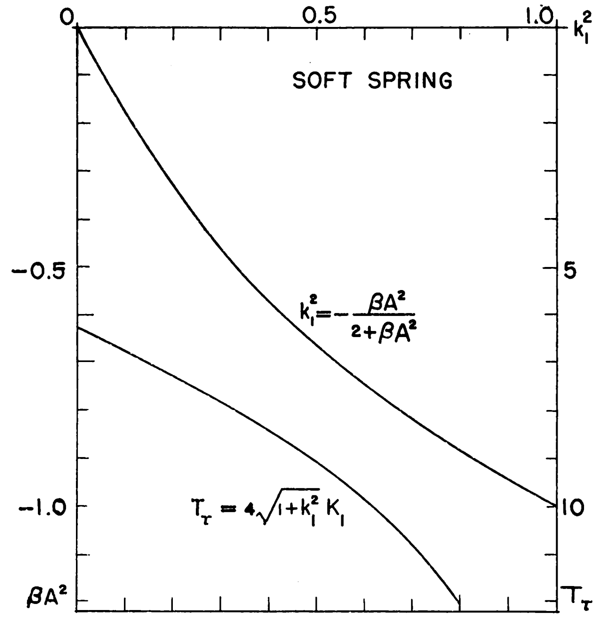

FIG. 2. Free vibration period and amplitude as functions of the modulus for a system with a soft spring.

When the non-linearity and the peak amplitude are such that $|\beta A^2|$ is small in comparison with unity, the modulus k will also be small. Expanding the quarter period K in powers of k^2 and substituting the expansion into (7) we find for the period

$$T_\tau = 2\pi[1 - \tfrac{3}{4}k^2 + 0(k^4)]. \tag{9}$$

The equivalent circular frequency $p_{e\tau}$ can be easily found as,

$$p_{e\tau}^2 = (2\pi/T_\tau)^2 = 1 + \tfrac{3}{4}\beta A^2 + 0(\beta^2 A^4). \tag{10}$$

This agrees with the result obtained by the perturbation method when $|\beta|$ is assumed small [8].

3. Some exact solutions of forced vibration. In this section we shall examine the response of a Duffing system to forced vibration. We assume that small quantities of positive damping are present in the physical system, but absent from the equation of motion. Consequently the periodic solutions of interest are those usually referred to as steady state vibrations. We shall assume here that the effect of small damping on the motion is slight.

Let us consider a system whose equation of motion is

$$\frac{d^2x}{d\tau^2} + x + \beta x^3 = F(\tau), \tag{11}$$

where $F(\tau)$ is the external forcing function divided by k_{s1}. The forcing function $F(\tau)$ in (11) is usually taken to be a simple harmonic function of an arbitrary circular frequency. Here, however, this is not the case. Instead we shall try to find out what type of external forcing function will excite a steady state response, which is an exact solution of the equation of motion, and which is expressible in a simple form.

3.1 Simple elliptic response. First let us ask under what condition the response x as a function of τ will be proportional to the forcing function, leaving the precise form of the function undetermined. Putting $F(\tau) = Bx$, the equation of motion becomes

$$\frac{d^2x}{d\tau^2} + (1 - B)x + \beta x^3 = 0. \tag{12}$$

The solution of this equation can be readily found to be

$$x = A \operatorname{cn}(p\tau, k), \tag{13}$$

where

$$p = (1 - B + \beta A^2)^{1/2}, \qquad k = [\beta A^2/2(1 - B + \beta A^2)]^{1/2}. \tag{13a}$$

We note that there are four parameters, A, B, p, and k involved in the solution and two relationships (13a) between them. Hence, any two of them may be arbitrarily chosen and the other two will be determined by (13a). Such a set of parameters will assure a response like (13). For example, if we take p and k as given we have

$$A = (2/\beta)^{1/2}pk, \qquad B = 1 - p^2(1 - 2k^2). \tag{14}$$

The response x and the necessary forcing function $F(\tau)$ are respectively

$$x = A \operatorname{cn}(p\tau, k) = (2/\beta)^{1/2}pk \operatorname{cn}(p\tau, k), \tag{15}$$

$$F(\tau) = BA \operatorname{cn}(p\tau, k) = (2/\beta)^{1/2}pk[1 - p^2(1 - 2k^2)] \operatorname{cn}(p\tau, k). \tag{16}$$

The parameters p and k in cn $(p\tau, k)$ play important geometrical roles. The modulus k controls the proportions of the higher harmonics in the Fourier analysis while p and k together determine the periodicity of the function. The period in τ is given by (6). Figure 3 shows curves of T_τ as functions of p and k.

A response consisting of only a single term of elliptic function like (13) will be called a *simple elliptic response.* It should be emphasized here that not every elliptic forcing excites a simple elliptic response. The amplitude of the elliptic forcing function must bear a definite relationship to the parameters p and k. We shall denote those forcing functions satisfying this relationship as *favored* elliptic forcing functions.

In case of a soft non-linear spring, $\beta < 0$, the solution of (12) is

$$x = A \operatorname{sn}(p_1\tau, k_1),$$

where

$$p_1 = (1 - B + \beta A^2/2)^{1/2}, \qquad k_1 = [-\beta A^2/2(1 - B + \beta A^2/2)]^{1/2}. \tag{17}$$

If we consider p_1 and k_1 as given we must have

$$B = 1 - p_1^2(1 + k_1^2), \qquad A = (-2/\beta)^{1/2}p_1k_1 \tag{18}$$

and the forcing function

$$F(\tau) = BA \operatorname{sn}(p_1\tau, k_1) = (-2/\beta)^{1/2}p_1k_1[1 - p_1^2(1 + k_1^2)] \operatorname{sn}(p_1\tau, k_1). \tag{19}$$

Equations (17) may also be obtained through the application of the first order transformation mentioned in Sec. 2. From here on we shall not, therefore, treat the cases

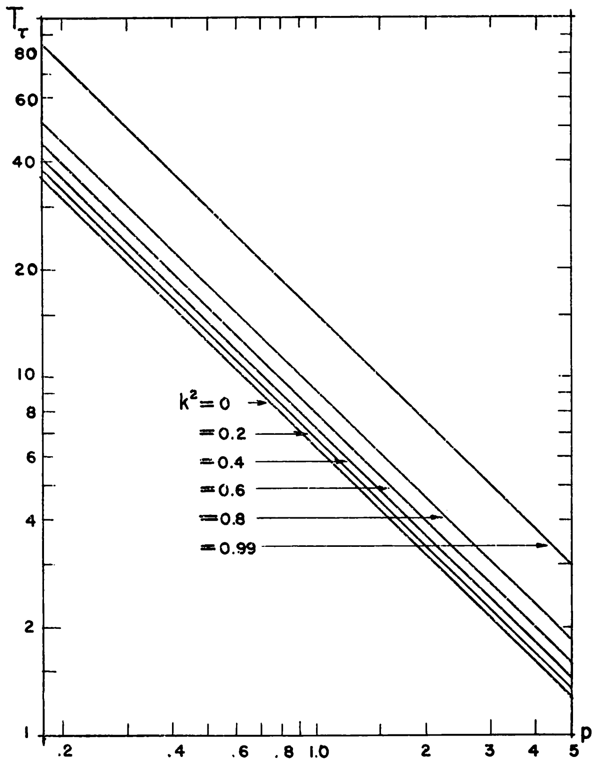

FIG. 3. Period in τ of the elliptic functions $cn(p\tau, k)$ and $sn(p\tau, k)$.

with soft springs separately. Whenever the modulus k appears outside the standard region of 0 to 1, one of the first order transformations must be used.*

The quantity $1/B$ is analogous to the well-known magnification factor—i.e., the ratio of response amplitude to forcing amplitude. In the linear case it gives the response amplitude for a forcing amplitude of unity. Near the point where $1/B$ approaches infinity resonance occurs. With non-linear systems it is better to take a somewhat different view. In linear cases it is customary to consider the forcing amplitude as given and the problem is to find the response amplitudes. In the non-linear case the response amplitude is intimately connected with the period, which is determined by p and k, and the wave form, which is determined by k. It is therefore desirable to consider the response amplitude as known and the problem is to find the necessary forcing amplitude. Hence, instead of $1/B$ we shall discuss the factor B, which is the ratio of forcing amplitude to response amplitude and will be called the *forcing amplitude factor*. Figures 4 and 5 show the curves of B *vs.* p with k^2 as a parameter for hard and soft non-linear springs respectively.

In Figs. 4 and 5 the curves of $k^2 = 0$ correspond to the usual linear case. B is given by $(1 - p^2)$ or the magnification factor $1/B = 1/(1 - p^2)$. Each of the curves in Fig.

*In a slightly different form, the simple elliptic response under one term elliptic forcing discussed here was treated by Helfenstein [3]. The material presented here was retained nevertheless because it is basic to all further development, and because Helfenstein's paper is not easily accessible.

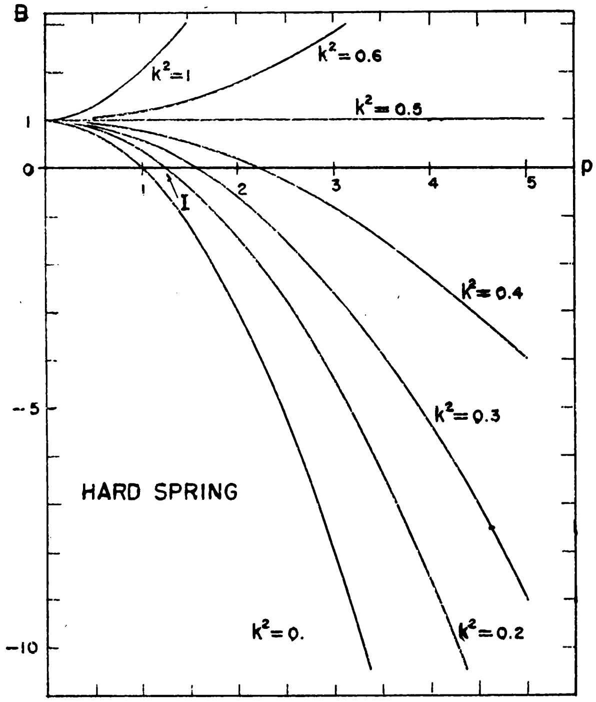

FIG. 4. Forcing amplitude factor B curves of simple elliptic response for a hard spring system.

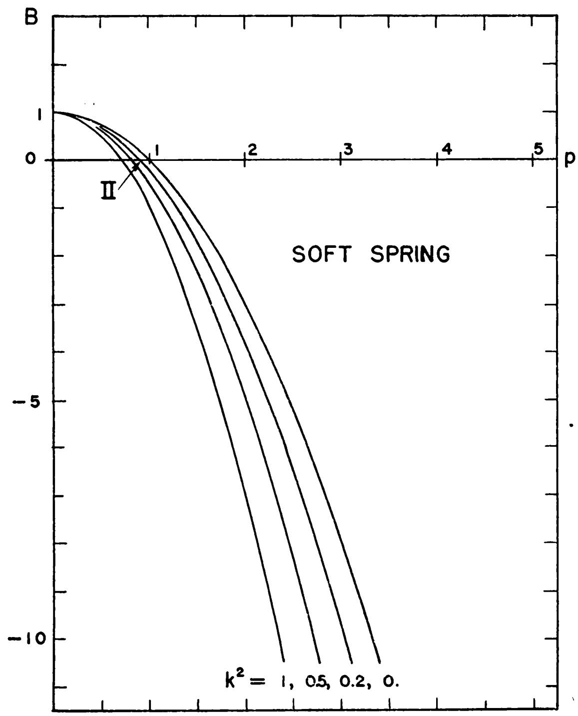

FIG. 5. Forcing amplitude factor B curves of simple elliptic response for a soft spring system.

5 and those in Fig. 4 with $k^2 < 0.5$ cross the p-axis once. Near the points of crossing, such as the points I and II, the value of B is extremely small. However, the response amplitude A is given by (14) and is finite. This means that near the crossing points extremely small external forcing amplitudes are capable of maintaining the response. This is the phenomenon of resonance. At any of the crossing points the external forcing required is zero. Here we have the case of free vibration. Thus, in non-linear systems such as those under discussion, the free vibrations emerge quite naturally as special cases of forced vibrations.

3.2 Two-term elliptic forcing. Results similar to those obtained in Sec. 3.1 can be readily found for other cases. For example, let us require that the forcing function $F(\tau)$ be $B_1x + B_3x^3$. The equation of motion may then be put as,

$$\frac{d^2x}{d\tau^2} + (1 - B_1)x + (\beta - B_3)x^3 = 0, \tag{20}$$

and its solution can be easily found. It is

$$x = A \operatorname{cn}(p\tau, k), \tag{21}$$

where

$$p = [1 - B_1 + (\beta - B_3)A^2]^{1/2}, \tag{21a}$$

$$k = \{(\beta - B_3)A^2/2[1 - B_1 + (\beta - B_3)A^2]\}^{1/2}. \tag{21b}$$

The favored forcing function is

$$F(\tau) = B_1 A \operatorname{cn}(p\tau, k) + B_3 A^3 \operatorname{cn}^3(p\tau, k). \tag{22}$$

Here we note that five parameters are involved in the solution, namely: A, B_1, B_3, p, and k, and there are two relationships between them. Therefore any three of the five, except the group of B_1, p, and k, can be arbitrarily chosen. A simple elliptic response like (21) is assured when the other two parameters are determined from (21a) and (21b). For the convenience of future applications we shall write out the favored elliptic forcing function, assuming that p, k and A have been chosen arbitrarily. This results in

$$F(\tau) = A[1 - p^2(1 - 2k^2)] \operatorname{cn}(p\tau, k) + A(\beta A^2 - 2p^2k^2) \operatorname{cn}^3(p\tau, k). \tag{23}$$

Another way of looking at the solution is the following: Let us denote $B_1 A$ by C_1 and B_3A^3 by C_3. They are respectively the coefficients of the linear and cubic terms in (23). Combining C_1 and C_3 by eliminating A we obtain

$$\beta^{1/2}C_3 = \frac{\beta^{1/2}C_1}{1 - p^2(1 - 2k^2)}\left\{\frac{\beta C_1^2}{[1 - p^2(1 - 2k^2)]^2} - 2p^2k^2\right\} \tag{24}$$

as the necessary relationship between C_1, C_3, p, and k of the forcing function in order to have a simple elliptic response.

It should be pointed out that although the exact solutions discussed here require a special class of forcing functions, that class itself is fairly general because of the freedom to choose any three of the parameters C_1, C_3, p, and k at will.

3.3 Multiple-term elliptic response. The inverse method employed to find exact solutions in the last two sections can be extended to those consisting of more than one elliptic term. For example, if we require the solution to be

$$x = A_1 \operatorname{cn}(p\tau, k) + A_3 \operatorname{cn}^3(p\tau, k), \tag{25}$$

the necessary forcing function can be found by substituting (25) into the left side of (11). Before doing this it is, however, convenient to perform a transformation on (11). Putting $y = cn\ (p\tau, k)$ and $X(y) = x(\tau)$, Eq. (11) becomes

$$p^2[(1 - k^2) - (1 - 2k^2)y^2 - k^2y^4]\frac{d^2X}{dy^2} + p^2[-(1 - 2k^2)y - 2k^2y^3]\frac{dX}{dy} + X + \beta X^3 = G(y), \tag{26}$$

where $G(y) = F(\tau)$.

In our case (25), $X = A_1y + A_3y^3$. When this is substituted in (26) and all terms are grouped in powers of y we get

$$G(y) = D_1y + D_3y^3 + D_5y^5 + D_7y^7 + D_9y^9, \tag{27}$$

where

$$D_1 = [1 - p^2(1 - 2k^2)]A_1 + 6p^2(1 - k^2)A_3\ ,$$

$$D_3 = [\beta A_1^2 - 2p^2k^2]A_1 + [1 - 9p^2(1 - 2k^2)]A_3\ ,$$

$$D_5 = 3[\beta A_1^2 - 4p^2k^2]A_3\ ,$$

$$D_7 = 3\beta A_1A_3^2\ ,$$

$$D_9 = \beta A_3^3\ .$$

Again, in this solution four parameters out of a total of nine may be arbitrarily chosen. An application of this exact solution will be discussed in the next section. It is evident how this result can easily be extended to cases where $X(y)$ is of power higher than three. In general, if $X(y)$ is of power $2n - 1$ there will be $4n + 1$ parameters and $3n - 1$ relations between them. Hence $n + 2$ parameters may be chosen arbitrarily.

4. Simple harmonic forcing. Up to now we have only discussed some exact solutions under external forcing consisting of favored elliptic functions. When the forcing function is simple harmonic the equation of motion is

$$\frac{d^2x}{d\tau^2} + x + \beta x^3 = P_0 \cos\frac{2\pi}{T}\tau = P_0 \cos\omega\tau, \tag{28}$$

where T is the forcing period, and ω the forcing frequency. In this instance an exact solution in a simple form is hardly to be expected. Various procedures are available to find the approximate steady state solution.

One method consists in assuming as an approximate solution a suitable function $z(\tau)$ with some undetermined parameters, and then defining an error function $e(\tau)$ as,

$$e(\tau) = \frac{d^2z}{d\tau^2} + z + \beta z^3 - P_0 \cos\frac{2\pi}{T}\tau \tag{29}$$

and, finally, determining the parameters such that the total error specified in some fashion be a minimum. From what we have learned in the preceding sections it seems natural to take a suitable elliptic function as the approximate solution. Let us put $z(\tau) = A\ cn\ (p\tau, k)$. The error function is then given by

$$e(\tau) = AB\ \text{cn}\ (p\tau, k) - P_0 \cos\frac{2\pi}{T}\tau \tag{30}$$

provided that Eqs. (14) are satisfied. Furthermore, to assure the same period for the response and the forcing function we must have

$$p = 4K(k)/T. \tag{31}$$

Here we have four parameters A, B, p, and k, but there are three conditions, Eqs. (14) and Eq. (31), to be satisfied. Only one of them is completely at our disposal. This one can be determined by using any of the several minimum total error criteria, such as the colocation method, the Galerkin's method, or the least square method. Solutions obtainable in this fashion were computed. When k^2 is small, which means small $|\beta A^2|$, these methods yield the well-known result

$$\omega^2 = 1 + \tfrac{3}{4}\beta A^2 - \frac{P_0}{A}.$$

When a multiple-term elliptic response is used for the approximate solution $z(\tau)$, proper formulation of the minimum total error can be made in the usual manner without difficulty, although the actual calculation becomes very involved, as usual.

We shall not reproduce these solutions here. Instead we wish to derive the approximate solutions of various orders of the equation (28) entirely from the various exact solutions obtained in Sec. 3. The central idea is to compare the orders of magnitude of the terms appearing in the exact solutions. It will be shown that approximate solutions of different orders will result quite naturally from this kind of analysis.

4.1 First order approximation. From Sec. 3.1 it is seen that the response will be $x = A\ cn\ (p\tau, k)$, provided that the forcing function is of the form of (16). This favored elliptic forcing function will be rewritten here as $F_1(\tau)$.

$$F_1(\tau) = A[1 - p^2(1 - 2k^2)]\ \mathrm{cn}\ (p\tau, k). \tag{32}$$

The function *cn* may be expanded into the Fourier series

$$\mathrm{cn}\ (p\tau, k) = \sum_{n=0}^{\infty} a_{2n+1} \cos \frac{(2n + 1)\pi p\tau}{2K}, \qquad a_{2n+1} = \frac{2\pi}{kK} \frac{q^{n+1/2}}{1 + q^{2n+1}}. \tag{33}$$

In this expansion it may be easily verified that the magnitudes of the various components are of the orders: $1, k^2, k^4, \cdots$ when k^2 is small in comparison with unity. Hence, outside of a common factor A the orders of magnitude of the leading parts of various harmonic components in $F_1(\tau)$ are respectively: $(1 - p^2)$, $(1 - p^2)k^2$, $(1 - p^2)k^4$, $\cdots$. From (14), βA^2 is known to be of the order k^2.

Next, let us see what error we commit if we replace the simple harmonic forcing function $P_0 \cos \omega\tau$ by $F_1(\tau)$. When $1 - p^2$ is restricted to be of the order of k^2, the first harmonic in $F_1(\tau)$ will be of the order k^2 and the rest will all be of higher order. By setting the coefficient of the first harmonic equal to P_0, we find

$$A[1 - p^2(1 - 2k^2)] = P_0 . \tag{34}$$

We insure an accuracy of k^2 if we replace $P_0 \cos \omega\tau$ by $F_1(\tau)$, provided of course that k^2 is small compared with unity. In order to have the same periodicity for the response and the forcing we must have

$$p = 2\omega K/\pi. \tag{35}$$

Equations (34), (35) and the first equation of (14) completely determine the values of

A, p, and k, thus the first order approximate solution, for any set of assigned values of β, ω, P_0. Eliminating p from these three equations and expanding K in powers of k^2, we get

$$\beta A^2 - 2k^2\omega^2[1 + \tfrac{1}{2}k^2 + O(k^4)] = 0, \tag{36}$$

$$(1 - \omega^2) - \frac{1}{16}k^2(1 - 25\omega^2) + O(k^4) = \frac{P_0}{A}. \tag{37}$$

Since, in this analysis, we limit $1 - p^2$ to the order of k^2, we find that $1 - \omega^2$ is also of the order k^2 and may be taken as αk^2 in a comparison of orders of magnitude, α being a quantity of the order unity. As the forcing function is approximated to the accuracy of k^2, only terms up to that order will be retained in (36) and (37). These equations now become

$$\beta A^2 - 2k^2 = 0, \tag{36a}$$

$$\alpha k^2 + \frac{3}{2}k^2 = \frac{P_0}{A}. \tag{37a}$$

From them we find

$$\omega^2 = 1 + \tfrac{3}{4}\beta A^2 - \frac{P_0}{A}, \quad \text{or,} \quad \omega = 1 + \tfrac{1}{2}\left[\tfrac{3}{4}\beta A^2 - \frac{P_0}{A}\right]. \tag{38}$$

The response is $x = A\ cn\ (p\tau, k)$. Again, retaining only terms up to k^2 in its Fourier expansion, one gets

$$x = A\left\{\cos\omega\tau + \frac{1}{32}\beta A^2(-\cos\omega\tau + \cos 3\omega\tau)\right\}. \tag{39}$$

Equations (38) and (39) are exactly those obtained from the perturbation method, [8]. Although this first order solution result is not new, the method through which it is obtained here is novel in that it is obtained as a special case from an exact solution of forced oscillation. Throughout the derivation one thing stands out. In the perturbation method the perturbation parameter usually used is the coefficient β of the non-linear term. The present analysis indicates that the controlling parameter is k^2 or $\beta A^2/2$ rather than β. Physically, this is to be expected. Even when β is not small, but A is restricted to to be small, the first order solution is still valid.

Because of the requirement that $1 - \omega^2$ be small, the solution obtained is only valid in the neighborhood of linear free vibrations. Let us now drop this requirement and assume that $1 - p^2$ may be of the order unity. In this case it is necessary to start with the exact solution found in Sec. 3.2. The response is again $x = A\ cn\ (p\tau, k)$ and the favored elliptic forcing function is given by (23) and will be denoted here as $F_2(\tau)$.

The function cn^3 may be expanded into a Fourier series.

$$\mathrm{cn}^3(p\tau, k) = \sum_{n=0}^{\infty} b_{2n+1}\cos\frac{(2n+1)\pi p\tau}{2K},$$
$$b_{2n+1} = \frac{1}{2k^2}\left\{2k^2 - 1 + (2n+1)^2\left(\frac{\pi}{2K}\right)^2\right\}\frac{2\pi}{kK}\frac{q^{n+1/2}}{1 + q^{2n+1}}. \tag{40}$$

When k^2 is small the magnitudes of the various components are: 1, 1, k^2, k^4, $\cdots$. It will be shown later *a posteriori* that when k^2 is small, βA^2 is of the order k^2. Keeping this in mind we find that, apart from a common factor A, the orders of magnitude of various harmonic components in $F_2(\tau)$ are:

$$\begin{array}{lllll} (1 - p^2), & (1 - p^2)k^2, & (1 - p^2)k^4, & \cdots & \text{from cn term.} \\ k^2 \quad , & k^2 \quad , & k^4 \quad , & \cdots & \text{from cn}^3 \text{ term,} \end{array}$$

If we again neglect terms of order k^4 or higher we have two harmonics from the cn and two from the cn^3. By reasoning similar to that used previously we can approximate the forcing function $P_0 \cos \omega\tau$ by $F_2(\tau)$ with an accuracy up to k^2, if we set

$$\begin{aligned} [1 - p^2(1 - 2k^2)]a_1 + (\beta A^2 - 2k^2p^2)b_1 &= P_0/A, \\ [1 - p^2(1 - 2k^2)]a_3 + (\beta A^2 - 2k^2p^2)b_3 &= 0. \end{aligned} \tag{41}$$

Another necessary condition is that concerning the periodicity given by (35). Equations (41) and (35) completely determine the approximate solution, provided that k thus found satisfies the order-of-magnitude requirement.

Substituting the coefficients a_1, a_3, b_1, and b_3 from (33) and (40) into (41), eliminating p by using (35), expanding K into powers of k^2, and retaining only terms up to k^2, we get

$$\begin{aligned} (1 - \omega^2) - (1 - 25\omega^2)k^2/16 + 3(\beta A^2 - 2k^2\omega^2)/4 &= P_0/A, \\ (1 - \omega^2)k^2/4 + (\beta A^2 - 2k^2\omega^2) &= 0. \end{aligned} \tag{42}$$

Eliminating k^2 from these two, we have

$$1 - \omega^2 - \frac{P_0}{A} + \beta A^2\left(\frac{7\omega^2 - 1}{9\omega^2 - 1}\right) = 0. \tag{43}$$

After A has been determined from (43) for given β, ω, and P_0, k^2 may be found from (42).

$$k^2 = 4\beta A^2/(9\omega^2 - 1). \tag{44}$$

The response is given by the following expression.

$$x = A\left\{\cos \omega\tau + \frac{k^2}{16}(-\cos \omega\tau + \cos 3\omega\tau)\right\}. \tag{45}$$

Near $\omega = 1/3$ the solution is probably not valid because of the large value of k^2 which invalidates the analysis. This solution also requires that $|\beta A^2|$ be small in comparison with unity. It follows from (43) that $1 - \omega^2 - P_0/A$ has to be small, implying small deviations from the linear forced response. The solution obviously represents a perturbation from the linear forced vibration. It is interesting to note that in most of the frequency ranges (43) does not differ very much from (38).

4.2 Higher order approximations. In the exact solution under two-term favored elliptic forcing we have three parameters completely at our disposal. However, in approximating a simple harmonic forcing by such an elliptic forcing function, one relation between these three is necessary on account of the periodicity requirement. With only two parameters at our disposal we can not hope to obtain any approximation beyond the zeroth order and the order of k^2. In order to get approximations of higher order we resort to the exact solutions of multiple-term elliptic response discussed in Sec. 3.3. For example, for the second order approximation we may use the response given by (25). The necessary forcing function is rewritten here as $F_3(\tau)$.

$$F_3(\tau) = D_1 \text{ cn} + D_3 \text{ cn}^3 + D_5 \text{ cn}^5 + D_7 \text{ cn}^7 + D_9 \text{ cn}^9, \tag{46}$$

where cn stands for $cn\ (p\tau, k)$ and the D's are given by (27).

Let us assume that βA^2 is of the order k^2 and A_3 is small in comparison with A_1 so that (A_3/A_1) is also of the order k^2. We shall denote (A_3/A_1) by $\alpha_3 k^2$ in order-of-magnitude calculations, α_3 being a quantity of order unity. Under these circumstances the coefficients D_7 and D_9 will be of the orders k^6A and k^8A respectively. If we retain only terms with powers of up to k^4, the D_7 and D_9 terms need not be considered. The orders of D_1 , D_3 , and D_5 are respectively A, k^2A and k^4A.

The function cn^5 may be expanded into a Fourier series. Let us denote the Fourier coefficients by c_{2n+1} . They are respectively of the order 1, 1, 1, k^2, $\cdots$ when k^2 is small. Again by comparing the orders of magnitude of various terms in $F_3(\tau)$ we can approximate the given forcing function $P_0 \cos \omega\tau$ by $F_3(\tau)$ to the accuracy of k^4, provided that we have

$$\begin{aligned} D_1a_1 + D_3b_1 + D_5c_1 &= P_0 , \\ D_1a_3 + D_3b_3 + D_5c_3 &= 0, \\ D_1a_5 + D_3b_5 + D_5c_5 &= 0. \end{aligned} \tag{47}$$

Retaining only the necessary powers of k in a's, b's and c's to maintain a uniform accuracy of k^4 in (47), we have

$$\begin{aligned} a_1 &= 1 - \frac{1}{16} k^2 - \frac{9}{256} k^4 + O(k^6), \\ a_3 &= \frac{1}{16} k^2 + \frac{1}{32} k^4 + O(k^6), \\ a_5 &= \frac{1}{256} k^4 + O(k^6), \\ b_1 &= \tfrac{3}{4} - \frac{3}{32} k^2 + O(k^4), \\ b_3 &= \tfrac{1}{4} + \frac{3}{64} k^2 + O(k^4), \\ b_5 &= \frac{3}{64} k^2 + O(k^4), \\ c_1 &= \frac{10}{16} + O(k^2), \\ c_3 &= \frac{5}{16} + O(k^2), \\ c_5 &= \frac{1}{16} + O(k^2). \end{aligned} \tag{48}$$

Substituting these coefficients and the expressions for D's from (27) into (47) and retaining only terms up to k^4, we get

$$1 - \omega^2 + \tfrac{3}{4}\beta A_1^2 + \frac{k^2}{16}(1 - \omega^2)(12\alpha_3 - 1) + \frac{3}{32}k^2\beta A_1^2(20\alpha_3 - 1) - \frac{3}{256}k^4(1 - \omega^2)(8\alpha_3 + 3) = P_0/A_1 ,$$

$$\beta A_1^2 + \tfrac{1}{4}k^2(1 - 9\omega^2)(1 + 4\alpha_3) + \frac{15}{4}\alpha_3 k^2\beta A_1^2 + \frac{1}{16}k^4(1 - 9\omega^2)(2 + 3\alpha_3) = 0, \tag{49}$$

$$k^2\beta A_1^2(1 + 4\alpha_3) + \frac{1}{12}k^4(1 - 25\omega^2)(1 + 12\alpha_3) = 0.$$

Here the periodicity requirement (35) has been used to eliminate p^2 in the D's. These three equations completely determine the second order approximation. In principle at least, the values of A_1, α_3, and k^2 can be determined for any set of assigned values of P_0, ω, and β. The solution is valid only when the resulting k^2 is small and α_3 is of the order unity. Once the values of A_1, α_3 and k^2 are known, the response is given with an accuracy up to k^4 by

$$x = A_1\left\{\left[1 + k^2\left(-\frac{1}{16} + \tfrac{3}{4}\alpha_3\right) + k^4\left(-\frac{9}{256} - \frac{3}{32}\alpha_3\right)\right]\cos\omega\tau + \left[k^2\left(\frac{1}{16} + \tfrac{1}{4}\alpha_3\right) + k^4\left(\frac{1}{32} + \frac{3}{64}\alpha_3\right)\right]\cos 3\omega\tau + \left[k^4\left(\frac{1}{256} + \frac{3}{64}\alpha_3\right)\right]\cos 5\omega\tau + O(k^6)\right\}. \tag{50}$$

For approximations of higher order than the second, similar analyses may be made by employing solutions with three or more terms in the response. No essential difficulties will arise except the complexity of the mathematical computation.

Before concluding this section we summarize the essential features of the analysis presented here. Our aim is to find a steady state solution of (28). The general approach given here is, first to find some exact solutions of slightly different problems; in the present problem, namely: elliptic responses under favored elliptic forcings. At the first sight one may think that these exact solutions are very restricted. In actuality they represent a very wide class of solutions because of the freedom to assign arbitrary values to the parameters available at our disposal. Indeed, approximate solutions of various orders of (28) are but special cases of these exact solutions when k^2 is small. Furthermore they can be obtained simply by comparing the orders of magnitude of various terms in the expansions without having to solve new differential equations which appear in the perturbation method.

It is believed that this type of approach may find application in solving other non-linear problems.

5. Subharmonic solutions and simple elliptic subharmonics. Consider again the simple elliptic response solution under two-term elliptic forcing, discussed in Sec. 3.2. The functions cn and cn^3 may be expanded into Fourier series according to (33) and (40). The lowest components are respectively:

$$B_1 A\,\frac{2\pi}{kK}\,\frac{q^{1/2}}{1 + q}\cos\frac{\pi p\tau}{2K}$$

$$B_3 A^3\,\frac{1}{2k^2}\left\{2k^2 - 1 + \left(\frac{\pi}{2K}\right)^2\right\}\frac{2\pi}{kK}\,\frac{q^{1/2}}{1 + q}\cos\frac{\pi p\tau}{2K}.$$

If we set these two equal in magnitude and opposite in sign, the fundamental component vanishes and there results

$$\frac{B_1}{B_3 A^2} = -\left\{\frac{2k^2 - 1}{2k^2} + \frac{1}{2k^2}\left(\frac{\pi}{2K}\right)^2\right\}. \tag{51}$$

Equation (51) expressed in terms of p, k, and A implies

$$\beta A^2 = 2k^2p^2 - 2k^2[1 - p^2(1 - 2k^2)]\Big/\left[2k^2 - 1 + \left(\frac{\pi}{2K}\right)^2\right]. \tag{52}$$

Under this condition the forcing function is devoid of the fundamental component and yet the same component in the response is predominant. This is the well-known *subharmonic phenomenon.*

When $k = 0$, we have

$$\begin{aligned} B_1 &= 1 - p^2, & B_3 &= -\beta A^2 = \frac{4}{3}(1 - p^2), \\ x &= A \cos p\tau, & F(\tau) &= -\tfrac{1}{3}A(1 - p^2)\cos 3p\tau, \end{aligned} \tag{53}$$

which is the case of simple subharmonic of order $1/3$.

Similar analyses can be made for solutions with the *sn* function. It is to be noted here that although (52) leads to subharmonic solutions which are neither "pure" nor "simple" as defined by Rosenberg [6], they are nevertheless *exact* solutions.

5.1 Simple elliptic subharmonic solutions. Analogous to the simple subharmonics we can define *simple elliptic subharmonic* solution of order $1/r$ as that $x = A\ cn\ (p\tau, k)$ excited by a forcing function $F(\tau) = C\ cn\ (rp\tau, k)$. The condition under which such solutions exist can again be obtained by either the inverse method [7] or the technique of transformation described in [4].

Let us call $cn\ (v, k)$ as z and define a function $H_n(z)$ according to

$$H_n(z) = \text{cn}\ (nv, k). \tag{54}$$

This is a generalization of the Tchebycheff's polynomials of the first kind. The general function $H_n(z)$ may be found by the addition theorems of elliptic functions.

Next let us consider a solution $\eta = cn\ (u/r, k)$. It satisfies the following differential equation

$$\frac{d^2\eta}{du^2} + \frac{1}{r^2}(1 - 2k^2)\eta + \frac{2k^2}{r^2}\eta^3 = 0. \tag{55}$$

If we add to this a trivial identity

$$gH_r(\eta) = g\ \text{cn}\ (u, k) \tag{56}$$

we get a differential equation

$$\frac{d^2\eta}{du^2} + \frac{1}{r^2}(1 - 2k^2)\eta + \frac{2k^2}{r^2}\eta^3 + gH_r(\eta) = g\ \text{cn}\ (u, k) \tag{57}$$

which represents a special non-linear forced oscillation under an elliptic forcing. The response $cn\ (u/r, k)$ is also a simple elliptic function but of a period r times as large as the forcing function. This is defined as a simple elliptic subharmonic of order $1/r$.

In general $H_n(z)$ will not be a finite polynomial but rather a more general rational function in z. For example, $H_3(z)$ is given by

$$H_3(z) = \frac{z[1 - 4s^2 + k^2(6s^4) + k^4(-4s^6 + s^8)]}{1 + k^2(4s^6 - 6s^4) + k^4(4s^6 - 3s^8)}, \tag{58}$$

where $s^2 = sn^2\,(v, k) = 1 - z^2$. When the modulus k^2 is small, $H_n(z)$ can however be expanded into a series in powers of k^2. Thus, for $n = 3$ we have

$$\begin{aligned} H_3(z) = z[(-3 + 4z^2) + 4k^2z^2(-1 + 6z^2 - 9z^4 + 4z^6) \\ + 4k^4z^2(1 - z^2)^3(-1 + 3z^2 + 12z^4 - 16z^6) + O(k^6)]. \end{aligned} \tag{59}$$

Substituting this into (57) with $r = 3$ and retaining only terms up to k^2, we get

$$\begin{aligned} \frac{d^2\eta}{du^2} + \left(\frac{1 - 2k^2}{9} - 3g\right)\eta + \left[\frac{2k^2}{9} + 4g(1 - k^2)\right]\eta^3 + 24k^2g\eta^5 \\ - 36k^2g\eta^7 + 16k^2g\eta^9 = g\ \mathrm{cn}\,(u, k), \end{aligned} \tag{60}$$

as the differential equation which possesses a simple elliptic subharmonic solution of the order 1/3. The solution is of the accuracy k^2, where k^2 is assumed to be small.

When $k = 0$, the function $H_n(z)$ reduces to the usual Tchebycheff's polynomials of the first kind and Eq. (57) becomes the one given by Rosenberg in [6].

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