HOMOGENEOUS SOLUTIONS IN ELASTIC WAVE PROPAGATION*

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Summary. Busemann's method of conical flows is formulated for two-dimensional elastic wave propagation. The equations of motion are reduced to either Laplace's equation in two dimensions or the wave equation in one dimension, and solutions then are obtained with the aid of complex variable or characteristics theory, respectively. Special attention is paid to that class of problems in which the hyperbolic domains (of the two-dimensional wave equation) are simple wave zones, in consequence of which the solutions may be continued into the elliptic domain (of Laplace's equation) without explicitly posing the boundary conditions on the boundary separating the two domains. The method is applied to the diffraction of P- and SV-pulses by a perfectly weak half-plane.

1. Introduction. The purposes of this paper are(a) an exposition of the use of homogeneous solutions[1] in problems of two-dimensional, elastic wave propagation and (b) a complete solution for the transient diffraction of dilatational or vertically polarized shear waves by a perfectly weak half-plane.

Homogeneous solutions to the wave equation appear to have been discussed originally by Green [1] and were developed extensively by Bateman [2], but the most powerful applications were initiated only much later by Busemann [3] in his method of conical flows. Busemann's essential contribution was the introduction of Chaplygin's transformation [4] to reduce the wave equation to Laplace's equation, thereby permitting the use of function theory; (we note a striking adumbration of Busemann's work in Donkin's formula [5]). Busemann's technique has been applied extensively to supersonic wing problems [6] and to transient diffraction by a half-plane [7, 8] and by a wedge [9, 10]. We also note that a somewhat earlier reduction of the wave equation in three dimensions to Laplace's equation in two dimensions was effected by Sobolev [11] through superposition of plane waves. Sobolev applied his technique to the impulsive loading of an elastic half-space, but he does not appear to have emphasized either the important role played by homogeneity or the potential generality.

[*Note added in press.* It appears that Sobolev may have given a more general treatment in Ch. XII of the 1937 Russian translation (and extension) of P. Frank and R. von Mises, *Differential and integral equations of mathematical physics*, a reference cited by M. M. Fridman, Dokl. Akad. Nauk. SSSR **66**, 21–24 (1949)].

We may anticipate a homogeneous solution to a physical problem if either the data of that problem contain no characteristic length or if the only characteristic length must be derived from a parameter to which the solution must be proportional. The former category, in which supersonic flow past a semi-infinite cone provides an example, requires

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[1]A function $f(x_1, x_2, \cdots, x_n)$ is said to be *homogeneous of degree* m if it may be expressed in the form $x_1^m f(x_2/x_1, \cdots, x_n/x_1)$.

no assumption of linearity, but in the latter category linearization of the boundary conditions generally is a necessary prerequisite[2] (although the differential equation usually need not be linearized). We also remark that solutions to the wave equation may be patched together along characteristics and that homogeneous solutions then may be used over limited regions; the essential requirement is invariance under a scale transformation of the region in question.

The immediate advantage of homogeneity is that the order of the governing partial differential equation may be reduced. This advantage proves especially great in those two-dimensional problems of elastic wave propagation that permit the displacement potentials for vertically polarized motion or the displacement for horizontally polarized motion to be posed as homogeneous functions of degree zero, for then we may apply Chaplygin's transformation directly (as in Sec. 3 below). To be sure, we could apply the same transformation if the vertically polarized displacements were homogeneous and of degree zero, but then we should find it necessary to impose appropriate compatibility relations (as in the typical conical flow problems, where the velocity components are of degree zero and the velocity potential of degree one; see reference [6]).

We consider in Sec. 2 the equations of motion for homogeneous wave functions of degree zero, but we remark that—insofar as homogeneity of any degree may be assumed—the restriction to degree zero is not essential; more general results may be constructed via Duhamel superposition by virtue of the assumed linearity. We then go on, in Sec. 3, to reduce the wave equations to either Laplace's equation in two dimensions or the wave equation in one dimension and to construct general solutions with the aid of analytic function or characteristics theory, respectively.

Having the general formulation, we consider as an example the problem of transient diffraction by a half-plane when both dilatational and shear waves are present. This problem also has been solved by de Hoop [12], who formulated it as a Wiener-Hopf integral equation.[3] A closely related problem, the sudden opening of a semi-infinite crack in a previously uniform tension field, was solved by Maue [13], whose work appeals to the homogeneity properties discussed herein but culminates in an integral equation that he solves by the Wiener-Hopf technique. Maue's results could be applied directly to the normal incidence on a half-plane of two, symmetrically disposed compression waves characterized by a step discontinuity in normal stress.

[*Note added in press.* An incomplete solution based on Sobolev's method (*l.c.a.*), assuming an incident wave proportional to the time-integral of (4.1) below, has been given by M. M. Fridman, Dokl. Akad. Nauk. SSSR **66**, 21–24 (1949). He attempts only to reduce the problem to quadratures, the completion of which would appear to be rather difficult].

Other problems to which the present method is applicable include the action of an impulsive line-load on a semi-infinite solid, for which many previous solutions exist [14], and the as yet unsolved problem of P- or SV-diffraction by a wedge-shaped cavity[4] (the corresponding SH problem is essentially that solved in references [9] and [10].

[2]Problems in which the solution is nonlinearly proportional to some input parameter also may be possible.

[3]Dr. de Hoop worked on the problem at the Institute of Geophysics, University of California, Los Angeles in 1956-7, His subsequent completion of the solution and the present work then were carried out independently.

[4]This problem is being attacked by one of Professor Knopoff's students.

EQUATIONS OF MOTION

2.1 The displacements. We consider those (essentially two-dimensional) wave motions in an isotropic, elastic medium for which the displacement vector $\mathbf{u}$ may be derived according to

$$\mathbf{u} = \nabla\phi + \nabla \times (\mathbf{k}\psi) + \mathbf{k}w, \tag{2.1}$$

where ϕ denotes the scalar potential for irrotational displacements (dilatational or *P-waves*) in a plane (the x, y- or r, θ-plane) normal to the unit vector $\mathbf{k}$ (along the z-axis), $\mathbf{k}\psi$ the vector potential for solenoidal displacements in this same plane (vertically polarized shear or *SV-waves*), and w a transverse displacement (horizontally polarized shear or *SH-waves*)[5]. It is known that ϕ, ψ, and w satisfy the wave equations [15]

$$c_1^2\nabla^2\phi = \phi_{tt}, \qquad c_2^2\nabla^2\psi = \psi_{tt}, \qquad c_2^2\nabla^2 w = w_{tt}, \tag{2.2a, b, c}$$

where c_1 and c_2 denote the velocities of dilatational and shear waves, respectively, and are given by

$$c_1^2 = (\lambda + 2\mu)/\rho \quad \text{and} \quad c_2^2 = \mu/\rho \tag{2.3a, b}$$

in terms of Lamé's constants λ and μ and the density ρ. We also introduce the speed ratio γ and the critical angle θ_c according to

$$\gamma = \cos\theta_c = c_2/c_1 = [\mu/(\lambda + 2\mu)]^{1/2}. \tag{2.4}$$

We now assume the boundary conditions to be such that ϕ, ψ, and w must be homogeneous functions of degree zero in the polar coordinates r and θ and the time t. We choose as our dimensionless, homogeneous coordinates the angle θ and either

$$\xi = c_1 t/r \quad \text{or} \quad \eta = c_2 t/r, \tag{2.5a, b}$$

in terms of which

$$\phi = \phi(\xi, \theta), \qquad \psi = \psi(\eta, \theta), \qquad w = w(\eta, \theta). \tag{2.6a, b, c}$$

The radial (u) and tangential (v) components of displacement then are found to be [note that $\xi(\partial/\partial\xi) \equiv \eta(\partial/\partial\eta) \equiv -r(\partial/\partial r)$]

$$u = r^{-1}(-\xi\phi_\xi + \psi_\theta) \quad \text{and} \quad v = r^{-1}(\phi_\theta + \eta\psi_\eta), \tag{2.7a, b}$$

where subscripts denote differentiation (as they do everywhere in this paper except in the following section, and, throughout, on the stress components τ_{ij}).

2.2. The stresses. The Cartesian stress tensor is given by

$$\tau_{ij} = \lambda\delta_{ij}\,\nabla\cdot\mathbf{u} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.8}$$

where u_i denotes a Cartesian component of the displacement with $i = 1, 2, 3$, δ_{ij} denotes

[5] We could achieve a more formal symmetry by deriving w from a second vector potential, but this would be unnecessarily circuitous.

the Kronecker delta, and the usual summation convention is implied. Transforming to cyclindrical polar coordinates and introducing ξ and η from (2.5a, b), we obtain

$$\tau_{rr} = 2(\lambda + \mu)r^{-2}\phi_{\xi\xi} - \tau_{\theta\theta} , \tag{2.9a}$$

$$\tau_{\theta\theta} = \mu r^{-2}\{[(\gamma^{-2} - 2\xi^2)\phi_\xi]_\xi + 2(\eta\psi)_{\eta\theta}\}, \tag{2.9b}$$

$$\tau_{zz} = \lambda r^{-2}\phi_{\xi\xi} , \tag{2.9c}$$

$$\tau_{r\theta} = \mu r^{-2}\{-2(\xi\phi)_{\xi\theta} + [(1 - 2\eta^2)\psi_\eta]_\eta\}, \tag{2.9d}$$

$$\tau_{rz} = -\mu r^{-1}\xi w_\xi , \quad \text{and} \quad \tau_{\theta z} = \mu r^{-1} w_\theta . \tag{2.9e, f}$$

We shall be interested especially in $\tau_{\theta\theta}$ and $\tau_{r\theta}$, which we also may express in the rather more symmetric forms

$$\tau_{\theta\theta} = \mu r^{-2}\frac{\partial}{\partial\eta}[(1 - 2\eta)^2\phi_\eta + 2\eta\psi_\theta] \tag{2.10a}$$

and

$$\tau_{r\theta} = \mu r^{-2}\frac{\partial}{\partial\eta}[-2\eta\phi_\theta + (1 - 2\eta^2)\psi_\eta]. \tag{2.10b}$$

2.3. The boundary conditions. We shall consider only boundaries that are either completely free (*weak boundary*) or completely constrained (*rigid boundary*). The corresponding boundary conditions at any point, where the components of the normal are n_i , are either

$$\tau_{ij}n_j = 0 \quad (\textit{weak}) \tag{2.11a}$$

or

$$u_i = 0 \quad (\textit{rigid}) \tag{2.11b}$$

provided that the curvature of the boundary is finite. The boundary conditions at a sharp edge may be inferred from the requirement that the strain energy in the neighborhood of the edge remain bounded, which implies

$$\tau_{ij}n_j = O(r^{-1/2}), \qquad r \to 0 \tag{2.12a}$$

or, equivalently,

$$u_i = O(r^{1/2}), \qquad r \to 0. \tag{2.12b}$$

Transformation of the Wave Equation

3.1. Wave equation in homogeneous variables. The end result of expressing $\nabla^2\phi$ in r and θ in (2.2a) and then posing the homogeneous solution (2.6a) is

$$(\xi^2 - 1)\phi_{\xi\xi} + \xi\phi_\xi + \phi_{\theta\theta} = 0. \tag{3.1}$$

We may obtain the corresponding equations for ψ and w simply by replacing ξ by η.

We remark that (3.1) is elliptic in $\xi > 1$ and hyperbolic in $\xi < 1$. If we regard ξ^{-1} and θ as polar coordinates (thereby referring all lengths to c_1t), the elliptic and hyperbolic domains correspond to the interior ($\xi > 1$) and exterior ($\xi < 1$) of the unit circle ($\xi = 1$).

The circle $\xi = 1$ or $r = c_1t$ evidently represents a singular wave front, across which we may expect discontinuities in the ξ-derivatives of sufficiently high order and in the neighborhood of which the solutions to (3.1) cannot be uniformly valid. Fortunately, this neighborhood proves to be extremely small for elastic solids, and we shall rest content with the statement that the solutions to (3.1) must be continuous across $\xi = 1$, while their first ξ-derivatives may be infinite like $(\xi - 1)^{-1/2}$ as $\xi \to 1+$. We also remark that the *second* derivatives of these solutions may be discontinuous across the characteristics of (3.1) in its hyperbolic domain.

3.2. The Chaplygin transformation. We may reduce (3.1) to Laplace's equation

$$\phi_{\theta\theta} + \phi_{ss} = 0 \tag{3.2a}$$

for points inside the unit circle through Chaplygin's transformation

$$s = -\cosh^{-1}\xi = \log\,[\xi - (\xi^2 - 1)^{1/2}]. \tag{3.3a}$$

Similarly, we obtain the one-dimensional wave equation

$$\phi_{\theta\theta} - \phi_{\sigma\sigma} = 0, \tag{3.2b}$$

with

$$\sigma = \cos^{-1}\xi. \tag{3.3b}$$

It follows that we may pose the solution for ϕ in the complementary forms

$$\phi = Rl F(\alpha), \qquad \xi > 1 \tag{3.4a}$$

with

$$\alpha = \theta + i\cosh^{-1}\xi, \tag{3.5a}$$

where $F(\alpha)$ is an analytic function of the complex variable α; or

$$\phi = F_+(\alpha_+) + F_-(\alpha_-), \tag{3.4b}$$

with

$$\alpha_{\pm} = \theta \pm \cos^{-1}\xi, \tag{3.5b}$$

where $F_{\pm}$ are arbitrary functions of the characteristic variables $\alpha_{\pm}$.

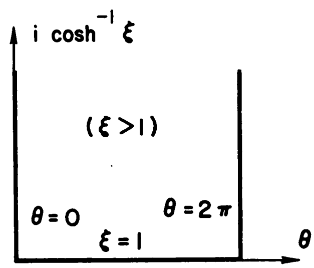


FIG. 1. The α-plane.

We may interpret the solution (3.4a) in terms of the mapping of the interior of the unit circle into the interior of a semi-infinite rectangle in the α-plane, as shown in Fig. 1 (we distinguish the planes $\theta = 0$ and $\theta = 2\pi$ in anticipation of the half-plane diffraction problem to be treated in Sec. 4; if a finite sector of the unit circle were excluded by the physical boundaries, as in the problem of diffraction by a wedge, we could subject α to further transformations). We also find it convenient—especially in discussing singularities—to introduce the transformation

$$z = \cos \alpha = \xi \cos \theta - i(\xi^2 - 1)^{1/2} \sin \theta. \tag{3.6}$$

The interior of the unit circle then maps on the exterior of the cut from -1 to $+1$, as shown in Fig. 2.

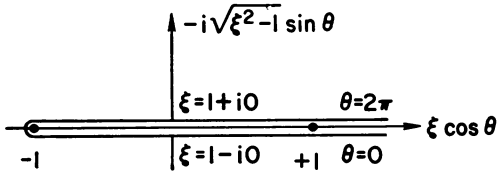


FIG. 2. The z-plane.

We may construct the characteristics of (3.5b) by drawing tangents to the unit circle in the clockwise (α_+) and counterclockwise (α_-) directions, as shown in Fig. 3. Any point in the hyperbolic domain ($\xi < 1$) then may be

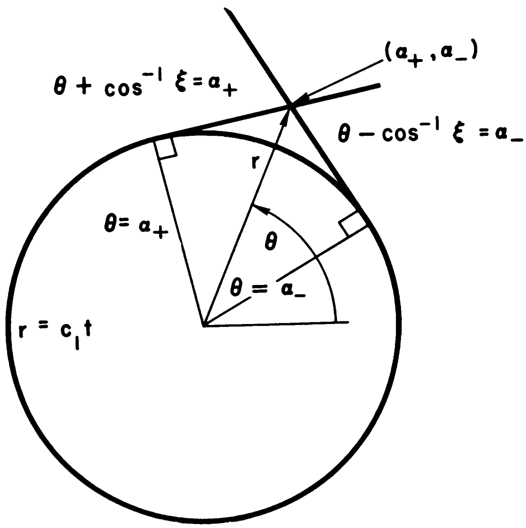


FIG. 3. Characteristics generated from the unit circle.

located by the intersection of two characteristics of opposite ($\pm$) families, and we may designate the coordinates of this point as (α_+ , α_-); if $\alpha_+ = \alpha_-$ the point is on the unit circle ($\xi = 1$), while if $\alpha_+ = \alpha_- \pm \pi$ the point is at infinity ($\xi = 0$).

3.3. Application to shear waves. We may apply the results of Secs. 3.1 and 3.2 equally to ψ and w if we replace ξ by η; thus,

$$\psi = R\, lG(\beta) \tag{3.7}$$

with

$$\beta = \theta + i \cosh^{-1} \eta, \qquad \eta > 1 \tag{3.8a}$$

or

$$\beta_{\pm} = \theta \pm \cos^{-1} \eta, \qquad \eta < 1, \tag{3.8b}$$

and similarly for w.

3.4. Simple wave zones. A *simple wave zone* is a region in which the solution to a hyperbolic equation depends on only one of the characteristic variables. The simple wave zones that we encounter in the present study are bounded by a characteristic, along which they are adjacent to a region of zero disturbance[6]; by an arc of the unit circle, along which they are adjacent to an elliptic region; and by a boundary on which appropriate conditions are prescribed, as shown in Fig. 4.

We infer from the foregoing definition and from the required continuity of $\phi_\theta = Rl\, F'(\alpha)$ that the solutions given by (3.4a) may be continued into a simple wave zone simply by continuing α as the variable (either α_+ or α_-) that is constant along the characteristic boundary; in so doing, we may drop the $+$ or $-$ subscript. We also may continue implicitly the prescribed boundary condition along the single family of characteristics to the unit circle and thereby seek the solutions in the elliptic and hyperbolic

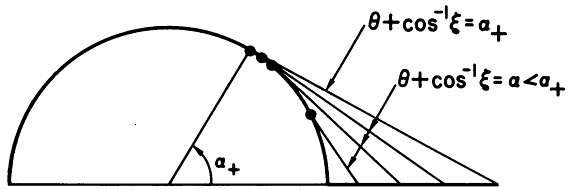


FIG. 4. Simple wave zone bounded by an α_+ characteristic, an arc of the unit circle ($0 \leq \theta \leq \alpha_+$), and a line ($\theta = 0$) of specified boundary condition.

domains of (3.1) simultaneously (whereas in a more general configuration it would be necessary to obtain the solution first in the hyperbolic domains in order to specify boundary conditions completely around the boundary of the elliptic domain).

We consider, as a particular example, a dilatational disturbance originating at $r = 0$ and $t = 0$ and moving out along a free boundary $\theta = 0$. In order to satisfy the boundary conditions there, shear waves generally must be excited by the dilatational wave front; in accordance with Huygens' principle, each of these may be regarded as a cylindrical wave having its center at $r = c_1\tau$ and $\theta = 0$ and having a radius $c_2\ (t - \tau)$, where $0 \leq \tau \leq t$. The envelope of these waves then consists of that portion of the characteristic $\theta + \cos^{-1} \eta = \theta_c$ intercepted between the wave front $r = c_2 t$ and the free boundary, as shown in Fig. 5. Such envelopes (which also arise in problems, elastic

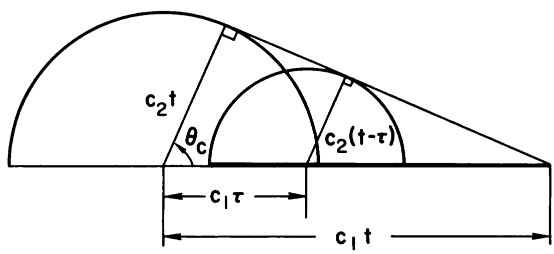


FIG. 5. Simple wave zone for shear waves excited by dilatational wave front moving along $\theta = 0$.

[6]It follows from characteristics theory that a simple wave zone is the only type of hyperbolic region that may adjoin a region of zero (or uniform) disturbance.

or acoustic, where two media of different wave speeds are contiguous) sometimes have been designated as *head waves*; however, following the terminology of characteristics theory in supersonic flow [16], we shall designate them as *Mach lines* or *Mach envelopes.*

3.5. Transformation of the displacements. We consider only the radial and tangential displacements u and v; the transverse displacement w is given directly by a solution of the type (3.7) and requires no further transformation. Substituting (3.4a) and (3.7) in (2.7a, b), we obtain

$$u = r^{-1}Rl[-i\xi(\xi^2 - 1)^{-1/2}F'(\alpha) + G'(\beta)] \tag{3.9a}$$

and

$$v = r^{-1}Rl[F'(\alpha) + i\eta(\eta^2 - 1)^{-1/2}G'(\beta)] \tag{3.9b}$$

for points in the elliptic domains; if α is continued as $\alpha_\pm$ we have only to replace $i(\xi^2 - 1)^{-1/2}$ by $\mp (1 - \xi^2)^{-1/2}$ and similarly for β.

The results (3.9a, b) may be continued into simple wave zones, but we shall find it more convenient to utilize the fact that the P- and SH-wave displacements must be normal and parallel to their respective characteristics (see Fig. 6). Designating these displacements as u^α and v^β (v^β is measured positive in the direction of increasing θ for both $\pm$ characteristics), we find

$$u^\alpha = \pm(r^2 - c_1^2t^2)^{-1/2}\phi_\alpha , \qquad \alpha = \theta \pm \cos^{-1} \xi \tag{3.10a}$$

and

$$v^\beta = \mp(r^2 - c_2^2t^2)^{-1/2}\psi_\beta , \qquad \beta = \theta \pm \cos^{-1} \eta. \tag{3.10b}$$

We emphasize that the total displacements generally contain both P- and SH-components and that the hyperbolic domains of these components do not coincide.

3.6. Solution procedure. The foregoing developments suggest the following, general procedure for the solution of those problems that admit homogeneous functions of degree zero and in which the hyperbolic domains may be identified as simple wave zones.

a. Pose the solutions in the form of (3.4a) and (3.7). We shall find it expedient, in so doing, to separate out prescribed components, such as incident waves [e.g., (4.2a, b)].

b. Convert the initial conditions to conditions on the wave fronts [e.g., (4.4a, b)].

c. Calculate the displacements and/or stresses in forms similar to (3.9a, b) and impose the prescribed boundary conditions [e.g., (4.6a, b)].

d. Express the coordinates that vary on the boundaries (ξ and/or η in Sec. 4, where the boundaries are prescribed by fixed θ) in terms of α and β there (e.g., $\xi = \cos \alpha$ on $\theta = 0$) and substitute in the boundary conditions to obtain equations that contain only α and/or β.

e. Express the prescribed functions (representing applied loads, displacements, or incident waves) in terms of α or β on the boundaries. We shall find (Appendix A) that the resulting expressions are characterized essentially by their singularities (generally poles) but contain arbitrary functions.

f. Determine the relation between α and β on the boundaries (e.g., $\cos \beta = \gamma \cos \alpha$ on $\theta = 0$) and eliminate (at least implicitly) either α or β from the boundary conditions. We remark that either the same or different relations between α and β may be obtained on different boundaries; thus, we shall find only a single relation in Sec. 4, but two different relations would be obtained in the problems of P- and SV-wave diffraction by a wedge.

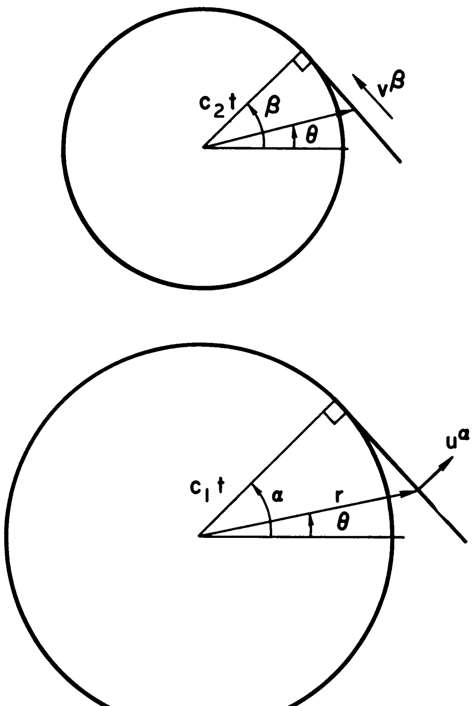


FIG. 6. The velocities in the hyperbolic domains.

g. Determine functions that satisfy the boundary conditions. We find that, in the problems to be considered subsequently, only the first derivatives of F and G enter the boundary conditions; accordingly, we may continue these conditions into either the α or β domains (see **f** above) and solve for F' and G'. If the form of the boundary conditions is different on different boundaries the resulting expressions for F' and G' must be developed as superpositions of the algebraic solution of the different conditions.

h. The solutions so obtained will contain the arbitrary functions introduced in **e**, and we must determine them in such a way that the final solutions satisfy the initial conditions, exhibit singularities on the physical boundaries only as physically appropriate (e.g., stress may be infinite like $r^{-1/2}$ at an edge), and are completely analytic in the interior of their elliptic domains. We find that the first two of these conditions may be met in a relatively straightforward manner (indeed, the initial conditions are likely to be satisfied automatically once the wave front geometry is determined). We also find that solutions satisfying the third condition of analyticity often may be obtained by inspection, but in Sec. 4 the removal of spurious singularities introduces appreciably greater complexity into the analysis. We find that the undesired singularities may be factored out through Cauchy integral-theorem representations, as in the solution of Wiener-Hopf integral equations.

DIFFRACTION BY HALF-PLANE

4.1. Formulation. We consider (see Fig. 7) the P-wave

$$\phi^i = H[\xi - \cos(\theta - \theta_1)] \tag{4.1}$$

to be incident on the weak half-plane $\theta = 0, 2\pi$; H denotes Heaviside's step function $[H(x) = 0,1$ as $x <, > 0]$ and θ_1 the angle of incidence. It would be equally simple, in principle, to provide for the simultaneous incidence of an SV-wave; this would almost double the length of the subsequent equations, however, and we shall rest content with sketching in the results by analogy (see Sec. 4.5).

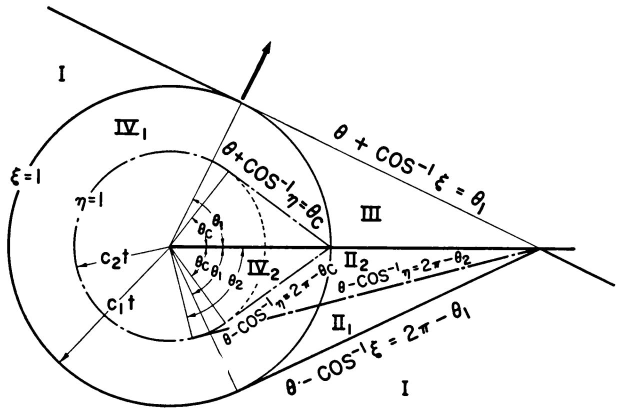


FIG. 7. Diffraction of P-wave by half-plane.

Our problem evidently admits no characteristic length other than that defined by the amplitude of the incident wave (which we take to be unity); accordingly, we may pose solutions in the form

$$\phi = \phi^i + Rl\, F(\alpha) \quad \text{and} \quad \psi = Rl\, G(\beta). \tag{4.2a, b}$$

We remark that $F(\alpha)$ and $G(\beta)$ will include specularly reflected P- and SV-waves emerging at the angles $2\pi - \theta_1$ and $2\pi - \theta_2$, respectively, where

$$\cos \theta_2 = \gamma \cos \theta_1 . \tag{4.3a}$$

If $0 < \theta_1 \leq \pi/2$ ϕ^i constitutes the entire disturbance for $t < 0$, but if $\pi/2 < \theta_1 < \pi$ these specular reflections, as given by (4.24b, c) below, must be included for $t < 0$ as well as $t > 0$. We also remark that (4.3a) may be generalized to read

$$\cos \beta = \gamma \cos \alpha, \qquad \theta = 0, \qquad 2\pi, \tag{4.3b}$$

since $\eta = \gamma\xi$ on the half-plane; (4.3b) then yields (4.3a) at the point of intersection of the plane-wave fronts (see Fig. 7).

Turning to the scattered waves, we first note that, since the P-wave disturbance from the edge at $r = 0$ originates at $t = 0$ and travels outward with speed c_1, the region of influence of this edge consists of the interior of the circle $\xi = 1$, which we then may identify as the elliptic domain for $F(\alpha)$. Similarly, the elliptic domain for $G(\beta)$ is the interior of the circle $\eta = 1$; but, in accordance with the discussion of Sec. 3.4, the SV-wave region of non-specular scattering also comprises the simple wave zones bounded by the Mach envelopes $\theta + \cos^{-1} \eta = \theta_c$ and $\theta - \cos^{-1}\eta = 2\pi - \theta_c$. Summing up, we designate the various zones according to (see Fig. 7):

I: incident wave zone,
II_1 : P-wave zone of specular reflection,

II_2 : SV-wave zone of specular reflection,
III: shadow zone,
IV_1 : P-wave scattering zone ($\xi > 1$),
IV_2 : SV-wave scattering zone;

this last zone comprises both elliptic and hyperbolic domains.

The initial conditions dictate: (a) $\phi = \phi^i$ on that portion of the P-wave scattering circle $\xi = 1$ intercepted between the incident wave front and the specularly reflected P-wave front and (b) $\psi = 0$ on the Mach envelope $\theta + \cos^{-1} \eta = \theta_c$ and on that portion of the SV-wave scattering circle intercepted between this Mach envelope and the specularly reflected SV-wave front (note that $\theta_2 > \theta_c$ for all θ_1). Imposing these conditions on (4.2a, b), we obtain

$$Rl\, F(\theta) = 0, \qquad \theta_1 < \theta < 2\pi - \theta_1 \tag{4.4a}$$

and

$$Rl\, G(\theta) = 0, \qquad \theta_c \le \theta < 2\pi - \theta_2\,, \tag{4.4b}$$

where the end-point $\theta = \theta_c$ comprises the aforementioned Mach envelope.

The boundary conditions require $\tau_{r\theta}$ and $\tau_{\theta\theta}$ to vanish on the half-plane; integrating (2.10a, b) from $\eta = 0$ (i.e., either $t = 0$ or $r = \infty$) yields

$$(1 - 2\eta^2)\phi_\eta + 2\eta\psi_\theta = 0 \tag{4.5a}$$

and

$$-2\eta\phi_\theta + (1 - 2\eta^2)\psi_\eta = 0, \qquad \theta = 0, \qquad 2\pi. \tag{4.5b}$$

Substituting (4.2a, b) in (4.5a, b), we obtain

$$\gamma^{-1}(1 - 2\eta^2)\{\delta(\xi - \cos\theta_1) + Rl[i(\xi^2 - 1)^{-1/2}F'(\alpha)]\} + 2\eta Rl[G'(\beta)] = 0 \tag{4.6a}$$

and

$$-2\eta\{-\sin\theta_1\,\delta(\xi - \cos\theta_1) + Rl[F'(\alpha)]\} + (1 - 2\eta^2) \cdot Rl[i(\eta^2 - 1)^{-1/2}G'(\beta)] = 0, \qquad \theta = 0, 2\pi. \tag{4.6b}$$

4.2. Solution for F′ and G′. We now rewrite (4.6a,b) in terms of α and β and represent the delta functions as in Appendix A to obtain

$$Rl[\csc\alpha\cos 2\beta F'(\alpha) + 2\gamma\cos\beta G'(\beta)] = Rl[A(\cos\alpha)/i\pi(\cos\alpha - \cos\theta_1)], \tag{4.7a}$$

$$Rl[-2\cos\beta F'(\alpha) + \csc\beta\cos 2\beta G'(\beta)] = -Rl[B(\cos\beta)/i\pi(\cos\alpha - \cos\theta_1)], \tag{4.7b}$$

$$A(\cos\alpha) = \pm\cos 2\theta_2\,, \qquad \alpha = \begin{matrix}\theta_1\\ 2\pi - \theta_1\end{matrix}\,, \tag{4.8a}$$

$$B(\cos\beta) = \pm\gamma\sin 2\theta_1\,, \qquad \beta = \begin{matrix}\theta_2\\ 2\pi - \theta_2\end{matrix}\,, \tag{4.8b}$$

and

$$\mathrm{Im}\, A(\xi) = \mathrm{Im}\, B(\eta) = 0. \tag{4.8c}$$

Our choice of arguments in A and B is dictated merely by convenience; in this connection, we note that $\beta = \theta_2$ is equivalent to $\alpha = \theta_1$ [see (4.3a, b)] in (4.7a, b).

We may continue (4.7a, b) into either the α- or β-domain by eliminating either β or α, respectively, through (4.3b) and removing the Rl operators [i.e., the equations so obtained yield solutions that satisfy the boundary conditions (4.3b) and (4.7a, b)]. Solving the resulting equations, we obtain

$$F'(\alpha) = \sin\alpha(A\cos 2\beta + \gamma B\sin 2\beta)/i\pi(\cos\alpha - \cos\theta_1)D(\gamma\cos\alpha) \tag{4.9a}$$

and

$$G'(\beta) = \sin\beta(\gamma A\sin 2\alpha - B\cos 2\beta)/i\pi(\cos\alpha - \cos\theta_1)D(\cos\beta), \tag{4.9b}$$

where

$$D = (2\gamma^2\cos^2\alpha - 1)^2 + 4\gamma^3\sin\alpha\cos^2\alpha(1 - \gamma^2\cos^2\alpha)^{\frac{1}{2}} \tag{4.9c}$$

$$= \cos^2 2\beta + 4\sin\beta\cos^2\beta(\gamma^2 - \cos^2\beta)^{1/2}, \tag{4.9d}$$

with $\cos\beta$ and $\cos\alpha$ defined by (4.3b) in (4.9a) and (4.9b), respectively.

The denominator $D(\gamma\cos\alpha)$, as given by (4.9c), has branch points at $\cos\alpha = \pm 1/\gamma$ and (at least for normal values of γ) zeros at $\cos\alpha = \pm\xi_R$, where

$$\xi_R = c_1/c_R, \tag{4.10a}$$

and c_R denotes the wave speed for Rayleigh surface waves. Similarly, $D(\cos\beta)$ has branch points at $\beta = \theta_c$ and $\pi - \theta_c$ ($\cos\beta = \pm\cos\theta_c$) and zeros at $\cos\beta = \pm\eta_R$, where

$$\eta_R = \gamma\xi_R = c_2/c_R. \tag{4.10b}$$

We choose the branch cuts so that they do not enter the domains of regularity, $\xi > 1$ and $\eta > 1$, and define the radicals to be positive and real for $\cos\alpha = 0$ and $\cos\beta = 0$.

We find it convenient, in the subsequent determination of A and B, to introduce the notation

$$z_1 = \cos\alpha \quad \text{and} \quad z_2 = \cos\beta, \tag{4.11a, b}$$

and to examine the singularities of F and G in z_1- and z_2-planes. The elliptic domains map onto the z_1- and z_2-planes cut from -1 to $+\infty$ (the cuts from -1 to 1 correspond to $\xi = 1$ and $\eta = 1$, while the cuts from $+1$ to $+\infty$ correspond to the half-plane separating $\theta = 0$ and $\theta = 2\pi$), and we require

$$\frac{dF}{dz_1} = -\frac{F'(\alpha)}{\sin\alpha} = \frac{A(z_1)(2\gamma^2 z_1^2 - 1) + 2\gamma^2 B(\gamma z_1)z_1(1 - \gamma^2 z_1^2)^{1/2}}{(-i\pi)(z_1 - \cos\theta_1)D(\gamma z_1)} \tag{4.12a}$$

and

$$\frac{dG}{dz_2} = -\frac{G'(\beta)}{\sin\beta} = \frac{2A(z_2/\gamma)z_2(\gamma^2 - z_2^2)^{1/2} - \gamma B(z_2)(2z_2^2 - 1)}{(-i\pi)(z_2 - \cos\theta_2)D(z_2)} \tag{4.12b}$$

to be regular in these cut planes. We also require

$$dF/dz_1 = O(z_1^{-1/2}), \qquad z_1 \to \infty \tag{4.12c}$$

and

$$dG/dz_2 = O(z_2^{-1/2}), \qquad z_2 \to \infty \tag{4.12d}$$

in order to meet the edge singularity requirements of (2.12), which imply (see Appendix B) $F'(\alpha)$, $G'(\beta) = O(r^{-1/2})$, $r \to 0$.

The requirement that dF/dz_1 be regular in the z_1-plane cut from -1 to $+\infty$ implies that $A(z_1)$ and $B(\gamma z_1)$ must cancel the singularities exhibited by $D(\gamma z_1)$ at $z_1 = -1/\gamma$ and $z_1 = -\xi_R$ [see (4.9c) and discussion following]; if we satisfy this requirement, it will follow that $A(z_2/\gamma)$ and $B(z_2)$ will cancel the singularities of $D(z_2)$ at $z_2 = -1$ and $z_2 = -\eta_R$, although only the latter cancellation may be posed as an *a priori* requirement (indeed, dG/dz_2 still will possess a branch point at $z_2 = -1$). We can achieve these cancellations by factoring D according to

$$D(z) = D_+(z)D_-(z), \tag{4.13}$$

such that $D_\pm(z)$ have singularities only in $Rlz \gtrless 0$.

At this point, our analysis parallels that of both de Hoop [12] and Maue [13], for $D(z)$ enters the transform kernel of the Wiener-Hopf integral equation and must be factored in the same way; we obtain[7]

$$D_\pm(z) = [2(1-\gamma^2)]^{1/2}(\eta_R \mp z)/L(\pm z), \tag{4.14}$$

where

$$L(z) = \exp\left\{\frac{1}{\pi}\int_\gamma^1 \frac{\chi(\zeta)}{\zeta - z}\, d\zeta\right\}, \tag{4.15}$$

and

$$\chi(z) = \tan^{-1}\left[\frac{z^2(1-z^2)^{1/2}(z^2-\gamma^2)^{1/2}}{(z^2-\frac{1}{2})^2}\right], 0 \le \chi \le \frac{\pi}{2} \tag{4.16}$$

is the phase angle of $D(z)$ for points on the bottom of the cut from $\zeta = 0$ to $\zeta = \gamma$. The path of integration in (4.15) must be indented over or under z if z tends to the real axis interval $(\gamma, 1)$ from $\operatorname{Im} z < 0$ or > 0, respectively, and we then may rewrite (4.15) according to

$$L(\cos\theta) = L_C(\cos\theta)e^{\mp i\chi(\cos\theta)}, \qquad \begin{matrix} 0 \le \theta \le \theta_c \\ 2\pi - \theta_c \le \theta \le 2\pi \end{matrix}, \tag{4.17}$$

where L_C denotes L based on the Cauchy principal value of the integral in (4.15)—viz.,

$$L_C(x) = \left(\frac{1-x}{x-\gamma}\right)^{\chi(x)/\pi} \exp\left\{\frac{1}{\pi}\int_\gamma^1 \frac{\chi(\zeta)-\chi(x)}{\zeta - x}\, d\zeta\right\}, \gamma \le x \le 1. \tag{4.18}$$

Numerical values of $\chi(z)$ and $L(z)$ have been calculated[8] for $\gamma = 3^{-1/2}$ ($\lambda = \mu$) and are plotted in Figs. 8 and 9. We remark that these results have application to other problems in elastic wave propagation (e.g., Maue's results, [13]).

Having the result (4.13), we can cancel the undesired singularities in D by including the factors $D_-(\gamma z_1)$ and $D_-(z_2)$ in A and B, respectively. Then, since $D_-(\gamma z_1)$ takes the same values at $\alpha = \theta_1$ and $2\pi - \theta_1$ ($z_1 = \cos\theta_1$) and is $O(z_1)$ at infinity, we require an additional factor in A that takes opposite values on $\alpha = \theta_1$ and $2\pi - \theta_1$ in consequence of (4.8a), is $O(z_1^{-1/2})$ in consequence of (4.12c), and is regular in the cut z_1-plane; we

[7]The details of the factorization may be found in Ref. [13]. The results were derived independently by the writer and therefore provide a check on those of Refs. [12] and [13].

[8]I am indebted to Professor Knopoff for aid with these and the subsequent calculations.

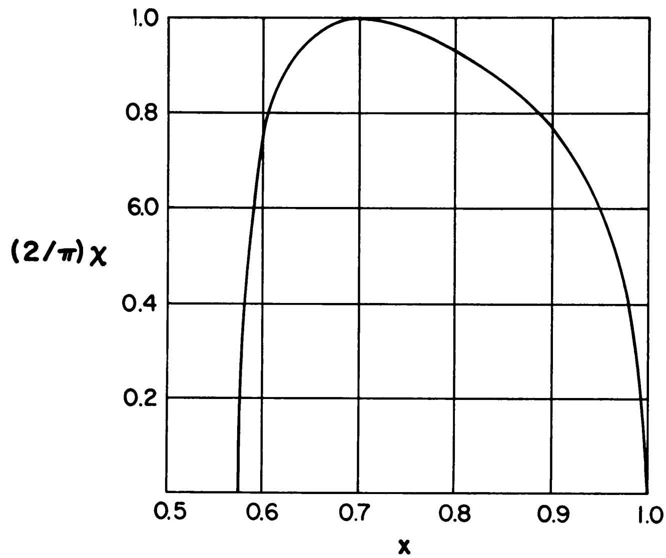


FIG. 8. $(2/\pi)\chi(x)$, as given by (4.16) for $\gamma = 3^{-1/2}$.

can satisfy these requirements with the factor $(1 + z_1)^{-1/2}$. Similar requirements apply to B, but $B(\gamma z_1)$ also must remove the branch point at $z_1 = -1/\gamma$ in the numerator of dF/dz_1; we can satisfy all of these requirements with the factor $(1 + z_2)^{-1/2}$. Introducing the normalization factors dictated by (4.8a, b), we conclude that

$$A(\cos\alpha) = \cos(\theta_1/2)\cos(2\theta_2)\sec(\alpha/2)D_-(\gamma\cos\alpha)/D_-(\cos\theta_2) \tag{4.19a}$$

and

$$B(\cos\beta) = \gamma\sin(2\theta_1)\cos(\theta_2/2)\sec(\beta/2)D_-(\cos\beta)/D_-(\cos\theta_2). \tag{4.19b}$$

Substituting (4.19a, b) in (4.9a, b) or (4.12a, b), we obtain the final results

$$F'(\alpha) = \frac{\{C_{11}\sin(\alpha/2)(2\gamma^2\cos^2\alpha - 1) + \gamma C_{21}\sin(2\alpha)[(1 - \gamma\cos\alpha)/2]^{1/2}\}L(\gamma\cos\alpha)}{i\pi\gamma(\xi_R - \cos\alpha)(\cos\alpha - \cos\theta_1)}, \tag{4.20a}$$

$$G'(\beta) = \frac{\{C_{11}\sin(2\beta)[\gamma(\gamma - \cos\beta)/2]^{1/2} - C_{21}\sin(\beta/2)\cos(2\beta)\}L(\cos\beta)}{i\pi(\eta_R - \cos\beta)(\cos\beta - \cos\theta_2)}, \tag{4.20b}$$

$$C_{11} = \cos(\theta_1/2)\cos(2\theta_2)C_0(\cos\theta_2), \tag{4.21a}$$

$$C_{21} = \gamma^2\sin(2\theta_1)\cos(\theta_2/2)C_0(\cos\theta_2), \tag{4.21b}$$

and

$$C_0(\cos\theta_2) = L(-\cos\theta_2)/(1 - \gamma^2)(\eta_R + \cos\theta_2), \tag{4.21c}$$

where L is defined by (4.15), α by (3.5), β by (3.8), γ by (2.4), ξ_R and η_R by (4.10a, b), and θ_2 by (4.3a). We observe that C_{11} and C_{21} are real for all θ_1 and that $C_{11} = -(1 - \gamma^2)^{-1/2}$ and $C_{21} = 0$ for normal incidence ($\theta_1 = \pi/2$). We also remark that the second (1) subscripts on C_{11} and C_{21} identify the excitation as a P-wave and that the results (4.20a, b) agree with those obtained by de Hoop [12].

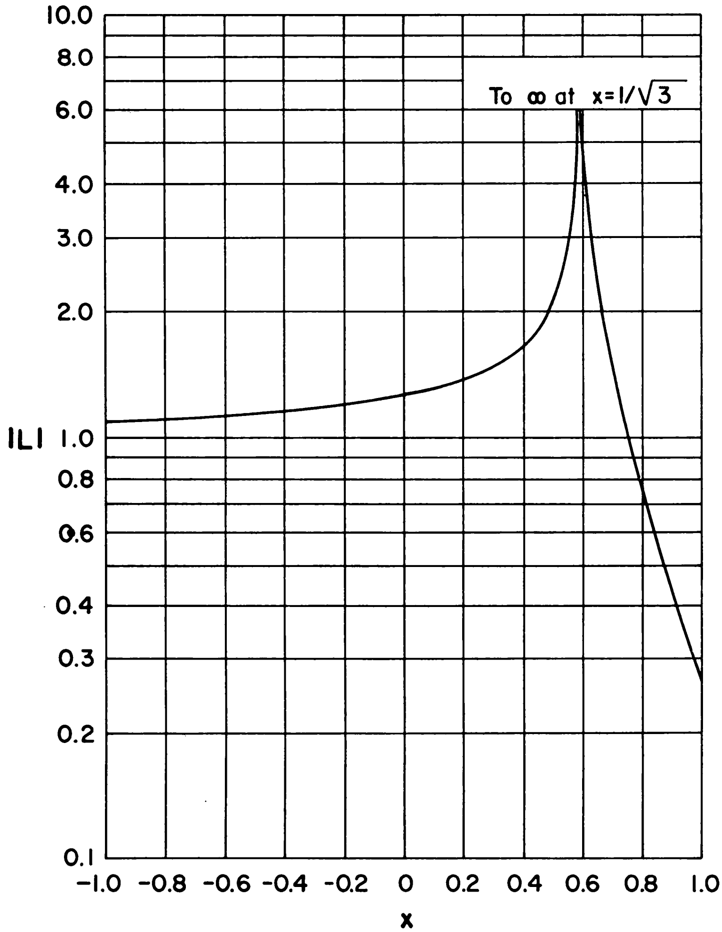


FIG. 9. The function $L(x)$, as given by (4.15) for $-1 \leq x < \gamma$ and by (4.18) for $\gamma < x \leq 1$ with $\gamma = 3^{-1/2}$.

4.3. The scattered wave fronts. We consider first the plane wave zones, where the solutions may be obtained directly from (4.8a, b) and (4.9a, b). Outside of the P-wave circle $\xi = 1$, both α and β are real, and the only contributions to $Rl\ F'(\alpha)$ and $Rl\ G'(\beta)$ must come from the poles at $\alpha = \theta_1$, $\alpha = 2\pi - \theta_1$, and $\beta = 2\pi - \theta_2$ (we remark that the numerator of G' vanishes at $\beta = \theta_2$, thereby cancelling the corresponding zero in the denominator); these poles yield

$$Rl\ F'(\alpha) = \sin\theta_1 \delta[\xi \cos\theta - (1 - \xi^2)^{1/2} \sin\theta - \cos\theta_1] \text{ in III}, \tag{4.22a}$$

$$Rl\ F'(\alpha) = R_{11} \sin\theta_1 \delta[\xi \cos\theta + (1 - \xi^2)^{1/2} \sin\theta - \cos\theta_1] \text{ in II}_1, \tag{4.22b}$$

and

$$Rl\ G'(\beta) = R_{21} \sin\theta_2 \delta[\eta \cos\theta + (1 - \eta^2)^{1/2} \sin\theta - \cos\theta_2] \text{ in II}_2, \tag{4.22c}$$

where

$$R_{11} = (-\cos^2 2\theta_2 + \gamma^2 \sin 2\theta_1 \sin 2\theta_2)/D(\cos\theta_2), \tag{4.23a}$$

$$R_{21} = 2\gamma^2 \sin 2\theta_1 \cos 2\theta_2/D(\cos\theta_2), \tag{4.23b}$$

and

$$D(\cos\theta_2) = \cos^2 2\theta_2 + \gamma^2 \sin 2\theta_1 \sin 2\theta_2 . \tag{4.23c}$$

Integrating (4.22a, b, c) and modifying the argument of the resulting step functions (only the locus across which this argument changes sign being important), we obtain the scattered waves

$$\phi^s = Rl\, F(\alpha) = -H[\theta_1 - (\theta + \cos^{-1}\xi)] \text{ in III}, \tag{4.24a}$$

$$\phi^s = Rl\, F(\alpha) = R_{11}H[(\theta - \cos^{-1}\xi) - (2\pi - \theta_1)] \text{ in } \mathrm{II}_1 , \tag{4.24b}$$

and

$$\psi = Rl\, G(\beta) = R_{21}H[(\theta - \cos^{-1}\eta) - (2\pi - \theta_2)] \text{ in } \mathrm{II}_2 . \tag{4.24c}$$

We have, then, the anticipated results that ϕ^s cancels ϕ^i in the shadow zone (III), and that ϕ^s and ψ in the zones of specular reflection ($\mathrm{II}_{1,2}$) agree with the known results [17] for reflection from an infinite, plane boundary.

Turning to the P-wave front, $\xi = 1-$, we find that $F'(\alpha)$ is imaginary there and yields only a radial velocity [see (3.9a, b)] that tends to infinity like $(\xi^2 - 1)^{-1/2}$ and then drops discontinuously to zero (except at $\theta = \theta_1$ and $2\pi - \theta_1$) for $\xi = 1+$. Setting $\xi = c_1t/r$ in the $(\xi^2 - 1)^{-1/2}$ factor and $\xi = 1$ elsewhere in the first term in (3.9a), we obtain

$$u \to (c_1^2t^2 - r^2)^{-1/2} f_{11}(\theta), \qquad r \to c_1t-, \tag{4.25a}$$

where

$$f_{11}(\theta) = Rl[-iF'(\theta)]. \tag{4.25b}$$

We may calculate the angular distribution function directly from (4.20a), since $iF'(\theta)$ is real; the result evidently will be singular at θ_1 and $2\pi - \theta_1$ in consequence of the plane waves of (4.22a, b). In the special case of normal incidence we obtain

$$f_{11}(\theta) = -\frac{\sin(\theta/2)(1 - 2\gamma^2\cos^2\theta)L(\gamma\cos\theta)}{\pi\gamma(1-\gamma^2)^{1/2}\cos\theta(\xi_R - \cos\theta)}, \quad \theta_1 = \pi/2. \tag{4.26}$$

We find a rather more complicated behavior at the SV-wave front ($\eta = 1$) in consequence of the simple wave zones in $r > c_2t$. We obtain for the (SV-contribution to the) tangential velocity just inside the semi-circular wave front

$$v \to (c_2^2t^2 - r^2)^{-1/2} f_{21}(\theta), \qquad r = c_2t-, \tag{4.27a}$$

where

$$f_{21}(\theta) = Rl[iG'(\theta)], \tag{4.27b}$$

but this component vanishes on $r = c_2t +$ only for $\theta_c \leq \theta \leq 2\pi - \theta_c$. The remaining intervals are in the simple wave zones, where (3.10b) yields

$$v^\beta = (r^2 - c_2^2t^2)^{-1/2} g_{21}(\theta \pm \cos^{-1}\eta), \tag{4.28a}$$

where

$$g_{21}(\theta) = \mp Rl[G'(\theta)], \qquad \begin{matrix} 0 < \theta < \theta_c \\ 2\pi - \theta_c < \theta < 2\pi \end{matrix}. \tag{4.28b}$$

We then may obtain v on $r = c_2t+$ by setting $\cos^{-1}\eta = 0$ in (4.28a), but we emphasize that the resulting approximation is not uniformly valid in the neighborhoods of $\beta = \theta_c$ and $2\pi - \theta_c$, where G' vanishes like $(\cos\beta - \cos\theta_c)^{1/2}$ in consequence of the boundary condition (4.4b). We may calculate f_{21} and g_{21} directly from (4.20b), making allowance for the fact that $L(\cos\theta)$ is real only for $\theta_c \le \theta \le 2\pi - \theta_c$; in the special case of normal incidence we obtain

$$f_{21}(\theta) = -h_{21}(\theta)L(\cos\theta), \qquad \theta_c \le \theta \le 2\pi - \theta_c, \tag{4.29a}$$

$$= -h_{21}(\theta)L_C(\cos\theta)\sin[\chi(\cos\theta)],$$

$$0 \le \theta \le \theta_c \quad \text{or} \quad 2\pi - \theta_c \le \theta \le 2\pi, \qquad \theta_1 = \pi/2, \tag{4.29b}$$

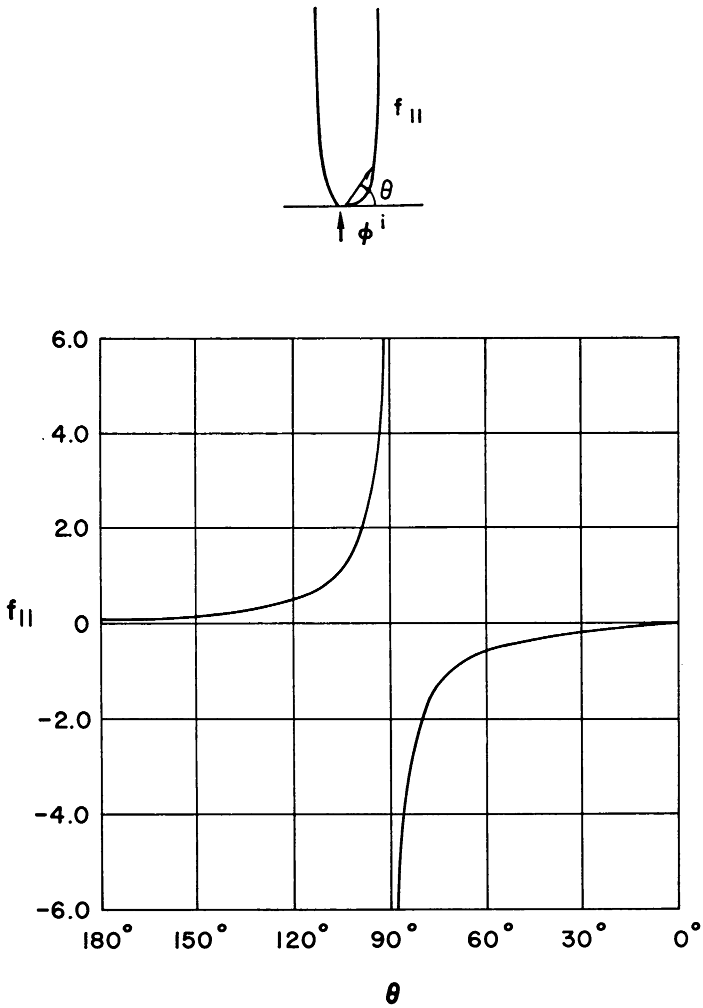


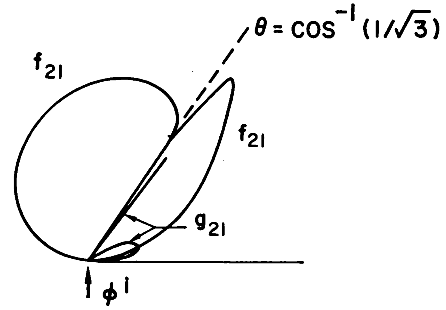
FIG. 10. The distribution of radial velocity along the P-wave circle ($r = c_1 t -$) resulting from normal incidence of a P-wave pulse; see (4.25a, b) and (4.26) with $\gamma = 3^{-1/2}$. The insert is a polar plot.

where

$$h_{21}(\theta) = \frac{1}{\pi}\left(\frac{2\gamma}{1-\gamma^2}\right)^{1/2}\frac{\sin\theta\,|\cos\theta-\gamma|^{1/2}}{(\eta_R-\cos\theta)}, \tag{4.29c}$$

and

$$g_{21}(\theta) = h_{21}(\theta)L_C(\cos\theta)\cos[\chi(\cos\theta)], \qquad \theta_1 = \pi/2. \tag{4.30}$$



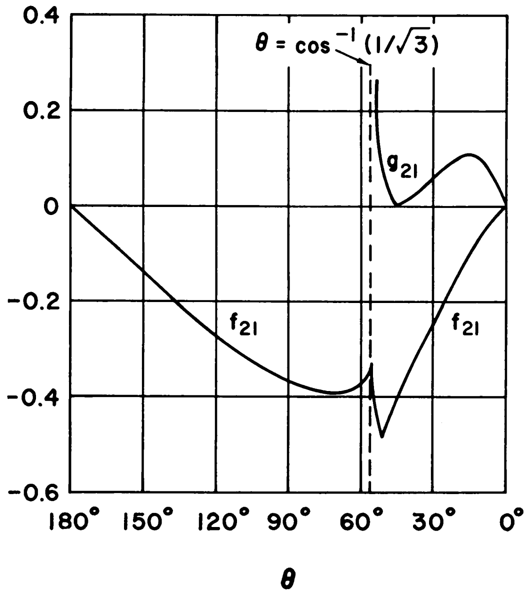


FIG. 11. The distribution of the tangential velocity along the S-wave circle resulting from the normal incidence of a P-wave pulse; see (4.27)-(4.30) with $\gamma = 3^{-1/2}$. The insert is a polar plot.

The numerical values of $f_{11}(\theta)$, $f_{21}(\theta)$, and $g_{21}(\theta)$, as given by (4.26), (4.29), and (4.30) with $\gamma = 3^{-1/2}$, are plotted in Figs. 10 and 11. We emphasize that the discontinuity in f_{11} at $\theta = \theta_1 = \pi/2$ is a direct consequence of that in ϕ^i.

4.4. Surface displacements. We may simplify the expressions for the surface displacements by introducing (4.3b) in (3.9a, b) to obtain

$$u = r^{-1}Rl\{\cot\alpha F'(\alpha) + G'(\beta)\} \tag{4.31a}$$

and

$$v = r^{-1}Rl\{F'(\alpha) - \cot\beta G'(\beta)\}, \qquad \theta = 0, 2\pi. \tag{4.31b}$$

Substituting (4.20a, b) in (4.31a, b), we obtain

$$u = Rl\left\{\frac{[(\gamma/2)C_{11}\sec(\alpha/2)\cos(2\beta - \alpha) + C_{21}\sin(\beta/2)]L(\cos\beta)}{i\pi r(\eta_R - \cos\beta)(\cos\beta - \cos\theta_2)}\right\} \tag{4.32a}$$

and

$$v = Rl\left\{\frac{[-\gamma C_{11}\sin(\alpha/2) + (\gamma/2)C_{21}\sec(\beta/2)\cos(2\beta - \alpha)]L(\cos\beta)}{i\pi r(\eta_R - \cos\beta)(\cos\beta - \cos\theta_2)}\right\}. \tag{4.32b}$$

We may reduce (4.32a, b) further if $r < c_2t$, for then the first and second terms in the numerator of (4.32a) are real and imaginary, respectively, and conversely for (4.32b). The real terms in the numerators make no contributions to the displacements except at the surface wave pole, where they yield delta functions. Noting that

$$\gamma\cos(2\beta - \alpha) = \cos 2\beta/2\cos\beta, \qquad \eta = \eta_R \tag{4.33}$$

in virtue of the evanescence of $D(\eta_R)$, we obtain

$$u = \mp\frac{C_{11}(2\eta_R^2 - 1)L(\eta_R)\delta(c_Rt - r)}{2^{3/2}(1 + \xi_R)^{1/2}\eta_R(\eta_R - \cos\theta_2)} \mp \frac{C_{21}r^{1/2}(c_2t - r)^{1/2}L(c_2t/r)}{\pi 2^{1/2}\eta_R(c_2t - r\cos\theta_2)(c_Rt - r)} \tag{4.34a}$$

and

$$v = \mp\frac{C_{21}(2\eta_R^2 - 1)L(\eta_R)\delta(c_Rt - r)}{2^{3/2}(1 + \eta_R)^{1/2}\eta_R^2(\eta_R - \cos\theta_2)} \pm \frac{C_{11}r^{1/2}(c_1t - r)^{1/2}L(c_2t/r)}{\pi 2^{1/2}\xi_R(c_2t - r\cos\theta_2)(c_Rt - r)}, \tag{4.34b}$$

$$\theta = \begin{matrix}0\\2\pi\end{matrix}, \qquad r < c_2t.$$

4.5. Incident SV-wave. We now assume an incident SV-wave in place of the incident P-wave of (4.1). Replacing (4.1) and (4.2a, b) by

$$\phi = Rl\,F(\alpha) \quad\text{and}\quad \psi = H[\eta - \cos(\theta - \theta_2)] + Rl\,G(\beta), \tag{4.35}$$

imposing the boundary conditions (4.5a, b), and then proceeding as in (4.6)–(4.20), we obtain solutions identical with (4.20a, b) if C_{11} and C_{21} therein are replaced by

$$C_{12} = \cos(\theta_1/2)\sin(2\theta_2)C_0 \quad\text{and}\quad C_{22} = -\cos(\theta_2/2)\cos(2\theta_2)C_0\,. \tag{4.36a, b}$$

We emphasize that, although θ_1 and θ_2 still are related by (4.3a), they now have somewhat different meanings—viz., θ_2 is the angle of incidence and has an admissible range $(0, \pi/2)$,[9] while θ_1, formally defined as the angle of reflection for the P-wave, can be real only if θ_2 lies in $(\theta_c, \pi/2)$. If θ_2 does lie in this interval the wave-front geometry differs from that of Fig. 7 only in that the incident wave zone (I) and the shadow

[9]No physical significance can be attached to $\pi/2 < \theta_2 \leq \pi$, for the incident SV-wave then would be preceded by a P-wave for all $t < 0$, and non-plane, scattered waves also would exist for all $t < 0$.

zone (III_2, say) will be bounded by those parts of $\beta_- = 2\pi - \theta_2$ and $\beta_+ = \theta_2$—rather than $\alpha_- = 2\pi - \theta_1$ and $\alpha_+ = \theta_1$—lying outside of the P-wave circle ($\xi = 1$). We then find

$$\psi^s = Rl\, G(\beta) = -H[\theta_2 - (\theta + \cos^{-1} \eta)] \text{ in } III_2, \tag{4.37a}$$

$$\phi = Rl\, F(\alpha) = R_{12}H[(\theta - \cos^{-1} \xi) - (2\pi - \theta_1)] \text{ in } II_1, \tag{4.37b}$$

$$\psi^s = Rl\, G(\beta) = R_{22}H[(\theta - \cos^{-1} \eta) - (2\pi - \theta_2)] \text{ in } II_2, \tag{4.37c}$$

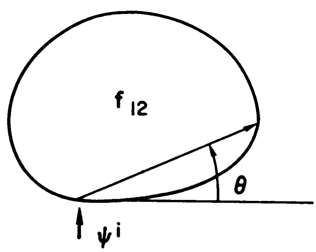
$$R_{12} = -2 \sin 2\theta_2 \cos 2\theta_2 / D(\cos \theta_2), \tag{4.38a}$$

and

$$R_{22} = R_{11}, \qquad \theta_c \leq \theta_2 \leq \pi/2, \tag{4.38b}$$

where R_{11} and D are given by (4.23a, c) for given $\theta_{1,2}$. The results (4.37b, c) agree with those for reflection from an infinite, plane boundary [18].

If θ_2 lies in the interval $(0, \theta_c)$ the zones of specular reflection no longer exist. The angle θ_1 then is complex, $\cos \theta_1$ is real but greater than unity, and the pole at $\alpha = 2\pi - \theta_1$ (F' remains regular at $\alpha = \theta_1$) lies in the P-wave circle on the illuminated side ($\theta = 2\pi$) of the half-plane, where it represents a surface wave moving with the dilatational wave speed c_1. The poles at $\beta = \theta_2$ and $2\pi - \theta_2$ lie inside the simple wave zones bounded by $\beta = \theta_c$ and $\beta = 2\pi - \theta_c$, repectively, and represent singular components of the complete disturbance there, although we remark that the pole at $\beta = \theta_2$ just cancels the incident wave in the angular interval $(0, \theta_2)$.



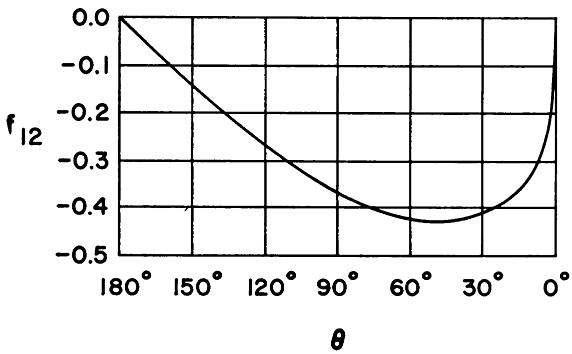


FIG. 12. The distribution of radial velocity on the P-wave circle ($r = c_1 t-$) resulting from normal incidence of an SV-wave pulse; see (4.39) with $\gamma = 3^{-1/2}$.

Turning to the non-plane wave fronts, the results (4.25), (4.27), and (4.28) remain valid if F' and G' are calculated on the basis of (4.36a, b); we may distinguish the results by writing f_{12}, f_{22}, and g_{22} in place of f_{11}, f_{21}, and g_{21}. In the special case of normal incidence we obtain

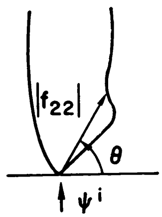
$$f_{12}(\theta) = -\frac{1}{\pi}\left(\frac{2}{1-\gamma^2}\right)^{1/2} \frac{\sin\theta(1-\gamma\cos\theta)^{1/2}L(\gamma\cos\theta)}{(\xi_R-\cos\theta)}, \quad \theta_2 = \frac{\pi}{2}, \tag{4.39}$$

$$\begin{aligned} f_{22}(\theta) &= -h_{22}(\theta)L(\cos\theta), \qquad \theta_c \le \theta \le 2\pi - \theta_c, \\ &= -h_{22}(\theta)L_C(\cos\theta)\cos[\chi(\cos\theta)], \end{aligned} \tag{4.40a}$$

$$0 \le \theta \le \theta_c \quad \text{or} \quad 2\pi - \theta_c \le \theta \le 2\pi, \qquad \theta_2 = \pi/2, \tag{4.40b}$$

where

$$h_{22}(\theta) = \frac{1}{\pi(1-\gamma^2)^{1/2}} \frac{\sin(\theta/2)\cos 2\theta}{\cos\theta(\eta_R - \cos\theta)}, \tag{4.40c}$$



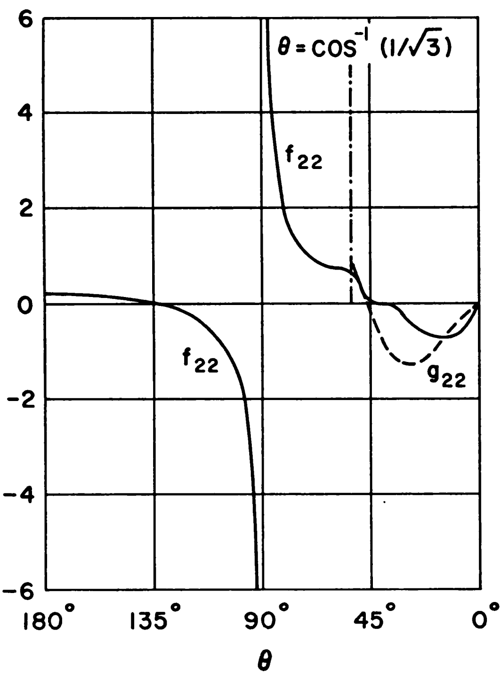


FIG. 13. The distribution of tangential velocity on the S-wave circle resulting from normal incidence of an SV-wave pulse; see (4.40) and (4.41) with $\gamma = 3^{-1/2}$.

and

$$g_{22}(\theta) = -h_{22}(\theta)L_C(\cos\theta)\sin[\chi(\cos\theta)], \qquad \theta_2 = \pi/2. \tag{4.41}$$

The numerical values of $f_{12}(\theta)$, $f_{22}(\theta)$, and $g_{22}(\theta)$ are plotted in Figs. 12 and 13.

4.6. Rigid half-plane. Assuming an incident P-wave and solutions in the form of (4.2a, b), we find that the requirement that u and v, as given by (2.7a, b) and (3.9a, b), vanish on the half-plane yields [see (4.7a, b)]

$$Rl\{\cot\alpha F'(\alpha) + G'(\beta)\} = \cos\theta_1\delta(\cos\alpha - \cos\theta_1), \tag{4.42a}$$

and

$$Rl\{F'(\alpha) - \cot\beta G'(\beta)\} = \sin\theta_1\delta(\cos\alpha - \cos\theta_1), \qquad \theta = 0, 2\pi. \tag{4.42b}$$

These equations may be solved as in (4.7)–(4.20), but the results are of rather limited interest, and we note here only that the counterpart of D is

$$D = \gamma\cos(\alpha - \beta) = z_2^2 + (1 - z_2^2)^{1/2}(\gamma^2 - z_2^2)^{1/2}, \tag{4.43}$$

which has no zeros in the cut plane (s).

Appendix A

Representation of $\delta(x)$. We require $f(z)$, $z = x + iy$, such that

$$Rl\, f(z) = \delta(x), \qquad y = 0\pm. \tag{A.1}$$

Noting that

$$\delta(x) = \pm\frac{1}{\pi}\lim_{y\to 0+}\left(\frac{y}{x^2 + y^2}\right), \tag{A.2}$$

we find that (A.1) is satisfied by

$$f(z) = a(z)/i\pi z, \tag{A.3a}$$

where $a(z)$ has no other poles on the real axis and satisfies

$$\operatorname{Im} a(x) = 0, \tag{A.3b}$$

and

$$a(x \pm i0) = \mp 1. \tag{A.3c}$$

We remark that this result also could have been deduced from Cauchy's residue theorem, which implies that $f(z)$ must have a simple pole with residue $\mp(i\pi)^{-1}$ at the origin and be real everywhere else on the real axis.

Appendix B

Asymptotic behavior at edge. We require the asymptotic behavior of the displacements given by (3.9a, b) in conjunction with the solutions of (4.9a, b). Expanding the radicals in (3.9a, b), we obtain

$$u = r^{-1}Rl\{-i[F'(\alpha) + iG'(\beta)] - \tfrac{1}{2}i\xi^{-2}F'(\alpha) + O(\xi^{-4}F')\} \tag{B.1a}$$

and

$$v = r^{-1}Rl\{[F'(\alpha) + iG'(\beta)] + \tfrac{1}{2}i\eta^{-2}G'(\beta) + O(\eta^{-4}G')\}. \tag{B.1b}$$

Now,

$$\cos \alpha \sim \xi e^{-i\theta}[1 + O(\xi^{-2})] \tag{B.2a}$$

and

$$\cos \beta \sim \eta e^{-i\theta}[1 + O(\eta^{-2})], \tag{B.2b}$$

so that $\cos \alpha$ and $\cos \beta$ are asymptotically related as in (4.3b). Substituting (B2a, b) in (4.9a, b), we find that

$$F'(\alpha) + iG'(\beta) = O(\xi^{-2}F', \eta^{-2}G'). \tag{B.3}$$

It then follows from (B.1a, b) that

$$u, v = r^{-1}O(\xi^{-2}F', \eta^{-2}G') = O(rF', rG'). \tag{B.4}$$

Imposing the requirement (2.12), we obtain

$$F', G' = O(r^{-1/2}) = O(\xi^{+1/2}, \eta^{+1/2}). \tag{B.5}$$

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