which exceed a certain critical value, damped oscillations will occur, but for values of α^2 less than this critical value, two aperiodic modes of decay appear. The solution of Eq. (19) at this critical point for the principal mode $l = 2$ is [3]

$$\sigma_{2;0}R^2/\nu = 3.69 \quad \text{and} \quad \sigma_{2;\nu}/\sigma_{2;0} = 0.968. \tag{22}$$

For a drop of water surrounded by air ($T_1 = 74$ dynes/cm) this gives a radius $R = 0.23$ mm. Drops larger than this critical radius will therefore tend to oscillate while smaller drops will be aperiodically damped.

References

1. H. Lamb, *Hydrodynamics*, 6th ed., Cambridge University Press, 1932
2. H. Lamb, *On the oscillations of a viscous spheroid*, Proc. London Math. Soc. (1), **13**, 51-66 (1881)
3. S. Chandrasekhar, *The oscillations of a viscous liquid globe*, Proc. London Math. Soc. (3), **9**, 141-149 (1959)

ON THE DIFFRACTION OF AN ARBITRARY PULSE BY A WEDGE OR A CONE*

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Abstract. By virtue of Green's Theorem, it is shown that for the diffraction of an arbitrary two-dimensional incident pulse by a wedge of angle μ, the ratio of the resultant velocity potential to the corresponding value of the incident pulse at the corner of the wedge at any instant is equal to $2\pi/(2\pi - \mu)$; and that for the diffraction of a three-dimensional pulse by a cone of solid angle ω, the ratio at the vertex of the cone is equal to $4\pi/(4\pi - \omega)$.

Two-dimensional space. The statement concerning diffraction of a pulse by a wedge is evidently true in the special case of an incident plane Heaviside pulse which was solved by Keller and Blank [1]. It therefore also follows for all incident pulses which are superpositions of plane Heaviside pulses, or limits of such superpositions. Since this includes all incident pulses it yields the preceding statement. However, these considerations depend upon knowing the exact solution in a special case which the following proof does not require.**

Let $t = 0$ be the instant at which the incident pulse $\varphi^{(i)}$ hits the corner of the wedge, which is located at the origin ($x_1 = 0$, $x_2 = 0$). Let $h(x_1, x_2)$ and $k(x_1, x_2)$ denote, respectively, $\varphi^{(i)}$ and $\varphi_t^{(i)}$ at an instant $t = -t_0 < 0$ if the corner is absent. If G represents the domain in the $x_1 - x_2$ plane outside which both h and k vanish, then the origin must lie outside G. When the wedge is present, the region G lies outside the wedge if the incident disturbance $\varphi^{(i)}$ has not hit either side of the wedge at $t = -t_0 < 0$. Then the resultant disturbance φ at any instant $t_1 > -t_0$ fulfills the wave equation and the same initial conditions as that of $\varphi^{(i)}$, i.e., in the region exterior to the wedge

$$\varphi(-t_0, x_1, x_2) = h(x_1, x_2) \quad \text{and} \quad \frac{\partial \varphi}{\partial t}(-t_0, x_1, x_2) = k(x_1, x_2). \tag{1}$$

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**This paragraph is based on a private communication from Prof. J. B. Keller, Institute of Mathematical Sciences, New York University.

In addition φ fulfills the boundary condition $\partial\varphi/\partial n = 0$ on the two sides of the wedge, where $\partial/\partial n$ means normal derivative.

To express the resultant velocity potential at the origin in terms of the initial data, it is necessary to reexamine the derivation of Volterra's formula in order to take care of the boundary condition. The derivation begins from Green's Theorem, which states [2]:

$$\int_A^* \left(n_t \frac{\partial v}{\partial t} - n_1 \frac{\partial v}{\partial x_1} - n_2 \frac{\partial v}{\partial x_2}\right) dA + \int_A^* v\left(n_t \frac{\partial \varphi}{\partial t} - n_1 \frac{\partial \varphi}{\partial x_1} - n_2 \frac{\partial \varphi}{\partial x_2}\right) dA = 0, \qquad (2)$$

where $v(t, x_1, x_2) = [(t_1 - t)^2 - x_1^2 - x_2^2]^{-1/2}$; and n_t, n_1, n_2 are the t —, x_1 — and x_2 components of the unit vector $\bar{n}$ normal to the surface A in the t, x_1, x_2 space. Without

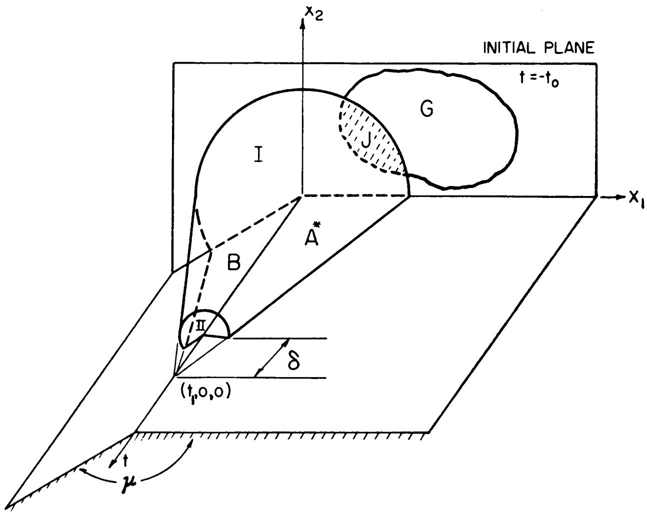

Fig. 1. Diffraction by a wedge

losing any generality the speed of sound is set equal to unity. The surface A consists of (Fig. 1):

(i) A^*, which is the part of the characteristic cone $1/v = 0$ lying in between the planes $t = -t_0$, $t = t_1 - \delta$ and the boundary planes.

(ii) I, which is the part of the plane $t = -t_0$, lying inside A^*.

(iii) II, which is the part of the plane $t = t_1 - \delta$, lying inside A^*.

(iv) B, which is the part of the boundary planes, lying inside A^* and between planes $t = -t_0$ and $t = t_1 - \delta$.

The finite part of both integrals over A^* vanishes as usual [2]. On the surface of the wedge the terms inside the brackets of the integrands become the normal derivative of v and φ, respectively. The former is shown to be zero by performing the differentiation, and the latter equals zero according to the boundary condition. Therefore, the integral over B vanishes.

As $\delta \to 0$, the finite part of the first integral over II approaches $(2\pi - \mu)\,\varphi(t_1, 0, 0)$ and the second integral approaches zero [2].

As a result, Eq. (2) together with the initial conditions, yields:

$$(2\pi - \mu)\varphi(t_1, 0, 0) = \int_J^* \left[kv - h\frac{\partial v}{\partial t}\right] dJ, \tag{3}$$

where J is the region of G which lies inside the cone A^*.

From Volterra's formula, the left side of the equation is equal to $2\pi\varphi^{(i)}(t_1, 0, 0)$; it follows that

$$\varphi(t_1, 0, 0) = \frac{2\pi}{2\pi - \mu}\varphi^{(i)}(t_1, 0, 0). \tag{4}$$

By the method of images it is shown that the relationship expressed in Eq. (4) is still valid, if the incident disturbance hits either or both sides of the wedge before it hits the corner [3].

Three-dimensional space. Let $t = 0$ be the instant the incident pulse $\varphi^{(i)}$ hits the cone* whose vertex is located at the origin of the x_1, x_2 — and x_3 — axes. In the absence of the cone, the incident potential at the origin can be expressed by Kirkchhoff's formula [4]

$$\varphi^{(i)}(t, 0, 0, 0) = -\frac{1}{4\pi}\int_S \left\{[\varphi^{(i)}]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{\partial\varphi^{(i)}}{\partial n}\right] - \frac{1}{r}\frac{\partial r}{\partial n}\left[\frac{\partial\varphi^{(i)}}{\partial t}\right]\right\} dS, \tag{5}$$

where r is the distance from the origin,

$\partial/\partial n$ means differentiation along the outward normal to the surface S, $[\varphi^{(i)}] =$

*Here the instant $t = 0$ is defined differently from the previous case. For, even if $\varphi^{(i)}$ hit a portion of the surface of the cone excluding the vertex, the resultant potential in general cannot be obtained from $\varphi^{(i)}$ by the method of images, as in the two-dimensional case.

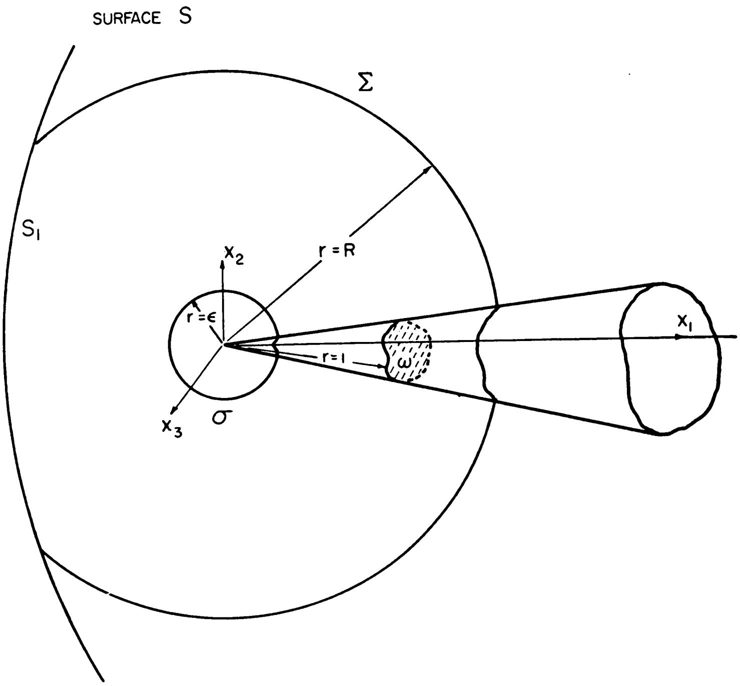

FIG. 2. Diffraction by a cone

$\varphi^{(i)}$ $(x_1, x_2, x_3, t - r)$ and $[\varphi]$ is called the retarded value of φ, and S denotes a surface whose shortest distance from the surface of the cone is greater than $T > 0$ (Fig. 2). The reason for doing so will become evident later.

To obtain the resultant velocity potential at the vertex of the cone, Green's Theorem is applied:

$$\int_{S_1+\sigma+\Sigma+B} \left\{ [\varphi] \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r} \frac{\partial[\varphi]}{\partial n} \right\} dS + \int_V \frac{1}{r} \nabla^2[\varphi]\, dV = 0, \tag{6}$$

where V is the volume bounded by the surface S, the two concentric spheres $r = \epsilon$ and $r = R$ and the surface of the cone, and S_1, σ, Σ and B denote, respectively, the surfaces of the volume V.

A repetition of the passages in the derivation of Kirkchhoff's formula [4] gives

$$\int_{S_1+\sigma+\Sigma+B} \left\{ \left(\frac{[\varphi]}{r^2} + \frac{1}{r}\left[\frac{\partial \varphi}{\partial t}\right] \right) \frac{\partial r}{\partial n} + \frac{1}{r}\left[\frac{\partial \varphi}{\partial n}\right] \right\} dS = 0. \tag{7}$$

On the surface of the cone $\partial r/\partial n = 0$, while the boundary condition gives $[\partial\varphi/\partial n] = 0$; therefore, the integral over B vanishes.

Since $\varphi \equiv \varphi^{(i)}$ for $t < 0$, the value of $[\varphi]$ and its derivative on the sphere $r = R$ is equal to that of $[\varphi^{(i)}]$ for $R > t$. For a finite value of t, the integral over Σ as $R \to \infty$ is identical with the corresponding integral, with φ replaced by $\varphi^{(i)}$. This integral tends to zero. This condition has been imposed on $\varphi^{(i)}$ in the derivation of Eq. (5).

As $\epsilon \to 0$, the integral over σ approaches $-(4\pi - \omega)\,\varphi(t, 0, 0, 0)$ and Eq. (7) becomes

$$\varphi(t, 0, 0, 0) = \frac{-1}{4\pi - \omega} \int_S \left\{ [\varphi] \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{\partial\varphi}{\partial n}\right] - \frac{1}{r}\frac{\partial r}{\partial n}\left[\frac{\partial \varphi}{\partial t}\right] \right\} dS. \tag{8}$$

Since the minimum distance between the surface S and the cone is greater than T, the value of $[\varphi]$ and its derivatives on S will be identical with that of $[\varphi^{(i)}]$ for any instant $t < 2T$. From Eqs. (8) and (5), it yields

$$\varphi(t, 0, 0, 0) = -\frac{4\pi}{4\pi - \omega}\, \varphi^{(i)}(t, 0, 0, 0) \quad \text{for} \quad t < 2T. \tag{9}$$

Since T is arbitrary, therefore, Eq. (9) is valid for any instant.

This relationship can be interpreted intuitively as if the space were contracted by the cone by a factor of $(4\pi - \omega)/4\pi$ in the vicinity of the vertex so that the velocity potential at the vertex is intensified by the reciprocal of the factor of contraction. An analogous interpretation can be given to the two-dimensional problem.

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References

1. J. B. Keller and A. Blank, *Diffraction and reflection of pulses by wedges and corners*, Communication in Pure and Applied Mathematics **4**, 75-94 (1951)
2. G. N. Ward, *Linearized theory of steady high-speed flow*, Cambridge University Press, England, 1955, pp. 55-58
3. L. Ting, *On the diffraction of an arbitrary pulse by a wedge or a cone*, PIBAL Report No. 502, Polytechnic Institute of Brooklyn, Feb. 1959
4. B. B. Baker and E. T. Copson, *The mathematical theory of Huygen's principle*, Oxford University Press, England, 1950, pp. 38-40