WAVE REFRACTION AT AN INTERFACE*

BY

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Abstract. A plane wave in one of two perfect gases moves toward the parallel plane interface between the gases. The wave is either continuous or headed by a shock front weak enough that entropy changes may be neglected. Using Riemann's solution of the appropriate hyperbolic partial differential equation, four equations are derived giving the details of the reflected and refracted wave motions. The equations are of first order integro-differential or implicit functional form depending on the boundary conditions and must be solved simultaneously for four functions of a single independent variable. The equations are suitable for numerical step-by-step solution.

I. Introduction. The refraction of a non-uniform wave at a fluid interface is obtained by solving a second order partial differential equation under conditions along boundaries whose locations are not known in advance. Heretofore a finite difference numerical scheme covering a two-dimensional net has been used to establish the location of the boundaries and solve the boundary value problem simultaneously. (See, for example, Courant and Friedrichs [1], Sec. 83.) This report describes an analytic method reducing the refraction problem to the solution of one-dimensional integro-differential equations. The reduction in dimension is the principal advantage of the analysis. The analytic form may also provide a convenient starting point for existence or uniqueness proofs. The method is an application of Riemann's solution of the partial differential equation involved (Courant and Hilbert [2], vol. II, Chap. 5, Sec. 4).

The mathematical formulation of a number of interface problems, including those appearing here, is given by Drummond [3]. The application of this fluid theory to the refraction of shocks in metal is discussed in Duvall and Zwolinski [4], Walsh and Christian [7] and Walsh, Shreffler and Willig [8]. An analysis similar to the present one is given by Heinz [9] for the reflection of a shock wave in air at a rigid wall.

II. Equations of motion. Each of the two fluids is supposed an inviscid, polytropic gas. It is further assumed that any shock discontinuities occurring are so weak that the entropy changes produced may be neglected. The initially homentropic fluids will then remain so.

With these assumptions the equations of one-dimensional flow of either fluid may be written in characteristic form (Ref. [1], Sec. 37). There are two families of characteristic curves which are referred to as the $C+$ and $C-$ characteristics in the (x, t) or physical plane and as the $\Gamma+$ and $\Gamma-$ characteristics in the (l, u) or hodograph plane. Along each characteristic of one family r is a constant, $C+: x_s = (u + c)t_s$, and $\Gamma+: l + u = 2r$. Along each characteristic of the other family s is a constant, $C-: x_r = (u - c)t_r$, and $\Gamma-: l - u = 2s$. Here

$$p = \alpha\rho^\gamma - \beta, \qquad c^2 = \frac{dp}{d\rho}, \qquad l = \frac{2}{\gamma - 1} c,$$

*Received November 10, 1958. Presented to the American Mathematical Society at the summer meeting in Seattle, Wash., August 1956.

literal subscripts mean partial differentiation, and the letters have their usual meanings:

x = distance from a fixed plane parallel to the interface,
t = time,
u = flow speed in x direction,
c = local speed of sound,
r, s = Riemann invariants,
l = convenient thermodynamic function,
p = pressure,
ρ = density, and
α, β, γ = given constants of the fluid ($\gamma > 1$).

The characteristic equations in the physical plane may be written

$$C+ : x_s = \left(\frac{\gamma + 1}{2} r + \frac{\gamma - 3}{2} s\right) t_s$$

$$C- : x_r = -\left(\frac{\gamma - 3}{2} r + \frac{\gamma + 1}{2} s\right) t_r$$

or, eliminating x,

$$t_{rs} + \lambda \frac{t_r + t_s}{r + s} = 0; \qquad \lambda = \frac{\gamma + 1}{2(\gamma - 1)}. \tag{A}$$

If t is found as a function of r and s satisfying equation (A), then the characteristic relations $C+$ and $C-$ determine x as a function of r and s. Inverting these relations, r and s are then known at each point (x, t) in the physical plane or, using the $\Gamma+$ and $\Gamma-$ relations, u and l are known there. Since l is a univalent function or p or ρ, knowing any one of l, p, ρ, or c at a point determines the other three.

The solution of equation (A) under quite general boundary conditions has been given by Riemann. For that solution one uses Riemann function $R = R(r, s; \xi, \eta)$ satisfying the following conditions:

(a) As a function of r and s, R is a solution of the differential equation adjoint to (A):

$$R_{rs} - \lambda\left[\left(\frac{R}{r + s}\right)_r + \left(\frac{R}{r + s}\right)_s\right] = 0.$$

(b) On $r = \xi$, $R_s - \lambda R/(\xi + s) = 0$, and
on $s = \eta$, $R_r - \lambda R/(r + \eta) = 0$.

(c) $R(\xi, \eta; \xi, \eta) = 1$.

It is true in general that if $R(r, s; \xi, \eta)$ is the Riemann function for a given differential equation, then $R(\xi, \eta; r, s)$ is the Riemann function for its adjoint. The Riemann function for the differential equation adjoint to (A) was obtained by Riemann and is given in [1, Sec. 82]. Interchanging parameter and variable points in that solution gives the Riemann function wanted here. The result may be checked of course by substitution into (a), (b), and (c) above.

$$R(r, s; \xi, \eta) = \left(\frac{r + s}{\xi + \eta}\right)^{\lambda} F\left[1 - \lambda, \lambda; 1; -\frac{(r - \xi)(s - \eta)}{(\xi + \eta)(r + s)}\right],$$

where F is the hypergeometric function. If λ is integral, F reduces to a polynomial.

It is shown in [1, Sec. 72], that the entropy change through a shock wave is of the third order in the shock strength. Thus shocks here must be weak enough that third and higher order terms are negligible. The velocity of a shock wave is then

$$U = u_0 \pm c_0 + \frac{\gamma + 1}{4}(u - u_0) \pm \frac{(\gamma + 1)^2}{32}\frac{(u - u_0)^2}{c_0},$$

where the upper signs are valid for a forward facing shock in which $U > u_0$, the lower signs for $U < u_0$, zero subscripts refer to conditions ahead of the shock, and no subscript to the conditions behind.

III. The transmitted wave. The two fluids are at rest initially except that a wave of known properties in the left-hand fluid, say, is moving toward the interface.

The initial state of the right-hand fluid may be represented by a point, A, in the (l', u) hodograph plane. (Primes are used for quantities in the right-hand fluid, no mark for those in the left; e.g. l' is the l function for the right-hand fluid.) If the transition from rest to motion is continuous, characteristics crossing the transition, the $\Gamma-$ characteristics here, must all go through A. Thus the whole flow of the right-hand fluid maps itself onto the $\Gamma-$ characteristic through A and so is a simple wave. The assumption of weak shocks implies that the simple wave flow is also correct in the discontinuous shock case [1, Sec. 72]. Letting s_1' be the constant value of s' in the simple wave, one has

$$l' - u = 2s_1' \tag{1}$$

throughout the right-hand fluid.

The solution process carried out below determines t, u, and l' consistent with Eq. (1) along the interface I. More definitely, t and u are determined as functions of a convenient parameter q along I. Then, since I is a particle path, $x = x(q)$ is found by integrating

$$\frac{dx}{dq} = u\frac{dt}{dq}.$$

The $C+$ characteristics in the simple wave satisfy

$$dx = \left(\frac{\gamma' + 1}{2}r' + \frac{\gamma' - 3}{2}s_1'\right)dt,$$

where different constant values of r' give the different characteristics. This equation may be integrated under the condition that $x = x(q)$, $t = t(q)$, and $r' = r'(q) = (1/2)[l'(q) + u(q)]$ where the characteristic crosses I. The integration gives

$$x - x(q) = \left[\frac{\gamma' + 1}{2}r'(q) + \frac{\gamma' - 3}{2}s_1'\right][t - t(q)], \tag{2}$$

where x and t designate an arbitrary point on the $C+$ characteristic.

If the flow is continuous it is completely described by Eq. (2). At any point (x, t), Eq. (2) determines the corresponding value of q. This gives $r' = r'(q)$. Also s' is known to be the constant s_1'. Then $u = r' - s_1'$ and $l' = r' + s_1'$ so that the flow is determined at that point.

The flow resulting from a sharp drop in pressure at initial point B, where the front of the oncoming wave in the left-hand fluid first reaches the interface, is also described by Eq. (2). The pressure drop means a corresponding range of values of r' is concentrated at B. This fits the algebra of (2) when a range of values of parameter q is assigned for which $t(q) = t_0$, the initial time, but $r'(q)$ is variable and covers the necessary range in r'.

The flow resulting from a sharp rise in pressure at B is discontinuous with a transmitted shock front preceding the simple wave region described by Eq. (2). In this case it remains to determine S_T, the path of the shock front transmitted into the right-hand fluid. This path, $x = X(q)$ and $t = T(q)$, is found by integrating

$$\frac{dX}{dq} = U_T \frac{dT}{dq}, \tag{3}$$

where U_T is the transmitted shock front speed. U_T is a known function of flow speeds and pressures on the two sides of the shock and so is a known function of q here. X and T are related by Eq. (2) with $x = X$ and $t = T$. Eliminating X between Eqs. (2) and (3) then gives a first order ordinary differential equation for $T = T(q)$.

$$U_T \frac{dT}{dq} - \frac{dx(q)}{dq} = \left[\frac{\gamma' + 1}{2} r'(q) + \frac{\gamma' - 3}{2} s_1'\right]\left[\frac{dT}{dq} - \frac{dt(q)}{dq}\right] + \frac{\gamma' + 1}{2} [T - t(q)] \frac{dr'(q)}{dq}.$$

Solving this gives $T(q)$. Equation (2) then gives $X(q)$ and so the shock front path S_T.

The above solution of the transmitted wave problem is valid so long as characteristics of the same kind do not intersect. For at such an intersection two distinct values of r' are given and so the flow is ambiguous. If the oncoming wave is such that the pressure along the interface decreases as time goes on, the intersection of $C+$ characteristics does not occur. For Eq. (2) may be written

$$x - x(q) = \frac{\gamma' + 1}{2} [l'(q) - 2s_1'][t - t(q)].$$

Since l' decreases as the pressure decreases, the slope of these straight lines increases with time and so they cannot intersect to the right of the interface in the (x, t) plane.

On the other hand, if the pressure increases along any section of the interface, the $C+$ characteristics through that section do eventually intersect. The flow ambiguity is eliminated by the formation of shock discontinuities. It is assumed here that the region of interest does not contain such intersections.

Since the flow in the simple wave is determined by the $C+$ relation, a single value for the slope of the $C-$ characteristic is obtained at each point. Only one $C-$ characteristic may then pass through each point. The $C-$ characteristics do not intersect.

IV. The interface. The interface lies along the Γ characteristic through A in the (l', u) hodograph plane. Now l' is a monotonically increasing function of p in the right-hand fluid with p in turn a similar function of l in the left-hand fluid. Thus, along I, l' is a function of l, $l' = f(l)$. Equation (1) valid throughout the right-hand fluid then becomes

$$f(l) - u = 2s_1',$$

the equation of the I curve in the (l, u) hodograph plane of the left-hand fluid. This one interface relation is equivalent to a pair of relations

$$u = u(q) \quad \text{and} \quad l = l(q)$$

or

$$r = r(q) \quad \text{and} \quad s = s(q)$$

for any convenient choice of parameter q.

The boundary condition along I is that the interface is a particle path:

$$\frac{dx}{dq} = u\,\frac{dt}{dq}.$$

This may be expanded in characteristic coordinates to

$$x_r\,\frac{dr}{dq} + x_s\,\frac{ds}{dq} = (r - s)\left[t_r\,\frac{dr}{dq} + t_s\,\frac{ds}{dq}\right].$$

Eliminating x by relations $C+$ and $C-$ finally gives

$$I\colon t_r\,\frac{dr}{dq} - t_s\,\frac{ds}{dq} = 0.$$

If B is the point at which the oncoming wave first meets the interface, the $C+$ characteristic through B is the front edge of the wave. This characteristic continues ahead through the vertex of a reflected rarefaction fan or through a shock front, as the case may be, on to the interface. At B_I , i.e., at B on the interface, conditions are those along the $C+$ characteristic through B. In the hodograph plane this means that B_I lies at the intersection of the $\Gamma+$ characteristic through B and the I curve.

The relative positions of A, B, and B_I determine the local nature of the discontinuities in both fluids at B, for the jumps from A and B to B_I in the hodograph plane represent those discontinuities. If B_I is to the right of A (as sketched in Fig. 1), l and therefore the pressure rise in going from A to B_I ; a shock front has been transmitted into the right-hand fluid. If B_I is to the left of A, a centered rarefaction wave has been transmitted. Similarly, if B_I is to the right of B (as in Fig. 1), a shock wave is reflected into the left-hand fluid. If B_I is to the left of B a rarefaction is reflected. The two types of reflection are discussed separately in the two following sections.

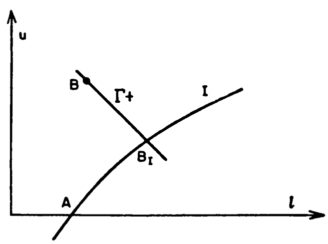

Fig. 1. Hodograph for left hand fluid.

For the case of a transmitted shock wave, with B_I to the right of A as in Fig. 1, D. C. Pack [6] has derived convenient analytic expressions for determining whether a shock or a rarefaction is reflected back into the left-hand fluid. One may rederive these relations geometrically as follows.

For a reflected shock, from Fig. 1, $l(B) < l(B_I)$. Since the pressure is an increasing function of l, $p(B) < p\,(B_I)$ and $p(B) - p(A) < p(B_I) - p(A)$. Since, also from Figure 1, $0 < u(B_I) < u(B)$, the pressure difference inequality is strengthened in the form

$$\frac{p(B) - p(A)}{u(B)} < \frac{p(B_I) - p(A)}{u(B_I)}.$$

The Rankine-Hugoniot relations for a shock of velocity U, where the subscripts 1 and 2 refer to the two sides of the shock, read

$$\rho_1(U - u_1) = \rho_2(U - u_2)$$

$$p_2 - p_1 = \rho_1(U - u_1)^2 - \rho_2(U - u_2)^2 = \rho_1(U - u_1)(u_2 - u_1).$$

Specializing the last relation to the present situation gives

$$p(B) - p(A) = \rho_0 U_0 u(B),$$

$$p(B_I) - p(A) = \rho_0' U_T u(B_I),$$

where ρ_0 and ρ_0' are the densities of the undisturbed fluids and U_0 and U_T are velocities of the oncoming and transmitted shocks, respectively. Substitution gives Pack's first inequality: For a reflected shock

$$\rho_0 U_0 < \rho_0' U_T .$$

Pack's second inequality refers to a strong oncoming detonation wave front for which $p(A)$ is negligible with respect to $p(B)$ and the detonation velocity, D, satisfies the Chapman-Jouguet condition $D = u(B) + c(B)$. These assumptions together with the Rankine-Hugoniot equations above and the speed of sound relation, $c^2 = \gamma p/\rho$, imply

$$D = (\gamma + 1)u(B),$$

$$p(B) = \rho_0 D u(B),$$

and

$$p(B)\left(\frac{1}{\rho_0'} - \frac{1}{\rho_f'}\right) = u_f^2 ,$$

equations needed below. Here u_f and ρ_f' are the particle velocity and density behind a fictitious shock front of pressure $p(B)$ propagated into the undisturbed right-hand fluid.

For the reflected wave to be a shock, Fig. 1 shows that point B lies above the shock polar of the right-hand fluid, the locus of points to which a shock could move point A. In particular B is above the point on the shock polar with $l = l(B)$. In symbols

$$u(B) > u_f .$$

Since $u(B)$ and u_f are both positive one has

$$[u(B)]^2 > u_f^2$$

and, with the equations listed above, this last may be reduced to Pack's second inequality: For a reflected shock,

$$(\gamma + 1)\rho_0\left(\frac{1}{\rho_0'} - \frac{1}{\rho_f'}\right) < 1.$$

V. The reflected rarefaction case. If B is to the right of B_I in the (l, u) hodograph plane, a centered rarefaction wave is reflected back into the left-hand fluid at the interface (see Fig. 2). The front of the oncoming wave, the $C+$ characteristic through B, goes on through the vertex of the centered rarefaction to B_I . The value of t is then known along BB_I in either the hodograph or the characteristic plane, this value being just the constant $t = t_0$ at B. If BD is the last $C-$ characteristic through B unaffected by the reflection, the value of t along BD is also known. (In the hodograph and characteristic planes D lies somewhere on the $C-$ characteristic through B. It is shown to the left of B in the hodograph sketch, Fig. 2(b), for definiteness.)

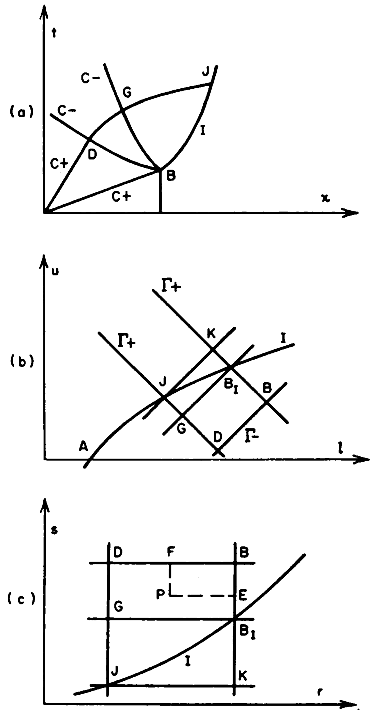

FIG. 2. Reflected rarefaction:
(a) Physical plane
(b) Hodograph
(c) Characteristic plane

The value of t, i.e., the solution of partial differential equation (A), at any point P in the characteristic rectangle determined by B, B_I , and D is then given following Reimann [2, Chap. 5, Sec. 4] by

$$t(P) = t(B)R(B;P) + \int_B^E R(J;P)\left(t_s + \frac{\lambda t}{r+s}\right) ds + \int_B^F R(J;P)\left(t_r + \frac{\lambda t}{r+s}\right) dr, \tag{4}$$

where E and F are the points of intersection of the characteristics through P and BB_I and BD, respectively, J is a general notation for the moving point of integration, and R is the Riemann function. In particular t may be determined along B_IG, the last characteristic in the centered rarefaction wave.

Another characteristic rectangle, B_IGJK, is determined by B_IG and interface curve I considered as a curved diagonal. (Since a characteristic such as B_IG may act as a "transition" line, a line along which the map of the physical plane into the hodograph plane is folded, there is no difficulty if I cuts back into rectangle B_IGDB. It is only necessary to treat separately regions where the slope of I is greater than or less than the slope of the $\Gamma-$ characteristic.)

Values of t at any point P in rectangle $B_I GJK$ are determined as before in terms of values of t along $B_I G$, where they are now known, and along $B_I K$, a characteristic in a fictitious region of the left-hand fluid continued across the interface. If E' and F' are the intersections of characteristics through P with $B_I K$ and $B_I G$, respectively, then

$$t(P) = t(B_I)R(B_I\,; P) + \int_{B_I}^{E'} R(J; P)\left(t_s + \frac{\lambda t}{r+s}\right) ds + \int_{B_I}^{F'} R(J; P)\left(t_r + \frac{\lambda t}{r+s}\right) dr.$$

The partial derivative of t with respect to r is obtained by computing the change in t from P to nearby point P' on the same $s =$ constant line.

$$t_r(P) = t(B_I)R_\xi(B_I\,; P) + R(F'; P)\left(t_r + \frac{\lambda t}{r+s}\right)(F')$$
$$+ \int_{B_I}^{E'} R_\xi(J; P)\left(t_s + \frac{\lambda t}{r+s}\right) ds + \int_{B_I}^{F'} R_\xi(J; P)\left(t_r + \frac{\lambda t}{r+s}\right) dr,$$

where parameter point P has coordinates (ξ, η) and

$$R_\xi = \left.\frac{\partial R}{\partial \xi}\right|_{r,s,\eta}.$$

Similarly

$$t_s(P) = t(B_I)R_\eta(B_I\,; P) + R(E'; P)\left(t_s + \frac{\lambda t}{r+s}\right)(E')$$
$$+ \int_{B_I}^{E'} R_\eta(J; P)\left(t_s + \frac{\lambda t}{r+s}\right) ds + \int_{B_I}^{F'} R_\eta(J; P)\left(t_r + \frac{\lambda t}{r+s}\right) dr.$$

Considering P to lie on the interface and substituting these forms into the interface boundary condition,

$$I\colon t_r(P)\,\frac{dr}{dq}\,(P) - t_s(P)\,\frac{ds}{dq}\,(P) = 0,$$

gives an equation to be solved for the unknown values of t along $B_I K$, that is the value of t at variable point E'. The equation is a first order integro-differential equation in one independent variable. For, the interface lies on the known curve

$$r = r(q), \qquad s = s(q).$$

Point P has this $r(q)$ and $s(q)$ for coordinates. Fixed point B_I has known coordinates r_I and s_I. Then F' has coordinates $r(q)$ and s_I, while E' has coordinates r_I and $s(q)$. The coordinates of all points and so the value of all functions thus depend on known constants and the one variable q. A step-by-step finite difference numerical process in one dimension can then be used to approximate the solution of the equation.

With t so determined along $B_I K$, Riemann's formula gives t at any point P within characteristic rectangle $GB_I KJ$. Thus t is known as a function of r and s in the reflected wave. From the values of t along I, t is known throughout the transmitted wave as was pointed out previously. Corresponding values of x are found by integrating the characteristic relations in either fluid. To obtain the path of the interface, the boundary condition

$$\frac{dx}{dq} = u\,\frac{dt}{dq}$$

is integrated to give $x = x(q)$ knowing $t = t(q)$.

VI. The reflected shock case. If B is to the left of B_I in the (l, u) hodograph plane, a shock discontinuity is reflected back into the left-hand fluid at the interface. The shock curve S_R in the (x, t) physical plane becomes two curves S_a and S_b in the (l, u) hodograph plane (see Fig. 3). S_a gives conditions just ahead of the shock and so starts at B. S_b gives conditions just behind the shock and so starts at B_I.

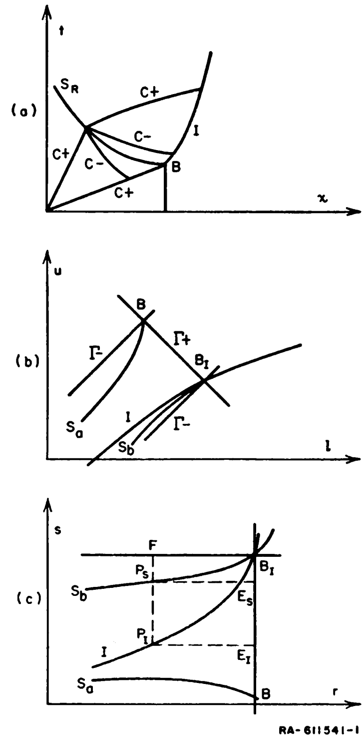

FIG. 3. Reflected shock:
(a) Physical plane
(b) Hodograph
(c) Characteristic plane

The homentropic assumption implies a weak shock across which the $C+$ characteristics are continuous. Thus r is the same on both sides of the shock and is a convenient parameter for the shock curve. The shock boundary condition then reads

$$\frac{dx}{dr} = U_R\,\frac{dt}{dr},$$

where U_R, the velocity of the reflected shock, is a function of flow speeds and pressures on both sides of the shock. If s_a is the value of s in the oncoming wave just ahead of the shock, for a given r, and s_b the corresponding value of s just behind the shock, U_R is the function of r, s_a, and s_b given in Sec. II above.

The oncoming wave flow is "known", i.e., x and t are known functions of r and s. The shock boundary condition becomes a differential equation for s_a:

$$S_1 : x_r + x_s \frac{ds_a}{dr} = U_R\left(t_r + t_s \frac{ds_a}{dr}\right),$$

where the partial derivatives x_r, etc., are all evaluated at (r, s_a).

Applying the shock condition just behind the front gives

$$x_r + x_s \frac{ds_b}{dr} = U_R\left(t_r + t_s \frac{ds_b}{dr}\right).$$

The partial derivatives are related by the $C+$ and $C-$ characteristic conditions so that x_r and x_s may be eliminated to give

$$S_2 : \left(U_R + \frac{\gamma - 3}{2} r + \frac{\gamma + 1}{2} s_b\right)t_r + \left(U_R - \frac{\gamma + 1}{2} r - \frac{\gamma - 3}{2} s_b\right)t_s \frac{ds_b}{dr} = 0.$$

Here t_r and t_s are to be evaluated at (r, s_b). However, the forms of t_r and t_s as functions of r and s_b are not yet known.

Now t may be viewed as a known function of r and s_a or as an unknown function of r and s_b. Equating these values of t along the shock then gives

$$S_3 : t(r, s_a) = t(r, s_b).$$

The problem restated is then to solve the partial differential equation (A) for t as a function of r and s in the angular region between I and S_b in the (r, s) characteristic plane subject to conditions I along the interface and S_1, S_2, S_3 along the shock (see Fig. 3(c)).

As was done with the reflected rarefaction, it is convenient to consider the flow continued outside the angular region between I and S_b to the characteristics through B_I. Forms for t, t_r, and t_s given in the previous section may be substituted in the various boundary conditions as before.

The substitution into interface condition I gives an integro-differential equation connecting t at F and t at E_I. Substitution into S_2 connects t at F with t at E_s. But in the step-by-step process by which the equations may be solved, t will already be known at E_s so that the S_2 equation actually relates t at F and function $s_b = s_b(r)$. (The r coordinate of F, r, is now the parameter q in terms of which all points are expressed.) Finally, substitution into S_3 gives an equation relating t at F, $s_a(r)$, and $s_b(r)$. The four equations I, S_1, S_2, and S_3 thus give the necessary number of relations to determine t along B_IF, t along B_IE, $s_a(r)$, and $s_b(r)$.

As in the case of the reflected rarefaction, knowledge of t along characteristics B_IF and B_IE and of the location of S_a and S_b in the (r, s) plane permits the determination of t everywhere and so, using either the characteristic relations or the appropriate boundary condition, the corresponding values of x everywhere.

The next section contains an application of the present method to a specific example.

VII. Refraction of a detonation. A problem of some interest is the determination of the flow of a material due to the detonation of a layer of high explosive on its surface. An idealized form of such a problem, amenable to the present methods, reads as follows.

A semi-infinite slab of material filling the space $x \geq 1$ is given. A layer of high explosive of unit thickness covers the slab. At time $t = 0$ the explosive is detonated everywhere on its free surface ($x = 0$) simultaneously. The gaseous combustion products are assumed to expand into a vacuum.

The detonation front burns into the high explosive at known speed D. It is assumed that the detonation is a Chapman-Jouguet process, i.e., that particle and sound speeds just behind the detonation are related by

$$u + c = D.$$

This and the equations for the conservation of mass and momentum determine all the flow quantities immediately behind the detonation front in terms of D and the known initial density of the high explosive. Assuming the combustion products form a polytropic gas with zero density at zero pressure fixes α, β, and γ, the constants in its equation of state.

The slab material is also assumed to be a polytropic gas with known equation of state. Some discussion of the experimental basis for this last assumption is given in [4, 6, and 7].

Since along the detonation front

$$\begin{aligned} dx &= D\,dt \\ &= (u + c)\,dt, \end{aligned}$$

this front lies on a $C+$ characteristic. The flow field behind the front is covered by $C-$ characteristics. The detonation front carries constant values of pressure and flow velocity and so is represented by the single point B in the (l, u) hodograph plane. The field behind it must then all lie on the $\Gamma-$ characteristic through B and so is a simple wave. In the wave $s = s_0$, a constant.

In the physical (x, t) plane the whole wave starts at the origin, $x = t = 0$, and so is a centered simple wave. Straight lines through the origin are all $C+$ characteristics and correspond to points on the $\Gamma-$ characteristic between B and V, the (vacuum) point where $\rho = l = 0$.

The $C+$ characteristic equations may be integrated here to give

$$x = \left(\frac{\gamma + 1}{2} r + \frac{\gamma - 3}{2} s_0\right) t. \tag{5}$$

This together with

$$s = s_0$$

describes the centered simple wave in the burned gas, the oncoming wave of the preceding theory.

Since a simple wave is a degenerate form, some simplifications of the reflected wave boundary conditions occur here. In this example it is assumed that B lies above interface curve I in the (l, u) hodograph plane so that the reflected shock considerations of Sec. VI apply. Points just ahead of this shock lie in the simple wave. The S_a curve in the hodograph plane thus coincides with $\Gamma-$ characteristic through B. Boundary condition S_1 is no longer needed to locate S_a. When the other conditions have been used to determine t as a function of r along the front, S_1, or its equivalent

$$\frac{dx}{dr} = U_R \frac{dt}{dr}, \tag{6}$$

may be used to find x as a function of r and so the location of the reflected shock in the (x, t) physical plane.

Boundary condition S_2 is needed as before to locate S_b, the hodograph curve of points just behind the reflected shock.

One may eliminate x between the shock condition, Eq. (6), and the equation valid in the oncoming simple wave, Eq. (5), to obtain

$$\frac{\gamma+1}{2}\,t + \left(\frac{\gamma+1}{2}\,t + \frac{\gamma-3}{2}\,s_0\right)\frac{dt}{dr} = U_R\,\frac{dt}{dr}.$$

By boundary condition S_3, the values of t at corresponding points of S_a and S_b are the same. Hence t and its derivatives in the equation may be considered to lie along S_b. The shock boundary condition then reads

$$S_4 : \frac{\gamma+1}{2}\,t = \left(U_R - \frac{\gamma+1}{2}\,r - \frac{\gamma-3}{2}\,s_0\right)\left(t_r + t_s\,\frac{ds_b}{dr}\right),$$

where t, t_r, and t_s are evaluated along S_b.

Finally the interface boundary condition, with parameter q identified with characteristic coordinate r of the left-hand fluid, reads

$$I : t_r = t_s\,\frac{ds_I}{dr}.$$

The forms for the partial derivatives of t to be substituted in S_2, S_4, and I may be conveniently written

$$t_r(P) = t_F'(F)R(F;P) + \Phi_3(E;P),$$

$$t_s(P) = t_E'(E)R(E;P) + \Phi_4(E;P),$$

where

$$t_F'(F) = t_r(F) = \frac{d}{dr}\,t(r, s_I),$$

$$t_E'(E) = t_s(E) = \frac{d}{ds}\,t(r_I, s),$$

$$\Phi_3(E;P) = \frac{\lambda t(F)}{r+s_I}\,R(F;P) + t(B_I)R_\xi(B_I;P) + \int_{B_I}^{E} R_\xi(J;P)\left(t_s + \frac{\lambda t}{r_I+s}\right)ds + \int_{B_I}^{F} R_\xi(J;P)\left(t_r + \frac{\lambda t}{r+s_I}\right)dr,$$

and

$$\Phi_4(E;P) = \frac{\lambda t(E)}{r_I+s}\,R(E;P) + t(B_I)R_\eta(B_I;P) + \int_{B_I}^{E} R_\eta(J;P)\left(t_s + \frac{\lambda t}{r_I+s}\right)ds + \int_{B_I}^{F} R_\eta(J;P)\left(t_r + \frac{\lambda t}{r+s_I}\right)dr.$$

In using these formulas, P is replaced by either P_S or P_I, points with the same r coordinate as F and lying on S_b or I (see Fig. 3(c)). Similarly E is replaced by E_S or E_I, points whose r coordinate is r_I and whose s coordinate agrees with P_S or P_I.

Making the substitutions into S_2, S_4, and I and then solving for the derivatives of the unknown functions gives

$$t_F'(F) = [\Phi_1 - (1 + \Phi_2)\Phi_3(E_S\,; P_S)]/(1 + \Phi_2)R(F; P_S) \tag{7}$$

$$t_E'(E_I) = \left[t_F'(F)R(F; P_I) + \Phi_3(E_I\,; P_I) - \Phi_4(E_I\,; P_I)\frac{ds_I}{dr} \right] \div \frac{ds_I}{dr} R(E_I\,; P_I) \tag{8}$$

$$\frac{ds_b}{dr}(P_S) = \Phi_1\Phi_2/(1 + \Phi_2)[t_E'(E_S)R(E_S\,; P_S) + \Phi_4(E_S\,; P_S)], \tag{9}$$

where

$$\Phi_1 = \frac{\gamma + 1}{2} t(P_S) \Big/ \left(U_R - \frac{\gamma + 1}{2} r - \frac{\gamma - 3}{2} s_0 \right)$$

$$\Phi_2 = \left(U_R + \frac{\gamma - 3}{2} r + \frac{\gamma + 1}{2} s_b \right) \Big/ \left(U_R - \frac{\gamma + 1}{2} r - \frac{\gamma - 3}{2} s_b \right)$$

and

$$U_R = (r - s_0) + \frac{\gamma - 1}{2}(r + s_0)\left[-1 + \frac{\gamma + 1}{4}\left(\frac{2}{\gamma - 1} \frac{s_0 - s_b}{r + s_b} \right) - \frac{(\gamma + 1)^2}{32}\left(\frac{2}{\gamma - 1} \frac{s_0 - s_b}{r + s_0} \right)^2 \right].$$

Equations (7), (8), and (9) then constitute three simultaneous, ordinary, first order, integro-differential equations for t_F, t_E, and s_b, the values of t along characteristics B_IF and B_IE and the location in the hodograph plane of points just behind the reflected shock. Using these, the value of t at any point of the reflected wave may be computed. It is necessary to calculate t along s_b using Eq. (4) as the integration proceeds for substitution into Φ_1. The computation of t along I may be carried out when convenient. Appropriate integrations then give x along s_b or I and so the path of the reflected shock or interface in the (x, t) plane. The path of the transmitted shock may then be computed as described in a previous section.

VIII. A numerical example. The theory of Sec. VII has been applied to determine the constants in an equation of state for the gaseous combustion products of a high explosive known as Composition B (See [7].) The equation is assumed to have the form

$$p = \alpha\rho^\gamma - \beta.$$

At the detonation front one has the Chapman-Jouguet condition,

$$u_1 + c_1 = D,$$

the conditions for conservation of mass and momentum,

$$\rho_1(u_1 - D) = -\rho_0 D,$$

and

$$p_1 = \rho_0 D u_1 ,$$

and the definition of the speed of sound,

$$c_1 = (\alpha\gamma\rho_1^{\gamma-1})^{1/2}.$$

These relations may be combined to give

$$\beta = [\rho_0 D^2 - (\gamma + 1)p_1]/\gamma$$

and

$$\alpha = (\rho_0^2 D^2/\gamma)[\gamma(\beta + \rho_0 D^2)/(\gamma + 1)\rho_0^2 D^2]^{\gamma+1}.$$

Thus two of the constants in the equation of state, α and β, are determined in terms of the third, γ, the initial density of the high explosive, ρ_0, and the known pressure and velocity of the detonation, p_1 and D.

For various values of γ, curves of U_T, the transmitted shock velocity, versus $(x - 1)$, the distance into the slab, may be plotted as in Fig. 4. These curves may then be compared with the experimental points given in Table I of Ref. [7]. The curve for $\gamma = 11/9$ is seen to agree fairly well with the experimental points near the surface of the slab. This value of γ is also suggested for combustion products in Ref. [1], Sec. 38. A more complete comparison of theory with experiment has been made by W. E. Drummond [10].

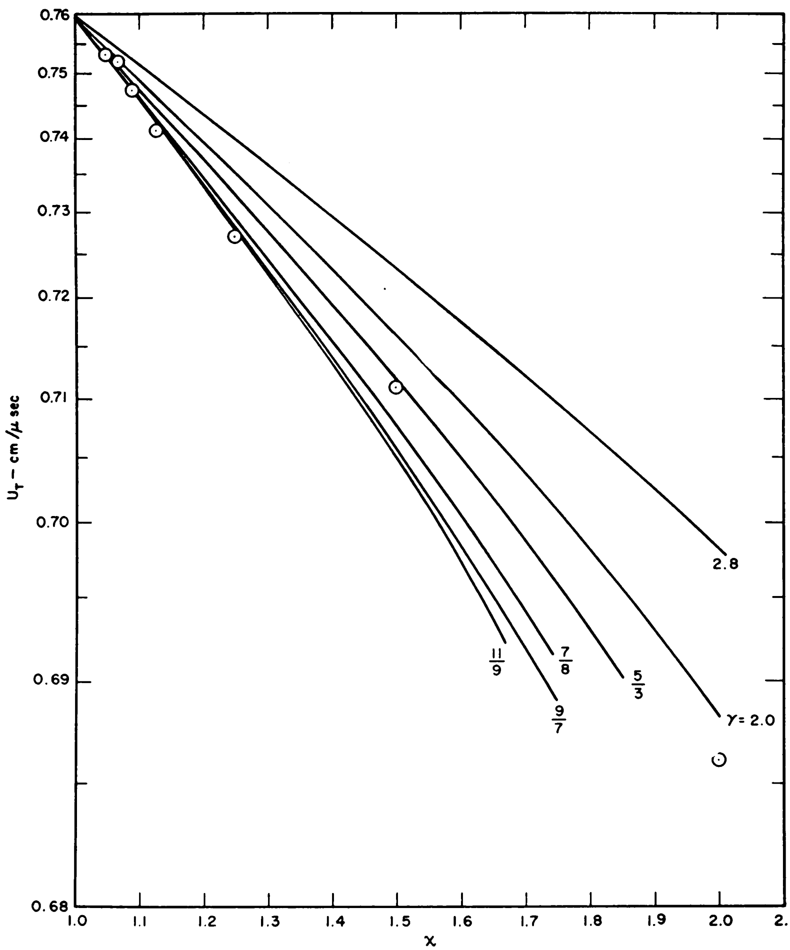

Fig. 4. Transmitted shock velocity *vs.* depth.

As γ varies, the (p, ρ) curves all pass through the same point (p_1, ρ_1) with the same slope, c_1^2. The equation of state has been taken in polytropic form to permit use of the Riemannian theory. For $\beta \neq 0$ the equation can only be an approximation valid in the neighborhood of $p = p_1$, $\rho = \rho_1$. Some indication of how large this neighborhood may be is given by $\rho(0)$, the density for which $p = 0$, since this density would be zero in a correct equation of state.

The equation of state of the metal slab is also in polytropic form. This form has been given some theoretical standing by Murnaghan in [5]. Constants used in the equation are those for aluminum given in [4].

The table below lists numerical values of interest. Densities are given in gm/cm^3, pressures in megabars, and velocities in cm/microsec.

$\rho_0 = 1.67$	$p_1 = 0.26$	$\gamma' = 4.267$
$D = 0.77$	$\rho_1 = 2.263$	$\beta' = 0.171$
	$c_1 = 0.567$	$\alpha' = 0.00214$

γ	β	α	$\rho(0)$
2.81	0	0.0263	0
2	0.104	0.0715	1.28
5/3	0.177	0.1124	1.31
7/5	0.261	0.168	1.37
9/7	0.307	0.201	1.39
11/9	0.336	0.222	1.41

The calculations have been ably carried through by T. C. Poulter, Jr., on an internally programmed digital computer. The work has been supported in part by the U. S. Army Bureau of Ordnance.

References

1. R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, Interscience Publishers, New York, 1948
2. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Julius Springer, Berlin, 1937
3. W. E. Drummond, *Interaction of nonuniform shock waves*, J. Appl. Phys. **28**, 76-85 (1957)
4. G. E. Duvall and B. J. Zwolinski, *Entropic equations of state and their application to shock wave phenomena in solids*, J. Acoust. Soc. Amer. **27**, 1054-8 (1955)
5. F. D. Murnaghan, *Finite deformation of an elastic solid*, John Wiley and Sons, New York, 1951
6. D. C. Pack, *The reflection and transmission of shock waves*. I. *The reflection of a detonation wave at a boundary;* II. *The effect of shock waves on an elastic target of finite thickness*, Phil. Mag. (8) **2**, 182-95 (1957)
7. J. M. Walsh and R. H. Christian, *Equation of state of metals from shock wave measurements*, Phys. Rev. (2) **97**, 1544-56 (1955)
8. J. M. Walsh, R. G. Shreffler and F. J. Willig, *Limiting conditions for jet formation in high velocity collisions*, J. Appl. Phys. **24**, 349-59 (1953)
9. C. Heinz, *Reflexion ebener Druckwellen an einer festen Wand*, Z. angew. Math. u. Mech. **37**, 63-73 (1957)
10. W. E. Drummond, *Explosive induced shock waves*. Part I, *Plane shock waves*, J. Appl. Phys. **28**, 1437-41 (1957)