WAVE REFRACTION AT AN INTERFACE*

BY

C. M. ABlOW

Stanford Research Institute, Menlo Park, Calif.

Abstract. A plane wave in one of two perfect gases moves toward the parallel plane interface between the gases. The wave is either continuous or headed by a shock front weak enough that entropy changes may be neglected. Using Riemann's solution of the appropriate hyperbolic partial differential equation, four equations are derived giving the details of the reflected and refracted wave motions. The equations are of first order integro-differential or implicit functional form depending on the boundary conditions and must be solved simultaneously for four functions of a single independent variable. The equations are suitable for numerical step-by-step solution.

I. Introduction. The refraction of a non-uniform wave at a fluid interface is obtained by solving a second order partial differential equation under conditions along boundaries whose locations are not known in advance. Heretofore a finite difference numerical scheme covering a two-dimensional net has been used to establish the location of the boundaries and solve the boundary value problem simultaneously. (See, for example, Courant and Friedrichs [1], Sec. 83.) This report describes an analytic method reducing the refraction problem to the solution of one-dimensional integro-differential equations. The reduction in dimension is the principal advantage of the analysis. The analytic form may also provide a convenient starting point for existence or uniqueness proofs. The method is an application of Riemann's solution of the partial differential equation involved (Courant and Hilbert [2], vol. II, Chap. 5, Sec. 4).

The mathematical formulation of a number of interface problems, including those appearing here, is given by Drummond [3]. The application of this fluid theory to the refraction of shocks in metal is discussed in Duvall and Zwolinski [4], Walsh and Christian [7] and Walsh, Shreffler and Willig [8]. An analysis similar to the present one is given by Heinz [9] for the reflection of a shock wave in air at a rigid wall.

II. Equations of motion. Each of the two fluids is supposed an inviscid, polytropic gas. It is further assumed that any shock discontinuities occurring are so weak that the entropy changes produced may be neglected. The initially homentropic fluids will then remain so.

With these assumptions the equations of one-dimensional flow of either fluid may be written in characteristic form (Ref. [1], Sec. 37). There are two families of characteristic curves which are referred to as the $C^+$ and $C^-$ characteristics in the $(x, t)$ or physical plane and as the $\Gamma^+$ and $\Gamma^-$ characteristics in the $(l, u)$ or hodograph plane. Along each characteristic of one family $r$ is a constant, $C^+: x_r = (u + c)t_r$, and $\Gamma^+: l + u = 2r$. Along each characteristic of the other family $s$ is a constant, $C^-: x_s = (u - c)t_s$, and $\Gamma^-: l - u = 2s$. Here

\[ p = \alpha \rho^\gamma - \beta, \quad c^2 = \frac{dp}{d\rho}, \quad l = \frac{2}{\gamma - 1} c, \]

---

*Received November 10, 1958. Presented to the American Mathematical Society at the summer meeting in Seattle, Wash., August 1956.
literal subscripts mean partial differentiation, and the letters have their usual meanings:

- \( x \) = distance from a fixed plane parallel to the interface,
- \( t \) = time,
- \( u \) = flow speed in \( x \) direction,
- \( c \) = local speed of sound,
- \( r, s \) = Riemann invariants,
- \( l \) = convenient thermodynamic function,
- \( p \) = pressure,
- \( \rho \) = density, and
- \( \alpha, \beta, \gamma \) = given constants of the fluid (\( \gamma > 1 \)).

The characteristic equations in the physical plane may be written

\[
C_+ : x = \left( \frac{\gamma + 1}{2} r + \frac{\gamma - 3}{2} s \right) t,
\]

\[
C_- : x = -\left( \frac{\gamma - 3}{2} r + \frac{\gamma + 1}{2} s \right) t,
\]

or, eliminating \( x \),

\[
(A) \quad t_r + \lambda \frac{t_r + t_s}{r + s} = 0; \quad \lambda = \frac{\gamma + 1}{2(\gamma - 1)}.
\]

If \( t \) is found as a function of \( r \) and \( s \) satisfying equation (A), then the characteristic relations \( C_+ \) and \( C_- \) determine \( x \) as a function of \( r \) and \( s \). Inverting these relations, \( r \) and \( s \) are then known at each point \((x, t)\) in the physical plane or, using the \( \Gamma_+ \) and \( \Gamma_- \) relations, \( u \) and \( l \) are known there. Since \( l \) is a univalent function or \( p \) or \( \rho \), knowing any one of \( l, p, \rho, \) or \( c \) at a point determines the other three.

The solution of equation (A) under quite general boundary conditions has been given by Riemann. For that solution one uses Riemann function \( R = R(r, s; \xi, \eta) \) satisfying the following conditions:

- (a) As a function of \( r \) and \( s \), \( R \) is a solution of the differential equation adjoint to (A):

  \[
  R_{rs} - \lambda \left[ \left( \frac{R}{r + s} \right)_r + \left( \frac{R}{r + s} \right)_s \right] = 0.
  \]

- (b) On \( r = \xi, R = \lambda R/(\xi + s) = 0 \), and

  on \( s = \eta, R_r - \lambda R/(r + \eta) = 0 \).

- (c) \( R(\xi, \eta; \xi, \eta) = 1 \).

It is true in general that if \( R(r, s; \xi, \eta) \) is the Riemann function for a given differential equation, then \( R(\xi, \eta; r, s) \) is the Riemann function for its adjoint. The Riemann function for the differential equation adjoint to (A) was obtained by Riemann and is given in [1, Sec. 82]. Interchanging parameter and variable points in that solution gives the Riemann function wanted here. The result may be checked of course by substitution into (a), (b), and (c) above.

\[
R(r, s; \xi, \eta) = \left( \frac{r + s}{\xi + \eta} \right)^\lambda F\left[ 1 - \lambda, \lambda; 1; -\frac{(r - \xi)(s - \eta)}{(\xi + \eta)(r + s)} \right],
\]

where \( F \) is the hypergeometric function. If \( \lambda \) is integral, \( F \) reduces to a polynomial.
It is shown in [1, Sec. 72], that the entropy change through a shock wave is of the third order in the shock strength. Thus shocks here must be weak enough that third and higher order terms are negligible. The velocity of a shock wave is then

\[
U = u_0 \pm c_0 + \frac{\gamma + 1}{4} (u - u_0) \pm \frac{(\gamma + 1)^2 (u - u_0)^2}{32 c_0},
\]

where the upper signs are valid for a forward facing shock in which \( U > u_0 \), the lower signs for \( U < u_0 \), zero subscripts refer to conditions ahead of the shock, and no subscript to the conditions behind.

III. The transmitted wave. The two fluids are at rest initially except that a wave of known properties in the left-hand fluid, say, is moving toward the interface. The initial state of the right-hand fluid may be represented by a point, \( A \), in the \((\theta', u)\) hodograph plane. (Primes are used for quantities in the right-hand fluid, no mark for those in the left; e.g. \( \theta' \) is the \( \theta \) function for the right-hand fluid.) If the transition from rest to motion is continuous, characteristics crossing the transition, the \( \Gamma \) characteristics here, must all go through \( A \). Thus the whole flow of the right-hand fluid maps itself onto the \( \Gamma \) characteristic through \( A \) and so is a simple wave. The assumption of weak shocks implies that the simple wave flow is also correct in the discontinuous shock case [1, Sec. 72]. Letting \( s'_i \) be the constant value of \( s' \) in the simple wave, one has

\[
l' - u = 2s'_i
\]

throughout the right-hand fluid.

The solution process carried out below determines \( t, u, \) and \( l' \) consistent with Eq. (1) along the interface \( I \). More definitely, \( t \) and \( u \) are determined as functions of a convenient parameter \( q \) along \( I \). Then, since \( I \) is a particle path, \( x = x(q) \) is found by integrating

\[
\frac{dx}{dq} = u \frac{dt}{dq}.
\]

The \( C^+ \) characteristics in the simple wave satisfy

\[
dx = \left(\frac{\gamma' + 1}{2} \theta' + \frac{\gamma' - 3}{2} s'_i \right) dt,
\]

where different constant values of \( \theta' \) give the different characteristics. This equation may be integrated under the condition that \( x = x(q) \), \( t = t(q) \), and \( \theta' = \theta'(q) = (1/2) \left[ l'(q) + u(q) \right] \) where the characteristic crosses \( I \). The integration gives

\[
x - x(q) = \left[ \frac{\gamma' + 1}{2} \theta'(q) + \frac{\gamma' - 3}{2} s'_i \right] (t - t(q)),
\]

where \( x \) and \( t \) designate an arbitrary point on the \( C^+ \) characteristic.

If the flow is continuous it is completely described by Eq. (2). At any point \( (x, t) \), Eq. (2) determines the corresponding value of \( q \). This gives \( \theta' = \theta'(q) \). Also \( s' \) is known to be the constant \( s'_i \). Then \( u = \theta' - s'_i \) and \( l' = \theta' + s'_i \) so that the flow is determined at that point.

The flow resulting from a sharp drop in pressure at initial point \( B \), where the front of the oncoming wave in the left-hand fluid first reaches the interface, is also described by Eq. (2). The pressure drop means a corresponding range of values of \( \theta' \) is concentrated
at \( B \). This fits the algebra of (2) when a range of values of parameter \( q \) is assigned for which \( t(q) = t_0 \), the initial time, but \( \tau'(q) \) is variable and covers the necessary range in \( \tau' \).

The flow resulting from a sharp rise in pressure at \( B \) is discontinuous with a transmitted shock front preceding the simple wave region described by Eq. (2). In this case it remains to determine \( S_T \), the path of the shock front transmitted into the right-hand fluid. This path, \( x = X(q) \) and \( t = T(q) \), is found by integrating

\[
\frac{dX}{dq} = U_T \frac{dT}{dq},
\]

where \( U_T \) is the transmitted shock front speed. \( U_T \) is a known function of flow speeds and pressures on the two sides of the shock and so is a known function of \( q \) here. \( X \) and \( T \) are related by Eq. (2) with \( x = X \) and \( t = T \). Eliminating \( X \) between Eqs. (2) and (3) then gives a first order ordinary differential equation for \( T = T(q) \).

\[
U_T \frac{dT}{dq} \frac{dx(q)}{dq} = \left[ \frac{\gamma' + 1}{2} \tau'(q) + \frac{\gamma'}{2} s'_i \right] \left[ \frac{dT}{dq} - \frac{dt(q)}{dq} \right] + \frac{\gamma' + 1}{2} [T - t(q)] \frac{dr'(q)}{dq}.
\]

Solving this gives \( T(q) \). Equation (2) then gives \( X(q) \) and so the shock front path \( S_T \).

The above solution of the transmitted wave problem is valid so long as characteristics of the same kind do not intersect. For at such an intersection two distinct values of \( \tau' \) are given and so the flow is ambiguous. If the oncoming wave is such that the pressure along the interface decreases as time goes on, the intersection of \( C+ \) characteristics does not occur. For Eq. (2) may be written

\[
x - x(q) = \frac{\gamma'}{2} \left[ \tau'(q) - 2s'_i \right] [t - t(q)].
\]

Since \( \tau' \) decreases as the pressure decreases, the slope of these straight lines increases with time and so they cannot intersect to the right of the interface in the \((x, t)\) plane.

On the other hand, if the pressure increases along any section of the interface, the \( C+ \) characteristics through that section do eventually intersect. The flow ambiguity is eliminated by the formation of shock discontinuities. It is assumed here that the region of interest does not contain such intersections.

Since the flow in the simple wave is determined by the \( C+ \) relation, a single value for the slope of the \( C- \) characteristic is obtained at each point. Only one \( C- \) characteristic may then pass through each point. The \( C- \) characteristics do not intersect.

IV. The interface. The interface lies along the \( \Gamma \) characteristic through \( A \) in the \((l', u)\) hodograph plane. Now \( l' \) is a monotonically increasing function of \( p \) in the right-hand fluid with \( p \) in turn a similar function of \( l \) in the left-hand fluid. Thus, along \( I \), \( l' \) is a function of \( l \), \( l' = f(l) \). Equation (1) valid throughout the right-hand fluid then becomes

\[
f(l) - u = 2s'_i,
\]

the equation of the \( I \) curve in the \((l, u)\) hodograph plane of the left-hand fluid. This one interface relation is equivalent to a pair of relations

\[
u = u(q) \quad \text{and} \quad l = l(q)
\]

or

\[
r = r(q) \quad \text{and} \quad s = s(q)
\]

for any convenient choice of parameter \( q \).
The boundary condition along $I$ is that the interface is a particle path:

$$\frac{dx}{dq} = u \frac{dt}{dq}.$$

This may be expanded in characteristic coordinates to

$$x_r \frac{dr}{dq} + x_s \frac{ds}{dq} = (r - s) \left[ t_r \frac{dr}{dq} + t_s \frac{ds}{dq} \right].$$

Eliminating $x$ by relations $C^+$ and $C^-$ finally gives

$$I: t_r \frac{dr}{dq} - t_s \frac{ds}{dq} = 0.$$

If $B$ is the point at which the oncoming wave first meets the interface, the $C^+$ characteristic through $B$ is the front edge of the wave. This characteristic continues ahead through the vertex of a reflected rarefaction fan or through a shock front, as the case may be, on to the interface. At $B_r$, i.e., at $B$ on the interface, conditions are those along the $C^+$ characteristic through $B$. In the hodograph plane this means that $B_r$ lies at the intersection of the $T^+$ characteristic through $B$ and the $I$ curve.

The relative positions of $A, B$, and $B_r$ determine the local nature of the discontinuities in both fluids at $B$, for the jumps from $A$ and $B$ to $B_r$ in the hodograph plane represent those discontinuities. If $B_r$ is to the right of $A$ (as sketched in Fig. 1), $I$ and therefore the pressure rise in going from $A$ to $B_r$; a shock front has been transmitted into the right-hand fluid. If $B_r$ is to the left of $A$, a centered rarefaction wave has been transmitted. Similarly, if $B_r$ is to the right of $B$ (as in Fig. 1), a shock wave is reflected into the left-hand fluid. If $B_r$ is to the left of $B$ a rarefaction is reflected. The two types of reflection are discussed separately in the two following sections.

For the case of a transmitted shock wave, with $B_r$ to the right of $A$ as in Fig. 1, D. C. Pack [6] has derived convenient analytic expressions for determining whether a shock or a rarefaction is reflected back into the left-hand fluid. One may rederive these relations geometrically as follows.

For a reflected shock, from Fig. 1, $l(B) < l(B_r)$. Since the pressure is an increasing function of $l$, $p(B) < p(B_r)$ and $p(B) - p(A) < p(B_r) - p(A)$. Since, also from Figure 1, $0 < u(B_r) < u(B)$, the pressure difference inequality is strengthened in the form

$$\frac{p(B) - p(A)}{u(B)} < \frac{p(B_r) - p(A)}{u(B_r)}.$$
The Rankine-Hugoniot relations for a shock of velocity \( U \), where the subscripts 1 and 2 refer to the two sides of the shock, read

\[
\rho_1(U - u_1) = \rho_2(U - u_2)
\]

\[
p_2 - p_1 = \rho_1(U - u_1)^2 - \rho_2(U - u_2)^2 = \rho_1(U - u_1)(u_2 - u_1).
\]

Specializing the last relation to the present situation gives

\[
p(B) - p(A) = \rho_0U_0u(B),
\]

\[
p(B_1) - p(A) = \rho_0U_Tu(B_1),
\]

where \( \rho_0 \) and \( \rho_0' \) are the densities of the undisturbed fluids and \( U_0 \) and \( U_T \) are velocities of the oncoming and transmitted shocks, respectively. Substitution gives Pack's first inequality: For a reflected shock

\[
\rho_0U_0 < \rho_0'U_T.
\]

Pack's second inequality refers to a strong oncoming detonation wave front for which \( p(A) \) is negligible with respect to \( p(B) \) and the detonation velocity, \( D \), satisfies the Chapman-Jouguet condition \( D = u(B) + c(B) \). These assumptions together with the Rankine-Hugoniot equations above and the speed of sound relation, \( c^2 = \gamma p/\rho \), imply

\[
D = (\gamma + 1)u(B),
\]

\[
p(B) = \rho_0 Du(B),
\]

and

\[
p(B)\left(\frac{1}{\rho_0} - \frac{1}{\rho'_f}\right) = u_f^2,
\]

equations needed below. Here \( u_f \) and \( \rho'_f \) are the particle velocity and density behind a fictitious shock front of pressure \( p(B) \) propagated into the undisturbed right-hand fluid.

For the reflected wave to be a shock, Fig. 1 shows that point \( B \) lies above the shock polar of the right-hand fluid, the locus of points to which a shock could move point \( A \). In particular \( B \) is above the point on the shock polar with \( l = l(B) \). In symbols

\[
u(B) > u_f.
\]

Since \( u(B) \) and \( u_f \) are both positive one has

\[
[u(B)]^2 > u_f^2
\]

and, with the equations listed above, this last may be reduced to Pack's second inequality: For a reflected shock,

\[
(\gamma + 1)\rho_0\left(\frac{1}{\rho_0} - \frac{1}{\rho'_f}\right) < 1.
\]

**V. The reflected rarefaction case.** If \( B \) is to the right of \( B_f \) in the \((l, u)\) hodograph plane, a centered rarefaction wave is reflected back into the left-hand fluid at the interface (see Fig. 2). The front of the oncoming wave, the \( C+ \) characteristic through \( B \), goes on through the vertex of the centered rarefaction to \( B_f \). The value of \( t \) is then
known along $BB_I$ in either the hodograph or the characteristic plane, this value being just the constant $t = t_0$ at $B$. If $BD$ is the last $C-$ characteristic through $B$ unaffected by the reflection, the value of $t$ along $BD$ is also known. (In the hodograph and characteristic planes $D$ lies somewhere on the $C-$ characteristic through $B$. It is shown to the left of $B$ in the hodograph sketch, Fig. 2(b), for definiteness.)

The value of $t$, i.e., the solution of partial differential equation (4), at any point $P$ in the characteristic rectangle determined by $B$, $B_I$ , and $D$ is then given following Reimann [2, Chap. 5, Sec. 4] by

$$t(P) = t(B)R(B; P) + \int_B^P R(J; P)\left(t_r + \frac{\lambda t}{r + s}\right) ds + \int_B^P R(J; P)\left(t_r + \frac{\lambda t}{r + s}\right) dr,$$

where $E$ and $F$ are the points of intersection of the characteristics through $P$ and $BB_I$ and $BD$, respectively, $J$ is a general notation for the moving point of integration, and $R$ is the Riemann function. In particular $t$ may be determined along $B_I G$, the last characteristic in the centered rarefaction wave.

Another characteristic rectangle, $B_I GJK$, is determined by $B_I G$ and interface curve $I$ considered as a curved diagonal. (Since a characteristic such as $B_I G$ may act as a "transition" line, a line along which the map of the physical plane into the hodograph plane is folded, there is no difficulty if $I$ cuts back into rectangle $B_I GDB$. It is only
necessary to treat separately regions where the slope of \( I \) is greater than or less than the slope of the \( T \) — characteristic.)

Values of \( t \) at any point \( P \) in rectangle \( B_tGJK \) are determined as before in terms of values of \( t \) along \( B_tG \), where they are now known, and along \( B_tK \), a characteristic in a fictitious region of the left-hand fluid continued across the interface. If \( E' \) and \( F' \) are the intersections of characteristics through \( P \) with \( B_tK \) and \( B_tG \), respectively, then

\[
t(P) = t(B_t)R_t(B_t; P) + \int_{B_t}^E R(J; P)\left(t_s + \frac{\lambda t}{r + s}\right) ds + \int_{B_t}^F R(J; P)\left(t_t + \frac{\lambda t}{r + s}\right) dr.
\]

The partial derivative of \( t \) with respect to \( r \) is obtained by computing the change in \( t \) from \( P \) to nearby point \( P' \) on the same \( s = \) constant line.

\[
t_r(P) = t(B_t)R_t(B_t; P) + R(F'; P)\left(t_t + \frac{\lambda t}{r + s}\right) + \int_{B_t}^E R(J; P)\left(t_s + \frac{\lambda t}{r + s}\right) ds + \int_{B_t}^F R(J; P)\left(t_t + \frac{\lambda t}{r + s}\right) dr,
\]

where parameter point \( P \) has coordinates \((\xi, \eta)\) and

\[
R_t = \frac{\partial R}{\partial \xi} \bigg|_{r, s, \eta}.
\]

Similarly

\[
t_s(P) = t(B_t)R_s(B_t; P) + \int_{B_t}^E R(J; P)\left(t_s + \frac{\lambda t}{r + s}\right) ds + \int_{B_t}^F R(J; P)\left(t_t + \frac{\lambda t}{r + s}\right) dr.
\]

Considering \( P \) to lie on the interface and substituting these forms into the interface boundary condition,

\[
I: t_s(P) \frac{dr}{dq}(P) - t_s(P) \frac{ds}{dq}(P) = 0,
\]

gives an equation to be solved for the unknown values of \( t \) along \( B_tK \), that is the value of \( t \) at variable point \( E' \). The equation is a first order integro-differential equation in one independent variable. For, the interface lies on the known curve

\[
r = r(q), \quad s = s(q).
\]

Point \( P \) has this \( r(q) \) and \( s(q) \) for coordinates. Fixed point \( B_t \) has known coordinates \( r_t \) and \( s_t \). Then \( F' \) has coordinates \( r(q) \) and \( s_t \), while \( E' \) has coordinates \( r_t \) and \( s(q) \). The coordinates of all points and so the value of all functions thus depend on known constants and the one variable \( q \). A step-by-step finite difference numerical process in one dimension can then be used to approximate the solution of the equation.

With \( t \) so determined along \( B_tK \), Riemann's formula gives \( t \) at any point \( P \) within characteristic rectangle \( GB_tKJ \). Thus \( t \) is known as a function of \( r \) and \( s \) in the reflected wave. From the values of \( t \) along \( I \), \( t \) is known throughout the transmitted wave as was pointed out previously. Corresponding values of \( x \) are found by integrating the character-
istic relations in either fluid. To obtain the path of the interface, the boundary condition

$$\frac{dx}{dq} = u \frac{dt}{dq}$$

is integrated to give $x = x(q)$ knowing $t = t(q)$.

VI. The reflected shock case. If $B$ is to the left of $B_I$ in the $(l, u)$ hodograph plane, a shock discontinuity is reflected back into the left-hand fluid at the interface. The shock curve $S_R$ in the $(x, t)$ physical plane becomes two curves $S_a$ and $S_b$ in the $(l, u)$ hodograph plane (see Fig. 3). $S_a$ gives conditions just ahead of the shock and so starts at $B$. $S_b$ gives conditions just behind the shock and so starts at $B_I$.

The homentropic assumption implies a weak shock across which the $C+$ characteristics are continuous. Thus $r$ is the same on both sides of the shock and is a convenient parameter for the shock curve. The shock boundary condition then reads

$$\frac{dx}{dr} = U_R \frac{dt}{dr},$$

where $U_R$, the velocity of the reflected shock, is a function of flow speeds and pressures on both sides of the shock. If $s_a$ is the value of $s$ in the oncoming wave just ahead of the
shock, for a given \( r \), and \( s_b \) the corresponding value of \( s \) just behind the shock, \( U_R \) is the function of \( r, s_a \), and \( s_b \) given in Sec. II above.

The oncoming wave flow is "known", i.e., \( x \) and \( t \) are known functions of \( r \) and \( s \). The shock boundary condition becomes a differential equation for \( s_a \):

\[
S_1 : x_r + x_s \frac{ds_a}{dr} = U_R \left( t_r + t_s \frac{ds_a}{dr} \right),
\]

where the partial derivatives \( x_r \), etc., are all evaluated at \((r, s_a)\).

Applying the shock condition just behind the front gives

\[
x_r + x_s \frac{ds_b}{dr} = U_R \left( t_r + t_s \frac{ds_r}{dr} \right).
\]

The partial derivatives are related by the \( C+ \) and \( C- \) characteristic conditions so that \( x_r \) and \( x_s \) may be eliminated to give

\[
S_2 : \left( U_R + \frac{\gamma - 3}{2} r + \frac{\gamma + 1}{2} s_a \right) t_r + \left( U_R - \frac{\gamma + 1}{2} r - \frac{\gamma - 3}{2} s_b \right) t_s \frac{ds_b}{dr} = 0.
\]

Here \( t_r \) and \( t_s \) are to be evaluated at \((r, s_b)\). However, the forms of \( t_r \) and \( t_s \) as functions of \( r \) and \( s_b \) are not yet known.

Now \( t \) may be viewed as a known function of \( r \) and \( s_a \) or as an unknown function of \( r \) and \( s_b \). Equating these values of \( t \) along the shock then gives

\[
S_3 : t(r, s_a) = t(r, s_b).
\]

The problem restated is then to solve the partial differential equation (4) for \( t \) as a function of \( r \) and \( s \) in the angular region between \( I \) and \( S_b \) in the \((r, s)\) characteristic plane subject to conditions \( I \) along the interface and \( S_1 \), \( S_2 \), \( S_3 \) along the shock (see Fig. 3(c)).

As was done with the reflected rarefaction, it is convenient to consider the flow continued outside the angular region between \( I \) and \( S_b \) to the characteristics through \( B_r \). Forms for \( t, t_r, \) and \( t_s \) given in the previous section may be substituted in the various boundary conditions as before.

The substitution into interface condition \( I \) gives an integro-differential equation connecting \( t \) at \( F \) and \( t \) at \( E_r \). Substitution into \( S_2 \) connects \( t \) at \( F \) with \( t \) at \( E_s \). But in the step-by-step process by which the equations may be solved, \( t \) will already be known at \( E_s \), so that the \( S_2 \) equation actually relates \( t \) at \( F \) and function \( s_b = s_b(r) \). (The \( r \) coordinate of \( F \), \( r \), is now the parameter \( q \) in terms of which all points are expressed.) Finally, substitution into \( S_3 \) gives an equation relating \( t \) at \( F \), \( s_a(r) \), and \( s_b(r) \). The four equations \( I, S_1, S_2, \) and \( S_3 \) thus give the necessary number of relations to determine \( t \) along \( B_r F \), \( t \) along \( B_r E \), \( s_a(r) \), and \( s_b(r) \).

As in the case of the reflected rarefaction, knowledge of \( t \) along characteristics \( B_r F \) and \( B_r E \) and of the location of \( S_a \) and \( S_b \) in the \((r, s)\) plane permits the determination of \( t \) everywhere and so, using either the characteristic relations or the appropriate boundary condition, the corresponding values of \( x \) everywhere.

The next section contains an application of the present method to a specific example.

VII. Refraction of a detonation. A problem of some interest is the determination of the flow of a material due to the detonation of a layer of high explosive on its surface. An idealized form of such a problem, amenable to the present methods, reads as follows.

A semi-infinite slab of material filling the space \( x \geq 1 \) is given. A layer of high ex-
plosive of unit thickness covers the slab. At time $t = 0$ the explosive is detonated everywhere on its free surface ($x = 0$) simultaneously. The gaseous combustion products are assumed to expand into a vacuum.

The detonation front burns into the high explosive at known speed $D$. It is assumed that the detonation is a Chapman-Jouguet process, i.e., that particle and sound speeds just behind the detonation are related by
\[ u + c = D. \]

This and the equations for the conservation of mass and momentum determine all the flow quantities immediately behind the detonation front in terms of $D$ and the known initial density of the high explosive. Assuming the combustion products form a polytropic gas with zero density at zero pressure fixes $\alpha$, $\beta$, and $\gamma$, the constants in its equation of state.

The slab material is also assumed to be a polytropic gas with known equation of state. Some discussion of the experimental basis for this last assumption is given in [4, 6, and 7].

Since along the detonation front
\[ dx = D \, dt = (u + c) \, dt, \]
this front lies on a $C+$ characteristic. The flow field behind the front is covered by $C-$ characteristics. The detonation front carries constant values of pressure and flow velocity and so is represented by the single point $B$ in the $(l, u)$ hodograph plane. The field behind it must then all lie on the $\Gamma-$ characteristic through $B$ and so is a simple wave. In the wave $s = s_0$, a constant.

In the physical $(x, t)$ plane the whole wave starts at the origin, $x = t = 0$, and so is a centered simple wave. Straight lines through the origin are all $C+$ characteristics and correspond to points on the $\Gamma-$ characteristic between $B$ and $V$, the (vacuum) point where $\rho = l = 0$.

The $C+$ characteristic equations may be integrated here to give
\[ x = \left( \frac{\gamma + 1}{2} r + \frac{\gamma - 3}{2} s_0 \right) t. \]  

This together with
\[ s = s_0 \]
describes the centered simple wave in the burned gas, the oncoming wave of the preceding theory.

Since a simple wave is a degenerate form, some simplifications of the reflected wave boundary conditions occur here. In this example it is assumed that $B$ lies above interface curve $I$ in the $(l, u)$ hodograph plane so that the reflected shock considerations of Sec. VI apply. Points just ahead of this shock lie in the simple wave. The $S_s$ curve in the hodograph plane thus coincides with $\Gamma-$ characteristic through $B$. Boundary condition $S_1$ is no longer needed to locate $S_s$. When the other conditions have been used to determine $t$ as a function of $r$ along the front, $S_1$, or its equivalent
\[ \frac{dx}{dr} = U_s \frac{dt}{dr}, \]
may be used to find \( x \) as a function of \( r \) and so the location of the reflected shock in the 
\((x, t)\) physical plane.

Boundary condition \( S_2 \) is needed as before to locate \( S_b \), the hodograph curve of 
points just behind the reflected shock.

One may eliminate \( x \) between the shock condition, Eq. (6), and the equation valid
in the oncoming simple wave, Eq. (5), to obtain

\[
\frac{\gamma + 1}{2} t + \left( \frac{\gamma + 1}{2} t + \frac{\gamma - 3}{2} s_0 \right) \frac{dt}{dr} = U_r \frac{dt}{dr}.
\]

By boundary condition \( S_3 \), the values of \( t \) at corresponding points of \( S_a \) and \( S_b \) are the
same. Hence \( t \) and its derivatives in the equation may be considered to lie along \( S_b \).

The shock boundary condition then reads

\[
S_4 : \frac{\gamma + 1}{2} t = \left( U_r - \frac{\gamma + 1}{2} r - \frac{\gamma - 3}{2} s_0 \right) \left( t_r + t_s \frac{ds_b}{dr} \right),
\]

where \( t, t_r, \) and \( t_s \) are evaluated along \( S_b \).

Finally the interface boundary condition, with parameter \( q \) identified with character-
istic coordinate \( r \) of the left-hand fluid, reads

\[
I : t_r = t_s \frac{ds_I}{dr}.
\]

The forms for the partial derivatives of \( t \) to be substituted in \( S_2, S_4, \) and \( I \) may be
conveniently written

\[
t_r(P) = t_s(F) R(F; P) + \Phi_3(E; P),
\]

\[
t_r(P) = t_s(E) R(E; P) + \Phi_4(E; P),
\]

where

\[
t_r(F) = t_s(F) = \frac{d}{dr} t(r, s_I),
\]

\[
t_r(E) = t_s(E) = \frac{d}{ds} t(r_I, s),
\]

\[
\Phi_3(E; P) = \frac{\lambda t(F)}{r + s_I} R(F; P) + t(B_1) R_s(B_1; P)
\]

\[
+ \int_{B_1}^F R_s(J; P) \left( t_r + \frac{\lambda t}{r + s_I} \right) ds + \int_{B_1}^F R_s(J; P) \left( t_s + \frac{\lambda t}{r + s_I} \right) dr,
\]

and

\[
\Phi_4(E; P) = \frac{\lambda t(E)}{r_I + s} R(E; P) + t(B_1) R_s(B_1; P)
\]

\[
+ \int_{B_1}^E R_s(J; P) \left( t_r + \frac{\lambda t}{r_I + s} \right) ds + \int_{B_1}^E R_s(J; P) \left( t_s + \frac{\lambda t}{r + s_I} \right) dr.
\]

In using these formulas, \( P \) is replaced by either \( P_s \) or \( P_I \), points with the same \( r \)
coordinate as \( F \) and lying on \( S_a \) or \( I \) (see Fig. 3(c)). Similarly \( E \) is replaced by \( E_s \) or \( E_I \),
points whose \( r \) coordinate is \( r_I \) and whose \( s \) coordinate agrees with \( P_s \) or \( P_I \).
Making the substitutions into $S_2$, $S_4$, and $I$ and then solving for the derivatives of the unknown functions gives

$$
t'_F(F) = \frac{[\Phi_1 - (1 + \Phi_2)\Phi_3(E_2; P_s)]/(1 + \Phi_2)R(F; P_s)}{1 + \Phi_2} \tag{7}
$$

$$
t'_B(E_B) = \left[ t'_F(F)R(F; P_I) + \Phi_3(E_B; P_I) - \Phi_4(E_B; P_I) \frac{ds_B}{dr} \right] + \frac{ds_B}{dr} R(E_I; P_I) \tag{8}
$$

$$
\frac{ds_B}{dr} (P_s) = \Phi_1\Phi_2/(1 + \Phi_2)[t'_B(E_B)R(E_B; P_s) + \Phi_4(E_B; P_s)], \tag{9}
$$

where

$$
\Phi_1 = \frac{\gamma + 1}{2} t(P_s) \left( U_R - \frac{\gamma + 1}{2} r - \frac{\gamma - 3}{2} s_0 \right)
$$

$$
\Phi_2 = \left( U_R + \frac{\gamma - 3}{2} r + \frac{\gamma + 1}{2} s_0 \right) \left( U_R - \frac{\gamma + 1}{2} r - \frac{\gamma - 3}{2} s_0 \right)
$$

and

$$
U_R = (r - s_0) + \frac{\gamma - 1}{2} (r + s_0) \left[ -1 + \frac{\gamma + 1}{4} \left( \frac{2}{\gamma - 1} \frac{s_0 - s_b}{r + s_0} \right) \right.
$$

$$
\left. - \frac{(\gamma + 1)^2}{32} \left( \frac{2}{\gamma - 1} \frac{s_0 - s_b}{r + s_0} \right)^2 \right].
$$

Equations (7), (8), and (9) then constitute three simultaneous, ordinary, first order, integro-differential equations for $t_F$, $t_B$, and $s_b$, the values of $t$ along characteristics $B_1F$ and $B_1E$ and the location in the hodograph plane of points just behind the reflected shock. Using these, the value of $t$ at any point of the reflected wave may be computed. It is necessary to calculate $t$ along $s_b$ using Eq. (4) as the integration proceeds for substitution into $\Phi_1$. The computation of $t$ along $I$ may be carried out when convenient. Appropriate integrations then give $x$ along $s_b$ or $I$ and so the path of the reflected shock or interface in the $(x, t)$ plane. The path of the transmitted shock may then be computed as described in a previous section.

**VIII. A numerical example.** The theory of Sec. VII has been applied to determine the constants in an equation of state for the gaseous combustion products of a high explosive known as Composition B (See [7]). The equation is assumed to have the form

$$
p = \alpha \rho^\gamma - \beta.
$$

At the detonation front one has the Chapman-Jouguet condition,

$$
u_1 + c_1 = D,
$$

the conditions for conservation of mass and momentum,

$$\rho_1(u_1 - D) = -\rho_0 \ D,
$$

and

$$p_1 = \rho_0 \ Dw_1,
$$

and the definition of the speed of sound,

$$c_1 = (\alpha \gamma \rho_1^{\gamma-1})^{1/2}.$$
These relations may be combined to give

\[ \beta = \frac{[\rho_0 D^2 - (\gamma + 1)p_1]}{\gamma} \]

and

\[ \alpha = \frac{(\rho_0 D^2/\gamma)[\gamma(\beta + \rho_0 D^3)/(\gamma + 1)\rho_0 D^2]}{\gamma + 1}. \]

Thus two of the constants in the equation of state, \( \alpha \) and \( \beta \), are determined in terms of the third, \( \gamma \), the initial density of the high explosive, \( \rho_0 \), and the known pressure and velocity of the detonation, \( p_1 \) and \( D \).

For various values of \( \gamma \), curves of \( U_T \), the transmitted shock velocity, versus \( (x - 1) \), the distance into the slab, may be plotted as in Fig. 4. These curves may then be compared with the experimental points given in Table I of Ref. [7]. The curve for \( \gamma = 11/9 \) is seen to agree fairly well with the experimental points near the surface of the slab. This value of \( \gamma \) is also suggested for combustion products in Ref. [1], Sec. 38. A more complete comparison of theory with experiment has been made by W. E. Drummond [10].

**Fig. 4.** Transmitted shock velocity vs. depth.
As $\gamma$ varies, the $(p, \rho)$ curves all pass through the same point $(p_1, \rho_1)$ with the same slope, $c_1^2$. The equation of state has been taken in polytropic form to permit use of the Riemannian theory. For $\beta \neq 0$ the equation can only be an approximation valid in the neighborhood of $p = p_1$, $\rho = \rho_1$. Some indication of how large this neighborhood may be is given by $\rho(0)$, the density for which $p = 0$, since this density would be zero in a correct equation of state.

The equation of state of the metal slab is also in polytropic form. This form has been given some theoretical standing by Murnaghan in [5]. Constants used in the equation are those for aluminum given in [4].

The table below lists numerical values of interest. Densities are given in gm/cm$^3$, pressures in megabars, and velocities in cm/microsec.

<table>
<thead>
<tr>
<th>$\rho_0$</th>
<th>$p_1$</th>
<th>$\gamma'$</th>
<th>$\rho(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.67</td>
<td>0.26</td>
<td>4.267</td>
<td>2.81</td>
</tr>
<tr>
<td>$D = 0.77$</td>
<td></td>
<td>$p_1 = 2.263$</td>
<td>2</td>
</tr>
<tr>
<td>$\beta' = 0.171$</td>
<td></td>
<td>$\alpha' = 0.00214$</td>
<td>5/3</td>
</tr>
<tr>
<td>$7/5$</td>
<td>0.177</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$9/7$</td>
<td>0.307</td>
<td></td>
<td>9/7</td>
</tr>
<tr>
<td>$11/9$</td>
<td>0.336</td>
<td></td>
<td>11/9</td>
</tr>
</tbody>
</table>

The calculations have been ably carried through by T. C. Poulter, Jr., on an internally programmed digital computer. The work has been supported in part by the U. S. Army Bureau of Ordnance.

REFERENCES