RESPONSE OF SHALLOW VISCOELASTIC SPHERICAL SHELLS TO TIME-DEPENDENT AXISYMMETRIC LOADS*

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1. Introduction. The quasi-static treatment of problems in the linear theory of viscoelasticity has received increasing attention in recent years. The method of solution employed in such problems rests on the use of the Laplace transform (to eliminate the dependence on time), and the correspondence principle—between the field equations and boundary conditions in the linear theories of homogeneous and isotropic elasticity and viscoelasticity—which, in the absence of thermal effects, has been established for incompressible media by Alfrey [1], and in general form by Lee [2]. The extension of Lee's analogy to problems involving time-dependent temperature fields has been very recently given by Sternberg [3]. Also, considerable attention has been given to oscillation and wave propagation problems of viscoelasticity in which the inertia terms have been included, e.g., [4, 5, 6], and additional references on the subject may be found in a recent survey by Lee [7].

Closely related to the scope of the present investigation is the recent work on vibrations of thin shallow elastic shells by E. Reissner [8], who, by utilizing the linear differential equations due to Marguerre [9], has shown that for *transverse vibrations* of shallow shells the longitudinal inertia terms (with negligible error) may be omitted; and hence, the formulation of the elastokinetic problems of shallow shells, as in the case of elastostatics, may be reduced to the determination of axial (or transverse) displacement and an Airy stress function. Subsequently, E. Reissner [10] dealt with transverse vibrations of axisymmetric shallow elastic spherical shells, and in particular, obtained the solution for an unlimited shell due to an oscillating point load (varying harmonically in time) at the apex.

The present paper is concerned with the response of shallow viscoelastic spherical shells to arbitrary time-dependent axisymmetric loads; the medium is assumed homogeneous and isotropic. Although emphasis is placed on unlimited shallow spherical shells, shallow spherical shell segments are also considered and discussed in Sec. 7. The solutions, employing the differential equations governing the transverse motion of thin shallow elastic shells, are obtained with the joint use of the Laplace and the Hankel transforms, which, by interchanging the order of the inversions, avoids an otherwise intricate task of contour integration in the complex Laplace transform-plane. Explicit results in integral form are deduced for viscoelastic shells under instantaneous pulse loading (including those uniformly distributed about and concentrated at the apex), and are particularized to the cases of Maxwell and Kelvin solids. The solution for a shallow elastic shell and

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for the case of a flat plate are also given as by-products of the general solution and comparison is made with known results [10]. It may be further noted that the transform technique employed here appears to be useful also in connection with other axisymmetric problems of stress wave propagation in viscoelastic solids.

2. Preliminary background. With reference to rectangular Cartesian coordinates x_i, the stress-strain law for an isotropic and homogeneous viscoelastic medium may be written as[1]

$$\begin{aligned} P_1(\theta)s_{ij} &= P_2(\theta)e_{ij} \\ P_3(\theta)\sigma_{ii} &= P_4(\theta)\epsilon_{ii}\ , \end{aligned} \tag{2.1}$$

where σ_{ij} and ϵ_{ij} are the components of the stress and the strain tensors s_{ij} and e_{ij} designate the deviatoric components of stress and strain, the operators $P_m(\theta)$ involving the constant coefficients $C_m^{(n)}$ $(m = 1, 2, 3, 4)$ are defined by

$$\begin{aligned} P_m(\theta) &= \sum_{n=0}^{N_m} C_m^{(n)}\theta^n; \qquad [C_m^{(N_m)} \neq 0] \\ \theta^n &\equiv \frac{\partial^n}{\partial t^n}\ , \end{aligned} \tag{2.2}$$

and t denotes time.

For future reference, we also recall that the Laplace transform with respect to t of a (suitably restricted) function $U(x, t)$ is given by[2]

$$U'(x, s) = L\{U(x, t); s\} \equiv \int_0^\infty e^{-st}U(x, t)\, dt, \tag{2.3}$$

where s is the transform parameter, and that the Hankel transform of order zero of the function $U'(x, s)$ is defined by[3]

$$U^*(\xi, s) = \int_0^\infty xJ_0(\xi x)U'(x, s)\, dx \tag{2.4}$$

provided that (i) the integral $\int_0^\infty U'(x, s)dx$ is absolutely convergent, and (ii) the function U' is of bounded variation over the region of interest. Furthermore, in connection with the Hankel transform of $\partial U'(x, s)/\partial x$, we need the property that (iii) $xU'(x, s)$ vanishes both at $x = 0$, and as $x \to \infty$; this ensures the existence of the inverse transform of U^*. Here it may be noted that the Hankel transform defined by (2.4) formally differs from that defined in [17]; however, with the transformation $U'(x, s) = x^{-1/2}V$ (V being the function corresponding to U' in [17]), after multiplying both sides of (2.4) by $\xi^{1/2}$ and setting $V^* = \xi^{1/2}U^*$, the two definitions are brought together.

Since in the theory of shells (and plates) the stress resultants and the stress couples are defined in terms of components of stress, rather than components of stress deviation, it is expedient for our present purposes to obtain an alternative form of (2.1). To this

[1]The Latin indices have, unless otherwise stated, the range of $i, j, = 1, 2, 3$, and the repeated indices imply the summation convention.

[2]See, for example, Churchill [11]; the argument x in U refers to the space variable.

[3]See, for example, Sneddon [12].

end, observing that the operators $P_m^{-1}(\theta)$ are in general noncommutative [13], we take the Laplace transform of (2.1) and express σ'_{ij} in terms of ϵ'_{ij} by[4]

$$\sigma'_{ij} = P_2(s)P_1^{-1}(s)\epsilon'_{ij} - \tfrac{1}{3}[P_2(s)P_1^{-1}(s) - P_4(s)P_3^{-1}(s)]\epsilon'_{kk}\,\delta_{ij}\,, \tag{2.5}$$
$$\sigma'_{ii} = P_4(s)P_3^{-1}(s)\epsilon'_{ii}\,,$$

where s_{ij} is the Kronecker delta.

It follows from the *correspondence principle* that the field equations and the boundary conditions governing the original viscoelastic problem are reducible to the field equations and boundary conditions of an associated problem in the linear theory of elasticity, with Young's modulus E and Poisson's ratio ν of the elastic solid replaced by

$$\begin{Bmatrix} E(s) \\ \nu(s) \end{Bmatrix} = [P_2(s)P_3(s) + 2P_1(s)P_4(s)]^{-1}\begin{Bmatrix} 3P_4(s)P_2(s) \\ [P_1(s)P_4(s) - P_2(s)P_3(s)] \end{Bmatrix}. \tag{2.6}$$

It is easily verified that the correspondence principle and the results (2.6) are also valid for any of the various (consistent) theories of thin shells.

We also note that in (2.6) the linear, homogeneous and isotropic elastic medium may be identified by allowing $E(s) \to E$ and $\nu(s) \to \nu$ (corresponding to $P_1(s) = P_2(s) = 1$, $P_3(s) = 2\mu$, $P_4(s) = 3K$, μ and K being the shear and the bulk moduli of the elastic solid, respectively). The operators $P_m(s)$, when associated with the Maxwell solid, are

$$P_1 = s + \tau^{-1}, \qquad P_2 = 2\mu s$$
$$P_3 = s, \qquad P_4 = 3Ks \tag{2.7a}$$

and (2.6) becomes

$$E(s) = \frac{s}{s + \frac{2}{3}(1+\nu)\tau^{-1}}E,$$
$$\nu(s) = \frac{s + \dfrac{1+\nu}{3\nu}\tau^{-1}}{s + \frac{2}{3}(1+\nu)\tau^{-1}}\nu, \tag{2.7b}$$

where $\tau = \eta/\mu$ is the relaxation time, η being the viscosity. Similarly, for the Kelvin solid,

$$P_1 = 1, \qquad P_2 = 2\mu(1 + \tau s)$$
$$P_3 = \tau s, \qquad P_4 = 3K\tau s \tag{2.8a}$$

and

$$E(s) = \frac{1 + \tau s}{1 + \dfrac{1-2\nu}{3}\tau s}E,$$
$$\nu(s) = \frac{1 - \dfrac{1-2\nu}{3\nu}\tau s}{1 + \dfrac{1-2\nu}{3}\tau s}\nu, \tag{2.8b}$$

where in (2.8b) τ denotes the retardation time.

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[4] It should be noted that unlike $P_m^{-1}(\theta)$ the operators $P_m^{-1}(s)$ are commutative.

3. Differential equations for transverse vibrations of shallow elastic spherical shells. Let H denote the rise of the shell segment, L the characteristic length (which for spherical shells may be conveniently taken as the radius of curvature R), h the shell thickness, ρ the density, and Γ the representative wave length. Then, for transverse vibrations of thin shallow elastic shells, with the stipulation that $(H/L)^2 \ll 1$, it has been shown by E. Reissner [8] that the effect of longitudinal inertia may be omitted (with negligible error) from the differential equations governing the motion of the shell as long as (Γ/L) is of order unity[5], i.e.,

$$\left(\frac{\Gamma}{L}\right) = 0(1) \tag{3.1a}$$

and when Γ is characterized by the following classification

(a) If $\left(\frac{H}{h}\right) = 0(1)$ or smaller, then

$$\Gamma^4 = \frac{1}{12(1 - \nu^2)} (h\gamma)^2, \tag{3.1b}$$

(b) If $\left(\frac{H}{h}\right) \gg 1$, then

$$\Gamma = \frac{1}{1 - \nu^2} \left(\frac{H}{L}\right)^2 \left(\frac{\gamma^2}{L}\right), \tag{3.1c}$$

where $\gamma^{-2} = \rho\omega^2/E$, ω being the circular frequency.

Thus, with reference to cylindrical polar coordinates (r designating the polar radius) and with omission of the longitudinal inertia term, the differential equations for the axisymmetric transverse vibrations of shallow elastic spherical shells are characterized by [10]

$$D\nabla^2\nabla^2 w + \frac{1}{R}\nabla^2 F = -\rho h \frac{\partial^2 w}{\partial t^2} + p(r, t)$$

$$\nabla^2\nabla^2 F - \frac{hE}{R}\nabla^2 w = 0, \tag{3.2}$$

and the various stress resultants and stress couples are given by

$$N_r = \frac{1}{r}\partial F/\partial r, \qquad N_\theta = \nabla^2 F - \frac{1}{r}\partial F/\partial r \tag{3.3a}$$

$$M_r = -D\left[\partial^2 w/\partial r^2 + \frac{\nu}{r}\partial w/\partial r\right], \qquad M_\theta = -D\left[\frac{1}{r}\partial w/\partial r + \nu\frac{\partial^2 w}{\partial r^2}\right] \tag{3.3b}$$

$$Q = -D\frac{\partial}{\partial r}(\nabla^2 w) \tag{3.3c}$$

where w is the axial displacement, F is the Airy stress function, p is the axial component of the surface load,

[5]Actually in [8], Reissner has further concluded that the neglect of longitudinal inertia terms is justified when (Γ/L) is of order of unity or smaller, but not when $(\Gamma/L) \gg 1$.

$$D = \frac{Eh^3}{12(1 - \nu^2)}, \quad \text{and} \quad \nabla^2(\ ) \equiv \frac{\partial^2}{\partial r^2}(\ ) + \frac{1}{r}\frac{\partial}{\partial r}(\ ).$$

It is also relevant to recall here that the steady state solution (for axisymmetric transverse vibrations of shallow elastic spherical shells) given by E. Reissner [10], where w and F are assumed to have the form

$$w = W(r) \exp(i\omega t), \qquad F = f(r) \exp(i\omega t), \qquad [i = (-1)^{1/2}],$$

involves Bessel functions (J_0, Y_0, I_0, K_0) of argument λr, where

$$\lambda^4 = -\frac{12(1 - \nu^2)}{(Rh)^2}\left[1 - \left(\frac{R}{\gamma}\right)^2\right], \qquad \gamma = \frac{(E/\rho)^{1/2}}{\omega}. \tag{3.4}$$

Indeed, since by (3.4) the case of $\gamma/R = 1$ (corresponding to $\lambda = 0$) is not admissible, for the axisymmetric vibrations treated in [10] two sets of solutions of (3.2) associated with the two frequency ranges $R/\gamma \gtrless 1$ exist.

The integration of the second of (3.2), together with the condition of vanishing circumferential displacement (which, as in the elastostatic solution of shallow spherical shells [14], demands the vanishing of the coefficient of the logarithmic term) leads to

$$\nabla^2 F = \frac{hE}{R} w + G_0(t). \tag{3.5}$$

If attention is confined to unlimited shallow shells, then since $w(\infty, t) = 0$ and $\nabla^2 F(\infty, t) = 0$ (the latter condition, by (3.3a), is due to independent vanishing of N_r and N_θ at $r = \infty$), it follows that in (3.5) $G_0(t) = 0$ and for unlimited shallow shells the system of differential equations (3.2) reduces to

$$\begin{aligned} D\nabla^2\nabla^2 w + \frac{1}{R}\nabla^2 F &= -\rho h \frac{\partial^2 w}{\partial t^2} + p(r, t) \\ \nabla^2 F &= \frac{hE}{R} w. \end{aligned} \tag{3.6}$$

Furthermore, for unlimited spherical shells, the remaining boundary conditions associated with (3.6) are

$$\frac{\partial w}{\partial r}(\infty, t) = \nabla^2 w(\infty, t) = 0, \tag{3.7}$$

the regularity requirements for oscillating distributed load are specified by

$$w(0, t), \qquad \frac{\partial w}{\partial r}(0, t), \qquad N_r(0, t), \qquad N_\theta(0, t); \quad \text{finite} \tag{3.8a}$$

$$M_r(0, t), \qquad M_\theta(0, t); \quad \text{finite}, \tag{3.8b}$$

and those appropriate for an oscillating point load are given by (3.8a).

4. Unlimited shallow viscoelastic spherical shells. The differential equations and the boundary conditions (as well as the regularity requirements) appropriate for an unlimited shallow viscoelastic spherical shell ($0 \leq r \leq \infty$), subjected to an arbitrary time-dependent axisymmetric load, following the application of the Laplace transform, and with an appeal to the correspondence principle (Sec. 2), are obtained from (3.6), (3.7), and (3.8) with the moduli E and ν (of the elastic solid) replaced by $E(s)$ and $\nu(s)$, respectively. In particular, with zero initial conditions, i.e.,

$$w(r, 0) = \frac{\partial w}{\partial t}(r, 0) = F(r, 0) = 0, \tag{4.1}$$

the differential equations of motion in the Laplace transform-plane read

$$\begin{aligned} D(s)\nabla^2\nabla^2 w' + \frac{1}{R}\nabla^2 F' + \rho h s^2 w' &= p' \\ \nabla^2 F' &= \frac{hE(s)}{R} w'. \end{aligned} \tag{4.2}$$

As the solution of the system of differential equations (4.2) involves Kelvin functions whose arguments are polynomials in s, thus prohibiting simple inversions, the determination of w and F will in general require cumbersome contour integration in the complex Laplace transform-plane. To overcome this difficulty, we consider the application of Hankel transform of order zero to (4.2), and require that w and its derivatives up to the fourth, and F' and its derivatives up to the first vanish to a suitable order at infinity such that all integrals employed in the following analysis exist.

Recalling the formula for the derivative of Hankel transform of order zero [12, p. 62], i.e.,

$$\int_0^\infty (\nabla^2 w')rJ_0(r\xi)\,dr = -\xi^2 w^*(\xi, s) \tag{4.3a}$$

and by iteration

$$\int_0^\infty (\nabla^2\nabla^2 w')rJ_0(r\xi)\,dr = \xi^4 w^*(\xi, s), \tag{4.3b}$$

then, with the aid of (4.3) and application of the Hankel transform of zero order to (4.2), we reach

$$\begin{aligned} D(s)\xi^4 w^* + \frac{1}{R}(\nabla^2 F)^* + \rho h s^2 w^* &= p^* \\ (\nabla^2 F)^* &= \frac{hE(s)}{R} w^*. \end{aligned} \tag{4.4}$$

Elimination of $(\nabla^2 F)^*$ from (4.4) results in

$$w^* = D^{-1}(s)[\xi^4 + \lambda_0^4(s)]^{-1} p^*(\xi, s), \tag{4.5}$$

where

$$\lambda_0^4 = \frac{12[1 - \nu^2(s)]}{(Rh)^2}\left[1 + R^2\left(\frac{\rho s^2}{E(s)}\right)\right] = l^{-4} + \frac{12[1 - \nu^2(s)]}{h^2 E(s)}\rho s^2, \tag{4.6a}$$

$$l^4 = \frac{(Rh)^2}{12[1 - \nu^2(s)]}. \tag{4.6b}$$

It is now clear that since the right-hand side of (4.5) involves only polynomials in ξ and S, the inversion of w^* in s is easily carried out, and its inversion in the Hankel transform parameter ξ (being a real variable) is also possible. Thus, by first taking the inverse Laplace transform of (4.5), followed by the inverse Hankel transform (which is a self-reciprocating transform), we obtain

$$w(r, t) = \int_0^\infty \xi J_0(r\xi)\, d\xi \int_0^t L^{-1}\{p^*(\xi, s); \zeta\} L^{-1}\{\psi; t - \zeta\}\, d\zeta, \tag{4.7a}$$

where

$$\psi = D^{-1}(s)[\xi^4 + \lambda_0^4(s)]^{-1}. \tag{4.7b}$$

It remains to determine the function F. However, since F does not necessarily conform to the requirements for the validity of its Hankel transform, while N_r and N_θ as given by (3.3a) are independently finite at $r = 0$ and vanish at $r = \infty$, we proceed instead to establish $\nabla^2 F$ and $1/r\, \partial F/\partial r$. To this end, we turn to the second of (4.4) and, with the aid of (4.5), write

$$(\nabla^2 F)^* = \frac{hE(s)}{R} p^* \left\{ D(s)\xi^4 + \rho h s^2 + \frac{hE(s)}{R^2} \right\}^{-1}. \tag{4.8}$$

Again, taking the inverse Laplace transform of (4.8), followed by the inverse Hankel transform, we obtain

$$\nabla^2 F(r, t) = \frac{h}{R} \int_0^\infty \xi J_0(r\xi)\, d\xi \int_0^t L^{-1}\{p^*(\xi, s); \zeta\} L^{-1}\{E(s)\psi(\xi, s); t - \zeta\}\, q\zeta, \tag{4.9a}$$

and by integration, since $N_r(0, t) = N_\theta(0, t) = \frac{1}{2} \nabla^2 F(0, t)$,

$$N_r = \frac{1}{r} \frac{\partial F}{\partial r} = \frac{1}{r^2} \int_0^r x \nabla^2 F(x, t)\, dx \tag{4.9b}$$

which completes the desired solution.

Before closing this section, we record some special types of loading, as well as their Hankel transforms, which will be of interest presently. For a pulse instantaneously applied and removed at $t = t_0$,

$$p(r, t) = q(r)\, \delta(t - t_0), \tag{4.10}$$

where the Dirac delta function is defined by

$$\begin{gathered} \delta(t - t_0) = 0, \qquad (t \neq t_0) \\ \int_{-\infty}^\infty \delta(t - t_0)\, dt = 1. \end{gathered} \tag{4.11}$$

If, on the other hand, the pulse applied at $t = t_0$ continues to act indefinitely, then it is only necessary to replace $\delta(t - t_0)$ in (4.10) with the Heaviside step function $H(t - t_0)$, defined by

$$H(t - t_0) = \begin{cases} 1 & (t - t_0 > 0) \\ \frac{1}{2} & (t = t_0). \\ 0 & (t < t_0) \end{cases} \tag{4.12}$$ [6]

[6]The use of the letter H with argument $(t - t_0)$ for the Heaviside step function should not be confused with the notation for the shell rise in Sec. 2.

Among the various forms which the function $q(r)$ may assume, we specifically cite those associated with (a) uniformly distributed pulse over $0 \leq r \leq a$ $(a/R \ll 1)$, and (b) point load pulse applied at $r = 0$, each given respectively by

$$q(r) = -p_0 H(a - r), \tag{4.13a}$$

$$q(r) = -\frac{P}{\pi r}\,\delta(r). \tag{4.13b}$$

Also, for future reference, the Hankel transforms of q specified by (4.13a) and (4.13b) are given respectively by [12, p. 88]

$$q^*(\xi) = -\frac{p_0 a}{\xi}\,J_1(a\xi), \tag{4.14a}$$

and by [15, p. 67]

$$q^*(\xi) = -\frac{P}{2\pi}. \tag{4.14b}$$

5. The elastic solution as a by-product. Reduction to known results. By allowing $E(s) \to E$ and $\nu(s) \to \nu$ in (4.5) and (4.8), and by taking their inverse transformations, we deduce the complete solution for the unlimited shallow elastic spherical shell under arbitrary time-dependent axisymmetric load. In particular, when the load is specified by (4.10) and (4.13a), then with the aid of (4.14a) and tables of integral transforms [16, Table 5.2] and [9, p. 323], there follows

$$w(r, t) = -\frac{p_0 a}{\rho h}\int_0^\infty J_0(r\xi)J_1(a\xi)\phi\,d\xi, \tag{5.1}$$

$$\begin{Bmatrix} \nabla^2 F(r, t) \\ \dfrac{\partial F}{\partial r}(r, t) \end{Bmatrix} = -\begin{Bmatrix} \dfrac{p_0 a E}{\rho R} \\ \dfrac{1}{r}\displaystyle\int_0^r x\,\dfrac{p_0 a E}{\rho R}\,dx \end{Bmatrix} \int_0^\infty J_0(r\xi)J_1(a\xi)\phi\,d\xi, \tag{5.2}$$

where

$$\phi = \frac{\sin\left\{\left(\dfrac{E}{\rho}\right)^{1/2}\dfrac{1}{R}\,[1 + (l\xi)^4]^{1/2}(t - t_0)\right\}}{\left(\dfrac{E}{\rho}\right)^{1/2}\dfrac{1}{R}\,[1 + (l\xi)^4]^{1/2}}. \tag{5.3}$$

As further specialization of the above solution, consider the case of a flat plate by letting $R \to \infty$ and setting $F = 0$. Thus, for an infinite elastic circular plate subjected to a pulse concentrated at the origin (allowing $a \to 0$ while $P = \pi a^2 p_0$), (5.1) reduces to

$$w(r, t) = -\frac{P}{2\pi\rho h}\left(\frac{\rho h}{D}\right)^{1/2}\int_0^\infty \xi J_0(r\xi)\,\frac{\sin\left\{\left(\dfrac{D}{\rho h}\right)^{1/2}\xi^2(t - t_0)\right\}}{\xi^2}\,d\xi \tag{5.4}$$

which, with $t_0 = 0$, the notation $r_0\ (E/\rho)^{1/2}t$, and the use of [17, Table 8.2)] becomes

$$w(r, t) = \frac{-P}{4\pi\rho h^2}\left[12(1 - \nu^2)\,\frac{\rho}{E}\right]^{1/2} si\left\{\frac{[12(1 - \nu^2)]^{1/2}}{4}\left(\frac{r}{h}\right)\left(\frac{r}{r_0}\right)\right\}, \tag{5.5a}$$

$$si(r) = -\int_r^\infty \frac{\sin x}{x}\,dx = -\tfrac{1}{2}\pi + Si(r), \tag{5.5b}$$

$Si(r)$ being the sine integral.

In the remainder of this section, we confine attention to shallow elastic spherical shells subjected to loading of the types

$$p(r, t) = -p_0 H(a - r)e^{i\omega t}, \tag{5.6a}$$

$$p(r, t) = -\frac{P}{\pi r}\,\delta(r)e^{i\omega t}, \tag{5.6b}$$

which vary harmonically in time. For such steady state solutions, it is more convenient to return to the viscoelastic solution (4.5) in the Laplace-Hankel transform-plane. Following Lee [6], we replace s by $i\omega$ in all quantities except in $p^*(\xi, s)$ which is replaced by[7] $p^*(\xi)$, and then take the inverse Hankel transform leading to the (real) steady state amplitude $W(r)$ for the axial displacement. When $p(r, t)$ is specified by (5.6a), through the process just described and with the aid of (4.14a), we obtain

$$W(r) = -p_0 a\, D^{-1} \int_0^\infty \frac{J_0(r\xi)J_1(a\xi)}{[\xi^4 + \lambda_0^4]}\,d\xi, \tag{5.7}$$

along with similar expressions for $\nabla^2 f$ and $\partial f/\partial r$, $f(r)$ being the amplitude of $F(r, t)$. Unfortunately, the solution (5.7) does not admit a closed representation; and probably for this case a more direct approach, involving the use of the elastic solution of the problem [10] and the Laplace transform, will (from a practical point of view) prove fruitful.

For the steady state solution due to an oscillating point load specified by (5.6b), with the aid of (4.14b), W and $\nabla^2 f$ are given by

$$W(r) = -\frac{P}{2\pi D} \int_0^\infty \frac{\xi J_0(r\xi)}{[\xi^4 + \lambda_0^4]}\,d\xi, \tag{5.8a}$$

$$\nabla^2 f(r) = -\frac{hE}{R}\,W(r). \tag{5.8b}$$

The integral in (5.8a), with λ_0^4 as positive, admits a closed representation [17, Table 8.2]. Hence,

$$W(r) = \frac{P}{2\pi D\lambda_0^2}\,kei(\lambda_0 r) \tag{5.9}$$

which is in exact agreement with the results given in [10] for

$$(R/\gamma)^2 > 1 \quad \text{or} \quad \omega^2 > (E/\rho)/R^2$$

(corresponding to $\lambda_0^4 > 0$); in (5.9), *kei* x (together with *ber* x, *bei* x, and *ker* x to be introduced presently) are the Kelvin functions.

On the other hand, if in (5.8) $\lambda_0^4 < 0$, i.e., for $(R/\gamma)^2 < 1$ or $\omega^2 < (E/\rho)/R^2$, we introduce the quantity λ through

$$\lambda_0^4 = \lambda^4 \exp\left(-i\frac{\pi}{4}\right), \tag{5.10}$$

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[7]Here, this process is equivalent to taking the inverse Laplace transform.

[where, by (4.6a) with s replaced by $i\omega$, λ is defined by (3.4)] and also employ the relations [18]

$$\begin{aligned} ker\, z = \frac{\pi}{4}\Big\{-iJ_0\Big[z \exp\Big(-i\frac{\pi}{4}\Big)\Big] - Y_0\Big[z \exp\Big(-i\frac{\pi}{4}\Big)\Big] \\ + iJ_0\Big[z \exp\Big(i\frac{\pi}{4}\Big)\Big] - Y_0\Big[z \exp\Big(i\frac{\pi}{4}\Big)\Big]\Big\}, \end{aligned} \tag{5.11a}$$

$$\begin{aligned} kei\, z = \frac{\pi}{4}\Big\{-J_0\Big[z \exp\Big(-i\frac{\pi}{4}\Big)\Big] + iY_0\Big[z \exp\Big(-i\frac{\pi}{4}\Big)\Big] \\ - J_0\Big[z \exp\Big(i\frac{\pi}{4}\Big)\Big] - iY_0\Big[z \exp\Big(i\frac{\pi}{4}\Big)\Big]\Big\} \end{aligned} \tag{5.11b}$$

whose argument $z = x + iy$ is complex. Thus, for $\lambda_0^4 < 0$, substitution of (5.11) and (5.10) into (5.8) and the use of [15, Table 8.2] lead to expressions whose real parts[8], i.e.,

$$W = \operatorname{Re}\left\{\frac{P}{4\pi D\lambda_0^2}\left[K_0(\lambda_0 r) + \frac{\pi}{2} Y_0(\lambda_0 r) - i\frac{\pi}{2} J_0(\lambda_0 r)\right]\right\}, \tag{5.12a}$$

$$f = \operatorname{Re}\left\{\frac{P}{4\pi D\lambda_0^2}\left(\frac{hE}{R}\right)\left[K_0(\lambda_0 r) - \frac{\pi}{2} Y_0(\lambda_0 r) + i\frac{\pi}{2} J_0(\lambda_0 r) + 2\ln\frac{r}{l}\right]\right\} \tag{5.12b}$$

agree with the corresponding results in [8] for $R/\gamma < 1$. Here, it may be of interest to note that with (5.10) and with the use of known relations of the type (see e.g. [18, p. 20])

$$Y_0(i^{1/2}x) = [bei\, x + i\, ber\, x] + \frac{2}{\pi}[-ker\, x + i\, kei\, x],$$

Eq. (5.12) for the range $R/\gamma > 1$ may be reduced to the form (5.9)[9].

6. Solutions for special viscoelastic materials. In this section, for simplicity's sake we limit ourselves to incompressible media ($\nu = \frac{1}{2}$), in which case (2.6) and (4.6) become

$$E(s) = \tfrac{3}{2}P_2(s)P_1^{-1}(s), \tag{6.1}$$

$$\lambda_0^4(s) = \frac{9}{h^2}\left[\frac{1}{R^2} + \frac{\rho s^2}{E(s)}\right],$$

and deduce explicitly solutions for the viscoelastic Maxwell and Kelvin solids.

(a) For the incompressible Maxwell solid with the aid of (2.7b), (6.1) reduces to

$$\begin{aligned} E(s) &= \frac{s}{s + \tau^{-1}} E, \\ \lambda_0^4(s) &= \frac{9\rho}{Eh^2}\left[s^2 + s\tau^{-1} + \frac{E}{\rho R^2}\right]. \end{aligned} \tag{6.2}$$

As tables for the inverse transforms of the functions involved here are available, we turn to (4.5) and write

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[8]It may be recalled that on account of the assumed form of the exponential time-dependence of the solution, the various quantities are in general complex, and their real parts give the desired results.

[9]In this connection, compare with [10, Eqs. (42) to (45)].

$$w^*(\xi, s) = \frac{9}{Eh^3}\frac{s+\tau^{-1}}{s}\frac{p^*(\xi, s)}{\xi^4 + \frac{9\rho}{Eh^2}\left[s^2 + s\tau^{-1} + \frac{E}{\rho R^2}\right]}$$
$$= \frac{p^*(\xi, s)}{oh}\frac{s+\tau^{-1}}{s[(s+(2\tau)^{-1})^2 + (\beta^2 - (2\tau)^{-2})]}, \tag{6.3a}$$

where

$$\beta = \left\{\left(\frac{E}{\rho}\right)\left(\frac{\tau}{R}\right)^2[1 + (l\xi)^4]\right\}^{1/2}\tau^{-1}. \tag{6.3b}$$

Then, for instantaneous pulse specified by (4.10), after recalling the Laplace transform of the δ-function [11, p. 27], as well as its inverse transform [11, p. 323], and using [16, Table 5.2], we take the inverse (Laplace followed by Hankel) transforms of (6.3a) to obtain

$$w(r, t) = \frac{1}{\rho h}\int_0^\infty [\xi q^*(\xi)J_0(r\xi)]\frac{\tau^{-1}}{\beta^2}\left\{1 - \exp\left(-\frac{t-t_0}{2}\tau^{-1}\right)\right.$$
$$\cdot\left\langle\cos\left[\left(\beta^2 - \frac{1}{4\tau^2}\right)^{1/2}(t-t_0)\right]\right. \tag{6.4}$$
$$\left.\left. - \frac{\tau\beta^2 - (2\tau)^{-1}}{\left(\beta^2 - \frac{1}{4\tau^2}\right)^{1/2}}\sin\left[\left(\beta^2 - \frac{1}{4\tau^2}\right)^{1/2}(t-t_0)\right]\right\rangle\right\} d\xi.$$

In a similar manner,

$$\nabla^2F(r, t) = \frac{E}{\rho R}\int_0^\infty [\xi q^*(\xi)J_0(r\xi)]$$
$$\cdot\left\{\frac{\exp\left(-\frac{t-t_0}{2}\tau^{-1}\right)}{\left(\beta^2 - \frac{1}{4\tau^2}\right)^{1/2}} \times \sin\left[\left(\beta^2 - \frac{1}{4\tau^2}\right)^{1/2}(t-t_0)\right]\right\} d\xi \tag{6.5}$$

and $\partial F/\partial r$, as in Sec. 4, is obtained by integration.

(b) For the incompressible Kelvin solid with the aid of (2.8b), (6.1) reads as

$$E(s) = (1 + \tau s)E,$$
$$\lambda_0^4(s) = \frac{9\rho}{Eh^2}\frac{s^2 + [E/(\rho R^2)](1+\tau s)}{(1+\tau s)}, \tag{6.6}$$

and again, to avoid the convolution integral in (4.7) and (4.9), we return to (4.5) and (4.8) and obtain

$$w(r, t) = \frac{1}{\rho h}\int_0^\infty [\xi q^*(\xi)J_0(r\xi)]\exp\left[-\frac{\tau\beta^2}{2}(t-t_0)\right]$$
$$\cdot\frac{\sin\left\{\beta\left[1 - \left(\frac{\tau\beta}{2}\right)^2\right]^{1/2}(t-t_0)\right\}}{\beta\left[1 - \left(\frac{\tau\beta}{2}\right)^2\right]^{1/2}} d\xi, \tag{6.7}$$

$$\nabla^2 F(r, t) = \frac{E\tau}{\rho R}\int_0^\infty [\xi q^*(\xi)J_0(r\xi)] \exp\left[-\frac{\tau\beta^2}{2}(t - t_0)\right]$$
$$\cdot\left\{\cos\left\langle\beta\left[1 - \left(\frac{\tau\beta}{2}\right)^2\right]^{1/2}(t - t_0)\right\rangle\right. \tag{6.8}$$
$$\left. + \frac{1 - \left(\frac{\tau\beta}{2}\right)^2}{\tau\beta\left[1 - \left(\frac{\tau\beta}{2}\right)^2\right]^{1/2}} \sin\left\langle\beta\left[1 - \left(\frac{\tau\beta}{2}\right)^2\right]^{1/2}(t - t_0)\right\rangle\right\} d\xi.$$

Since the integrals in the solutions (6.4) and (6.5), as well as (6.7) and (6.8), converge rapidly, the numerical evaluation of the results is feasible. In addition, it may be noted that as $t \to \infty$, the second term in the integrand of (6.4) diminishes exponentially while the first remains finite. Hence, under instantaneous loading, the shell medium for the Maxwell solid will assume a permanent deformation; no such effect is present in the solution (6.7) for the Kelvin solid. That this observation is not unexpected becomes evident by merely recalling the absence and presence of a restoring force in the one-dimensional Maxwell and Kelvin models, respectively. On the other hand, for an oscillatory load which itself (unlike an instantaneous load) supplies the restoring force, no permanent deformation takes place in the Maxwell medium.

7. Shallow viscoelastic shell segments. The solutions for shallow viscoelastic spherical shell segments, in principle, may be obtained in a manner similar to those for unlimited shallow shells (Sec. 4); the chief difference, however, is the use of finite Hankel transform (in place of Hankel transform) together with some manipulations necessary to accommodate the edge boundary conditions of the shell segment. Although the method of solution (to be discussed presently) permits the treatment of shell segments with various boundary conditions, such as those considered in [14, Sec. 11], for simplicity's sake we confine attention to the case of shallow spherical shell segments ($0 \leq r \leq r_0$) supported at $r = r_0$ by means of a ring which is restrained against rotation and non-resistant to axial force. Here, the regularity requirements at $r = 0$ are identical with (3.8), and the boundary conditions are given by

$$w(r_0, t) = N_r(r_0, t) = M_r(r_0, t) = 0. \tag{7.1}$$

Before proceeding further, we recall that the finite Hankel transform of a (suitably restricted) function $U'(r, s)$, as well as its inverse transform in the interval $0 \leq r \leq r_0$ are defined, respectively, by [12, p. 83]

$$U^*(\xi_i, s) = \int_0^{r_0} rJ_0(r\xi_i)U'(r, s)\, dr \tag{7.2a}$$

and

$$U'(r, s) = \frac{2}{r_0^2}\sum_i U^*(\xi_i, s)\frac{J_0(r\xi_i)}{[J_1(r_0\xi_i)]^2}, \tag{7.2b}$$

where ξ_i are the roots of the transcendental equation

$$J_0(r_0\xi_i) = 0 \tag{7.2c}$$

and in (7.2b), the summation is intended over all positive roots of (7.2c).

After introducing the new variables v and G

$$v = w - \frac{r^2 - r_0^2}{4} v_0, \qquad v_0 = \nabla^2 w(r_0, t),$$
$$G = \nabla^2 F - G_0, \qquad G_0 = \nabla^2 F(r_0, t), \tag{7.3}$$

defined to satisfy

$$v(r_0, t) = \nabla^2 v(r_0, t) = G(r_0, t) = 0, \tag{7.4}$$

we return to the system of differential equations defined by the first of (3.2) and (3.5). Then, following the application of Laplace transform (with zero initial conditions) and with an appeal to the correspondence principle, we deduce

$$D(s)\nabla^2\nabla^2 v' + \frac{1}{R}(G' + G_0') = p'(r, s) - \rho h s^2\left[v' + \frac{r^2 - r_0^2}{4} v_0'\right],$$
$$G' = \frac{hE(s)}{R}\left[v' + \frac{r^2 - r_0^2}{4} v_0'\right]. \tag{7.5}$$

From the finite Hankel transform of (7.5), there eventually follows

$$v^*(\xi_i, s) = [\xi_i^4 + \lambda_0^4(s)]^{-1}\left\{\frac{r_0}{\xi_i^3}\lambda_0^4(s)J_1(r_0\xi_i)v_0'(s) - \frac{r_0}{R\,D(s)}\frac{J_1(r_0\xi_i)}{\xi_i}G_0'(s) + \frac{p^*(\xi_i, s)}{D(s)}\right\}, \tag{7.6a}$$

$$G^*(\xi_i, s) = \frac{hE(s)}{R}[\xi_i^4 + \lambda_0^4(s)]^{-1}\left\{-r_0\xi_i J_1(r_0\xi_i)v_0'(s) - \frac{r_0}{R\,D(s)}\frac{J_1(r_0\xi_i)}{\xi_i}G_0'(s) + \frac{p^*(\xi_i, s)}{D(s)}\right\}. \tag{7.6b}$$

The two above equations may be put in the form

$$v^*(\xi_i, s) = v_0' f_1^*(\xi_i, s) - G_0' f_2^*(\xi_i, s) + v_1^*(\xi_i, s), \tag{7.7a}$$

$$G^*(\xi_i, s) = v_0' g_1^*(\xi_i, s) - G_0' g_2^*(\xi_i, s) + G_1^*(\xi_i, s), \tag{7.7b}$$

where the functions f_1^*, f_2^*, g_1^*, g_2^*, v_1^* and G_1^* are defined by direct comparison with (7.6). It may be noted here that, except for v_1^* and G_1^*, all functions involved in (7.7) are polynomials in the Laplace parameter s.

In order to eliminate the functions v_0' and G_0', we take the inverse Hankel transform of (7.7) and obtain

$$v'(r, s) = v_0' f_1'(r, s) - G_0' f_2'(r, s) + v_1'(r, s), \tag{7.8}$$

$$G'(r, s) = v_0' g_1'(r, s) - G_0' g_2'(r, s) + G_1'(r, s), \tag{7.8b}$$

and also by the second of (7.3),

$$r\frac{\partial F'}{\partial r}(r, s) = G_0' \int_0^r \eta[1 - g_2'(\eta, s)]\,d\eta + v_0' \int_0^r \eta g_1'(\eta, s)\,d\eta + \int_0^r \eta G_1'(\eta, s)\,d\eta. \tag{7.8c}$$

In view of the functional form of M_r in (3.3), we next introduce the operator

$$\Lambda \equiv \frac{\partial^2}{\partial r^2} + \frac{\nu}{r}\frac{\partial}{\partial r}, \tag{7.9}$$

and with the aid of (7.3), by virtue of the last two of the boundary conditions (7.1) when applied to (7.8), determine v_0' and G_0' as functions of f_1', f_2', g_1', g_2', v_1', and G_1', i.e.,

$$v_0'\left\{1 + [\Lambda f_2'(r_0, s)][\Lambda f_1'(r_0, s)]^{-1} \int_0^{r_0} \eta g_1' \, d\eta \Big/ \int_0^{r_0} \eta(1 - g_2') \, d\eta\right\} \tag{7.10a}$$

$$= [\Lambda f_1'(r_0, s)]^{-1}\left\{\Lambda v_1'(r_0, s) - \Lambda f_2'(r_0, s) \int_0^{r_0} \eta G_1' \, d\eta \Big/ \int_0^{r_0} \eta(1 - g_2') \, d\eta\right\}$$

and

$$G_0' = -\left[\int_0^{r_0} \eta(1 - g_2') \, d\eta\right]^{-1}\left\{v_0' \int_0^{r_0} \eta g_1' \, d\eta + \int_0^{r_0} \eta G_1' \, d\eta\right\}. \tag{7.10b}$$

Next, substituting (7.10) into (7.8) and taking the inverse Laplace transform followed by inverse Hankel transform, we obtain

$$v(r, t) = \frac{2}{r_0^2} \sum_i \{L^{-1}[v_0' f_1^*(\xi_i, s); t] - L^{-1}[G_0' f_2^*(\xi_i, s); t] + L^{-1}[v_1^*(\xi_i, s); t]\} \frac{J_0(r\xi_i)}{[J_1(r_0\xi_i)]^2}, \tag{7.11}$$

$$G(r, t) = \frac{2}{r_0^2} \sum_i \{L^{-1}[v_0' g_1^*(\xi_i, s); t] - L^{-1}[G_0' g_2^*(\xi_i, s); t] + L^{-1}[G_1^*(\xi_i, s); t]\} \frac{J_0(r\xi_i)}{[J_1(r_0\xi_i)]^2}, \tag{7.12}$$

which, together with (7.3) and (7.10), formally completes the solution sought.

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