PLATE DESIGN FOR MINIMUM WEIGHT*

By

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Summary. A discussion of the equations determining the minimum weight design of transversely loaded sandwich plates is given. By means of an inverse method, the solution for an elliptical plate clamped around its edge is derived.

1. Introduction. Apart from a paper by Prager [1]†, previous work [2-5] on the minimum weight or minimum volume design of plates has been confined to circular plates. In the present work, no assumption of circular symmetry is made. The basic equations are formulated and the four different types of solution are discussed. An inverse method is developed to obtain the minimum volume design for plates with built-in edge conditions. The method is used to obtain the design for an elliptical plate.

2. Basic equations. It is required to design a plate of given shape and with prescribed conditions of simple or built-in support along its edge. The design is to be such that the plate supports by bending stresses a given distribution of pressure and in addition the consumption of material is to be kept to a minimum. Here it is assumed that the plate is a sandwich plate consisting of a core of a given constant thickness H and identical face sheets of (variable) thickness h, where $h \ll H$. The core carries shear force only and a bending moment across a section of the plate is supplied by direct stresses in the face sheets. The material of the face sheets is assumed to be elastic-perfectly plastic and to obey the Tresca yield condition of constant maximum shearing stress during plastic flow. The principal bending moments M_1, M_2 at a point of the plate are then restricted to lie within or on the hexagon of Fig. 1 [6]. The maximum bending moment M_0 is equal to $\sigma_0 Hh$, where σ_0 is the yield stress of the face sheet material. Plastic flow can occur only for states of stress represented by a point on the yield hexagon, and during plastic flow the normality condition applies. This states that the vector with components proportional to the principal plastic curvature rates κ_1, κ_2 is normal to the side of the hexagon for stress points on a side, and lies in the fan bounded by normals to the adjacent sides for stress states represented by a corner of the hexagon.

Because of the normality condition, the rate D of dissipation of energy due to plastic action per unit area of the middle surface of the plate, given by

$$D = M_1 \kappa_1 + M_2 \kappa_2 , \tag{1}$$

is uniquely determined by the curvature rates κ_1, κ_2. It has been shown [3, 7, 8] that minimum volume of the face sheet material will be involved if the plate is designed to be just at the point of collapse under the given loads in a deformation mode such that

$$D/h = \text{constant} \tag{2}$$

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*Received June 23, 1959. The results presented in this paper were obtained in the course of research sponsored by the Office of Ordnance Research, Department of the Army, under Contract No. DA-19-020-ORD-4795.

†Numbers in brackets refer to the list of references at the end of the paper.

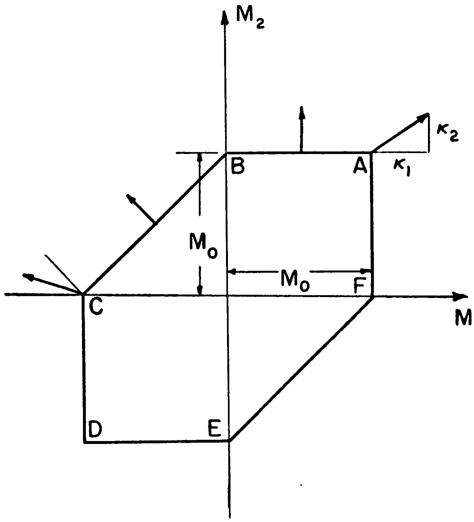

FIG. 1. Yield condition.

over the plate if the weight of the plate is neglected. The result is not restricted to plates in bending but also applies to shells [9] and weight can be taken into account. The term collapse is used here, as in limit analysis [10], to denote conditions under which plastic flow can occur under constant loads, changes in geometry being neglected. It should be noted that the dissipation rate D is directly proportional to the face sheet thickness h so that condition (2) does not involve h, and is a condition on the plastic curvature rates κ_1 , κ_2 only. However the form of condition (2) does depend upon the position of the stress point on the yield hexagon.

During collapse, the elastic strains are constant [10] so that the plastic curvature rates are determined directly from the transverse deflection rate w of the middle surface of the plate by the equations

$$\left.\begin{aligned} \kappa_1 &= \frac{\partial^2 w}{\partial s_1^2} + \frac{1}{\rho_1}\frac{\partial w}{\partial s_2}, \\ \kappa_2 &= \frac{\partial^2 w}{\partial s_2^2} + \frac{1}{\rho_2}\frac{\partial w}{\partial s_1}. \end{aligned}\right\} \tag{3}$$

In these equations, s_1 and s_2 denote distance along the (orthogonal) lines of principal curvature rate which coincide with the lines of principal bending moment. The quantities ρ_1 and ρ_2 denote, with due regard to sign, the radii of curvature of the lines of principal bending moment, Fig. 2, and we have

$$\rho_1^{-1} = -\frac{\partial\varphi}{\partial s_1}, \qquad \rho_2^{-1} = \frac{\partial\varphi}{\partial s_2}, \tag{4}$$

where φ is the angle between the first principal moment direction and the x-axis. The radii ρ_1 , ρ_2 satisfy

$$\frac{\partial}{\partial s_1}\left(\frac{1}{\rho_2}\right) + \frac{\partial}{\partial s_2}\left(\frac{1}{\rho_1}\right) + \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} = 0 \tag{5}$$

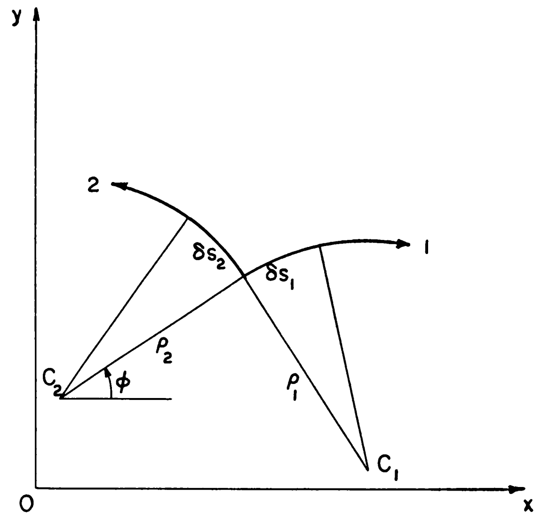

FIG. 2. Principal directions.

and also

$$\frac{\partial^2}{\partial s_1\,\partial s_2} - \frac{1}{\rho_1}\frac{\partial}{\partial s_1} = \frac{\partial^2}{\partial s_2\,\partial s_1} - \frac{1}{\rho_2}\frac{\partial}{\partial s_2}. \tag{6}$$

Because the s_1 , s_2 directions are principal directions the curvature rate κ_{12} is zero, so that

$$\frac{\partial^2 w}{\partial s_1\,\partial s_2} = \frac{1}{\rho_1}\frac{\partial w}{\partial s_1}, \qquad \frac{\partial^2 w}{\partial s_2\,\partial s_1} = \frac{1}{\rho_2}\frac{\partial w}{\partial s_2}. \tag{7}$$

Equations (3)–(7) can be used to show that

$$\frac{\partial \kappa_1}{\partial s_2} = \frac{1}{\rho_1}(\kappa_2 - \kappa_1), \tag{8}$$

$$\frac{\partial \kappa_2}{\partial s_1} = \frac{1}{\rho_2}(\kappa_1 - \kappa_2). \tag{9}$$

As these equations are obtained by eliminating w between Eq. (3), they are the compatibility conditions on κ_1 , κ_2 .

The bending moments M_1 , M_2 and shear forces Q_1 , Q_2 per unit length acting on an element of the plate with faces normal to the principal directions are shown in Fig. 3.

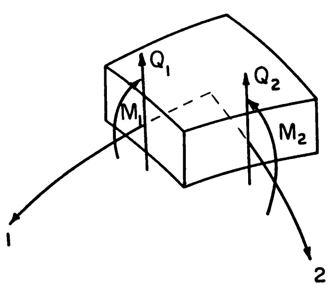

FIG. 3. Positive senses of shear forces and moments.

For definiteness we assume that the plate is horizontal and that the pressure p acts on the upper surface of the plate. Positive values of M_1, M_2 stress the lower face of the plate in tension and the shear forces are positive in the senses shown in the figure. To correspond, the deflection rate w is positive in the upwards direction. For equilibrium

$$\frac{\partial Q_1}{\partial s_1} + \frac{\partial Q_2}{\partial s_2} + \frac{1}{\rho_2} Q_1 + \frac{1}{\rho_1} Q_2 = p, \tag{10}$$

and

$$\left.\begin{aligned} \frac{\partial M_1}{\partial s_1} + \frac{1}{\rho_2}(M_1 - M_2) + Q_1 = 0, \\ \frac{\partial M_2}{\partial s_2} + \frac{1}{\rho_1}(M_2 - M_1) + Q_2 = 0. \end{aligned}\right\} \tag{11}$$

Referred to the rectangular cartesian coordinates (x, y), the equations corresponding to Eqs. (3), (10), (11) are

$$\kappa_x = \frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = \frac{\partial^2 w}{\partial y^2}, \qquad \kappa_{xy} = \frac{\partial^2 w}{\partial x\, \partial y}, \tag{12}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = p, \tag{13}$$

$$\left.\begin{aligned} \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + Q_x = 0, \\ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} + Q_y = 0. \end{aligned}\right\} \tag{14}$$

Equations (13) and (14) can be combined to give

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\, \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p. \tag{15}$$

3. The four types of solution. The condition (2) for minimum volume imposes a restriction on the deformation w. The form of the restriction depends upon the position of the stress point on the yield hexagon of Fig. 1. There are only four essentially different types of plastic flow which can occur. For example plastic flow with moments represented by corner A of Fig. 1 differs in character from plastic states at D by only a change in sign, and stress points on BC become stress points on EF if the suffices 1, 2 are interchanged. As typical of the four types of plastic flow, we shall consider in turn moment states represented by corners A and C and points on the sides AB and BC, not including the end points. For convenience in the algebra, the constant in condition (2) will be taken throughout to be $k\sigma_0 H$, where k is a positive constant. It will be seen that regimes AB and BC lead to restricted forms for the deflection rate and it is unlikely that these regimes will play a large part in the solution of particular problems.

Regime A. For the corner A, $M_1 = M_2 = M_0 = \sigma_0 Hh$ and the curvature rates κ_1 and κ_2 are positive. The condition (2) then requires $\kappa_1 + \kappa_2 = k$, or

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = k. \tag{16}$$

Equilibrium requires

$$\frac{\partial^2 M_0}{\partial x^2} + \frac{\partial^2 M_0}{\partial y^2} = -p. \tag{17}$$

Regime C. For the corner C, M_2 is zero and $M_1 = -M_0$. The curvature rates must satisfy the inequalities

$$-\kappa_1 \geq \kappa_2 \geq 0, \tag{18}$$

and the constancy of D/h requires

$$\kappa_1 = -k. \tag{19}$$

From Eqs. (8) and (19) it follows that

$$\frac{1}{\rho_1}(\kappa_2 + k) = 0. \tag{20}$$

As κ_2 is positive and k is by definition a positive constant, this equation requires $\rho_1 = \infty$ and the lines of principal moment in the s_1-direction are straight. Thus the orthogonal net of principal moment trajectories consists of families of straight lines and parallel curves.

It is convenient at this point to introduce new coordinates, Fig. 4. One of the curved

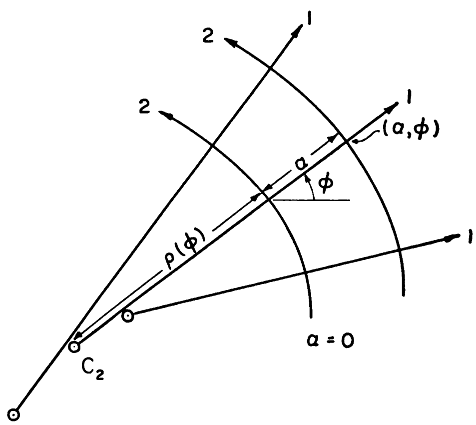

FIG. 4. Coordinate system for regime C.

principal moment lines is chosen as a base curve and distance from this curve along the straight lines is denoted by α. The inclination φ and the distance α are used as coordinates, and we have

$$ds_1 = d\alpha, \qquad ds_2 = \rho_2\, d\varphi, \qquad \rho_2 = \rho + \alpha, \tag{21}$$

where $\rho = \rho(\varphi)$ is the radius of curvature of the base curve $\alpha = 0$. Equation (19) is simply

$$\frac{\partial^2 w}{\partial \alpha^2} = -k \tag{22}$$

and therefore

$$w = -\tfrac{1}{2}k\alpha^2 + \alpha f(\varphi) + g(\varphi). \tag{23}$$

The functions f and g are not completely arbitrary but must be chosen to satisfy Eqs. (7). The final result is

$$w = -\tfrac{1}{2}k\alpha^2 + \alpha f(\varphi) + \int \rho(\varphi)f'(\varphi)\, d\varphi, \tag{24}$$

where the prime denotes differentiation with respect to φ. In addition, inequalities (18) must be satisfied.

It is easily shown that the equilibrium equations and the yield conditions $M_1 = -M_0$, $M_2 = 0$ give

$$Q_1 = \frac{\partial M_0}{\partial \alpha} + \frac{M_0}{\rho_2}, \qquad Q_2 = 0, \tag{25}$$

$$M_0 = C(\varphi) + \frac{D(\varphi)}{\rho_2} + \int \rho_2 p\, d\alpha - \frac{1}{\rho_2}\int \rho_2^2 p\, d\alpha, \tag{26}$$

where C and D are functions of φ only.

In the particular case when both families of principal moment lines are straight, it is convenient to take the x, y-axes in the s_1, s_2-directions. We then have

$$\frac{\partial^2 w}{\partial x^2} = -k, \qquad \frac{\partial^2 w}{\partial x\, \partial y} = 0, \tag{27}$$

so that

$$w = -\tfrac{1}{2}kx^2 + ax + g(y). \tag{28}$$

Here a is a constant and $g(y)$ must be such that

$$k \geq g''(y) \geq 0 \tag{29}$$

in order to satisfy inequalities (18).

Regime AB. For moment states represented by points on the side AB of Fig. 1, $M_2 = M_0$ and $M_0 > M_1 > 0$. The curvature rate κ_1 is zero and condition (2) then requires $\kappa_2 = k$. As both κ_1 and κ_2 are constant, Eqs. (8) and (9) require $\rho_1 = \rho_2 = \infty$ and both families of principal moment lines are straight. If the x, y-axes are taken in the s_1, s_2-directions, we have

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial x\, \partial y} = 0, \qquad \frac{\partial^2 w}{\partial y^2} = k, \tag{30}$$

and therefore

$$w = \tfrac{1}{2}ky^2 + ax + by + c, \tag{31}$$

where a, b, c are constants.

The moment M_{xy} is zero, $M_y = M_0$ and $M_0 > M_x > 0$, and the equilibrium equation (15) requires

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_0}{\partial y^2} = -p. \tag{32}$$

Regime BC. In this case $\kappa_1 = -\kappa_2 = -k$ and again both families of principal moment lines are straight. With the x, y-axes parallel to the s_1, s_2-directions, the deflection rate is given by

$$w = -\tfrac{1}{2}k(x^2 - y^2) + ax + by + c. \tag{33}$$

The moments satisfy

$$M_{xy} = 0, \qquad M_y - M_x = M_0 , \qquad M_0 > M_y > 0, \tag{34}$$

and for equilibrium

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} = -p. \tag{35}$$

4. Simply supported plate. When the plate is simply supported at its edge, the deflection rate w and the normal bending moment M_n must be zero at the boundary of the plate. For a circular plate, the minimum volume design is obtained by the use of regime A only. This is also true for a certain class of shapes. If the plate shape is such that the deflection rate w determined from Eq. (16) and the condition $w = 0$ at the edge gives non-negative curvature rates κ_1 and κ_2, then the deflection rate satisfying $D/h =$ constant has been found. The thickness distribution for any given distribution of pressure is obtained by solving Eq. (17) for M_0 subject to $M_0 = 0$ at the edge.

For example, for the plate bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{36}$$

the solution of (16) satisfying $w = 0$ at the edge is

$$w = \frac{ka^2b^2}{2(a^2 + b^2)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right), \tag{37}$$

and κ_1, κ_2 are non-negative. For *constant* pressure over the plate the thickness distribution is given by

$$\sigma_0 Hh = M_0 = \frac{pa^2b^2}{2(a^2 + b^2)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{38}$$

The case of the simply supported plate has been discussed previously by Prager [1].

5. Built-in plate. For the plate with a built-in or clamped edge, the deflection rate w and its normal derivative $\partial w/\partial n$ must be zero at the edge. The edge is therefore a line of (zero) principal curvature rate. The solution for the circular plate involves regime A in a central circular region and regime C in the remaining annular region [5]. Here an inverse method is developed for obtaining solutions similar in character to that for the circular plate.

It is assumed that in an inner region, which is bounded by the curve γ, regime A applies, and in the region between γ and the edge Γ of the plate, regime C applies, see Fig. 5. The curve γ is as yet undetermined. The lines of principal moment in the region between γ and Γ are straight lines normal to Γ and curves parallel to Γ, since the boundary is a line of principal moment. The trajectories are indicated by the dotted lines in Fig. 5. The coordinate system of Fig. 4 will be used, with the base curve $\alpha = 0$ taken to be

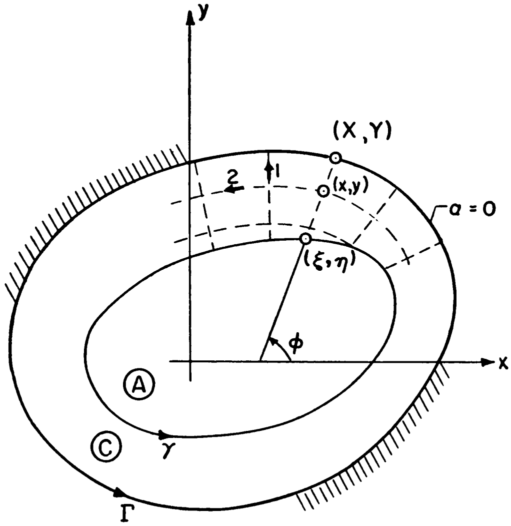

FIG. 5. Plate with built-in edge.

the edge Γ of the plane. The direction of α increasing is taken in the direction of the outward normal to Γ, so that α is negative in the region between γ and Γ.

The deflection rate w_C in the annular region has the form (23) and in order to have $w = 0$ and $\partial w/\partial\alpha = 0$ on Γ (where $\alpha = 0$) we must have

$$w_C = -\tfrac{1}{2}k\alpha^2. \tag{39}$$

The curve γ and the deflection rate w_A inside γ are to be determined from Eq. (16) which holds inside γ and the continuity of w and $\partial w/\partial n$ across γ. Assuming that the solution exists (with non-negative κ_1 and κ_2 inside γ), the determination of γ may prove to be difficult, and for this reason an inverse method of solution is developed here.

From Eq. (39), we have

$$\frac{\partial w_C}{\partial \alpha} = -k\alpha = (-2kw_C)^{1/2}. \tag{40}$$

Also since w_C depends on α only, $\partial w_C/\partial\alpha$ is equal to the magnitude of the gradient of w_C. It follows that the deflection rate (39) is such that

$$2kw_C + \left(\frac{\partial w_C}{\partial x}\right)^2 + \left(\frac{\partial w_C}{\partial y}\right)^2 = 0. \tag{41}$$

This equation holds in the region between γ and Γ and in particular Eq. (41) holds on the curve γ. Thus the deflection rate w_A satisfying Eq. (16) inside γ must satisfy (41) on γ because of the continuity of w and its first derivatives across γ, that is

$$2kw_A + \left(\frac{\partial w_A}{\partial x}\right)^2 + \left(\frac{\partial w_A}{\partial y}\right)^2 = 0 \quad \text{on} \quad \gamma. \tag{42}$$

The inverse method is therefore to choose a function w_A which satisfies Eq. (16). The curve γ is then determined as the locus of points satisfying Eqs. (42). In order to determine the curve Γ which defines the boundary of the plate, we again use the continuity of w and its first derivatives. From (39), the (variable) value of α on γ is determined by

$$-\alpha_\gamma = \left(-\frac{2w_A}{k}\right)_\gamma^{1/2}, \tag{43}$$

where the suffix γ means that the value on γ is to be taken. Also in the region between γ and Γ, the inclination φ of the straight principal moment trajectories is given by

$$\tan \varphi = \frac{\partial w_C/\partial y}{\partial w_C/\partial x}, \tag{44}$$

since these lines are normal to the curves w = constant. Thus the value of φ on γ can be determined from the values of $\partial w_A/\partial x$ and $\partial w_A/\partial y$ on γ,

$$\tan \varphi_\gamma = \left(\frac{\partial w_A/\partial y}{\partial w_A/\partial x}\right)_\gamma. \tag{45}$$

Knowing the values of α and φ on γ, the curve Γ can be found. The coordinates of a point on γ will be denoted by (ξ, η) and the point on Γ obtained by dropping a perpendicular from (ξ, η) to Γ will be denoted by (X, Y). The points (ξ, η) and (X, Y) then are distant $-\alpha_\gamma$ apart and the line joining them is a principal moment line. It follows that

$$\left.\begin{aligned} X &= \xi - \alpha_\gamma \cos \varphi_\gamma, \\ Y &= \eta - \alpha_\gamma \sin \varphi_\gamma. \end{aligned}\right\} \tag{46}$$

With Eqs. (42), (43) and (45), Eqs. (46) can be written as

$$\begin{aligned} X &= \xi + \frac{1}{k}\left(\frac{\partial w_A}{\partial x}\right)_\gamma, \\ Y &= \eta + \frac{1}{k}\left(\frac{\partial w_A}{\partial y}\right)_\gamma, \end{aligned} \tag{47}$$

and these equations are the parametric equations of the curve Γ, remembering that the point (ξ, η) lies on γ.

It should be remembered that κ_1 and κ_2 must be non-negative inside γ and the inequalities (18) must be satisfied in the annular region. With w given by Eq. (39),

$$\kappa_1 = -k, \qquad \kappa_2 = -\frac{k\alpha}{\rho_2}, \tag{48}$$

and since α is negative the inequalities (18) will be satisfied provided that between γ and Γ,

$$-\frac{\alpha}{\rho_2} \leq 1. \tag{49}$$

It is only necessary to check this inequality on γ as the left hand side attains its maximum on γ. Remembering the definition of ρ_2 [Eq. (21)], inequality (49) can be interpreted as requiring that the radius of curvature of Γ at (X, Y) be not less than twice the distance $-\alpha_\gamma$ between the points (ξ, η) on γ, (X, Y) on Γ.

The thickness distribution, or equivalently the values of M_0, throughout the plate can be determined for any given distribution of pressure when the curves γ and Γ are known. At the junction γ of the regimes A and C, equilibrium requires the bending moments M_n, M_{nt} to be continuous and the shear force Q_n must be continuous. In order to make the bending moments continuous, M_0 must be zero on γ. This condition and Eq. (17) then determine M_0 in the region bounded by γ. The value of the shear force Q_n on γ can then be found. The values of M_0 in the annular region are then given by Eq. (26) where the functions $C(\varphi)$ and $D(\varphi)$ are determined by the condition $M_0 = 0$ on γ and the known value of the shear force on γ.

We remark that if the pressure distribution $p(x, y)$ is non-zero at points inside γ, then the thickness h is non-zero everywhere in the plate except for points on the curve γ. However, if p is identically zero inside γ, then h is zero in this region, as h satisfies Laplace's equation inside γ and is zero on γ. In effect, the load produced by pressure at a point in the annular region is carried by a "cantilever beam" from the point to the nearest point of the edge of the plate. Loads in the inner region are carried by the whole boundary of the plate. This property of the minimum volume design for a built-in plate has been discussed previously for the case of the circular plate [5].

6. Elliptical plate with built-in edge support. If it is assumed that inside γ the deflection rate w is given by

$$w_A = -\tfrac{1}{2}\frac{kr^2q^2}{(r^2+q^2)}\left\{1 - \frac{x^2}{r^2} - \frac{y^2}{q^2}\right\}, \tag{50}$$

where r and q are constants, the solution for the elliptical plate is obtained The deformation rate (50) satisfies Eq. (16) and κ_1, κ_2 are positive. The curve γ is determined by substituting (50) into Eq. (42) and the result is the ellipse

$$\gamma:\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{51}$$

where the semi-axes a and b are given by

$$a^2 = \frac{r^2(r^2+q^2)}{r^2+2q^2}, \qquad b^2 = \frac{q^2(r^2+q^2)}{2r^2+q^2}. \tag{52}$$

The values of α and φ on γ can be obtained from (50) and (43), (45). The substitution of (50) into Eqs. (47) gives the coordinates (X, Y) of a point on Γ in terms of the coordinates (ξ, η) of the point of intersection of the curve γ with the normal to Γ at (X, Y),

$$X = \frac{\xi(r^2+2q^2)}{(r^2+q^2)}, \qquad Y = \frac{\eta(2r^2+q^2)}{(r^2+q^2)}. \tag{53}$$

As the coordinates (ξ, η) satisfy (51), the equation for the boundary Γ of the plate can be written

$$\Gamma:\ \frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1, \tag{54}$$

where

$$A^2 = \frac{r^2(r^2+2q^2)}{r^2+q^2}, \qquad B^2 = \frac{q^2(2r^2+q^2)}{r^2+q^2}. \tag{55}$$

Equations (55) can be solved for r^2 and q^2 to give

$$3r^2 = 2A^2 - B^2 + (A^4 - A^2B^2 + B^4)^{1/2}, \quad 3q^2 = 2B^2 - A^2 + (A^4 - A^2B^2 + B^4)^{1/2}. \tag{56}$$

The deflection rate w_C at the point (x, y) in the annular region is given by

$$w_C = -\tfrac{1}{2}k\alpha^2 = -\tfrac{1}{2}k[(X - x)^2 + (Y - y)^2], \tag{57}$$

where the normal to Γ at (X, Y) passes through (x, y), Fig. 5. Because the line joining (x, y) to (X, Y) is normal to Γ,

$$\frac{Y - y}{X - x} = \frac{YA^2}{XB^2}. \tag{58}$$

For given (x, y), X and Y can be determined from Eqs. (54) and (58) and substitution in (57) then yields the value of w_C at (x, y). It can be shown that inequality (49) is satisfied in the annular region, verifying that regime C applies in the annular region.

In Fig. 6 the values of a/A and b/A are plotted against B/A for the range $0 \leq B/A \leq 1$.

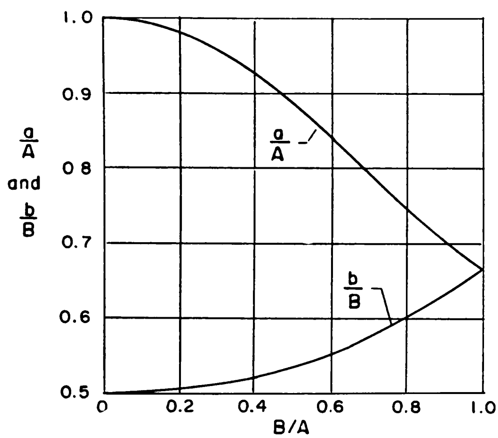

FIG. 6. Semi-axes a and b of curve γ for elliptical plate with semi-axes A and B.

When the plate is circular ($A = B$), the curve γ is a circle of radius two-thirds of the radius of the plate, as found previously [5]. For a plate which is long compared to its width ($B/A \to 0$), the ratio a/A tends to unity and b/B tends to one-half. This can be compared with the minimum volume design for a sandwich beam, built-in at the ends, for which the sections of zero thickness, which correspond to the curve γ, are distant one-fourth the length of the beam from each end.

Thickness distributions for specific pressure distributions may now be obtained as described in the previous section, and for illustration we consider the case of constant pressure p. The notation M_{0A} and M_{0C} will be used to denote the value of $M_0 = \sigma_0 Hh$ in the inner and annular regions respectively. As M_0 is zero on γ and M_0 satisfies Eq. (17) inside γ it follows that the value of M_0 at points inside γ is given by

$$M_{0A} = \tfrac{1}{2}p\,\frac{a^2b^2}{(a^2 + b^2)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{59}$$

In the annular region, M_0 is given by Eq. (26) and as p is constant and $M_0 = 0$ on γ it follows that

$$M_{0C} = \frac{1}{6} p(\rho_2^2 - \rho_3^2) + D(\varphi)\left(\frac{1}{\rho_2} - \frac{1}{\rho_3}\right), \tag{60}$$

where $\rho_3(\varphi)$ is the value of ρ_2 on γ. The value of $D(\varphi)$ is to be determined by the condition that the shear force Q_n be continuous across γ. It is convenient in the calculations to introduce an auxiliary variable θ defined by

$$X = A \cos \theta, \qquad Y = B \sin \theta. \tag{61}$$

Substitution of these values into Eqs. (53) shows that the coordinates (ξ, η) are given by

$$\xi = a \cos \theta, \qquad \eta = b \sin \theta. \tag{62}$$

The inclination φ of the normal at (X, Y) to Γ and the inclination θ_n of the normal at (ξ, η) to γ are given by

$$\tan \varphi = \frac{A}{B} \tan \theta, \qquad \tan \theta_n = \frac{a}{b} \tan \theta. \tag{63}$$

The radius of curvature $\rho(\varphi)$ of the curve Γ is

$$\rho = A^2B^2/(A^2 \cos^2 \varphi + B^2 \sin^2 \varphi)^{3/2}. \tag{64}$$

The value of ρ_3 is $\rho + \alpha_\gamma$ and here

$$\alpha_\gamma = -A(A - a)/(A^2 \cos^2 \varphi + B^2 \sin^2 \varphi)^{1/2}. \tag{65}$$

In the inner region, $M_x = M_y = M_{0A}$ and M_{0A} is zero on γ so that

$$Q_n = -\frac{\partial M_{0A}}{\partial n}. \tag{66}$$

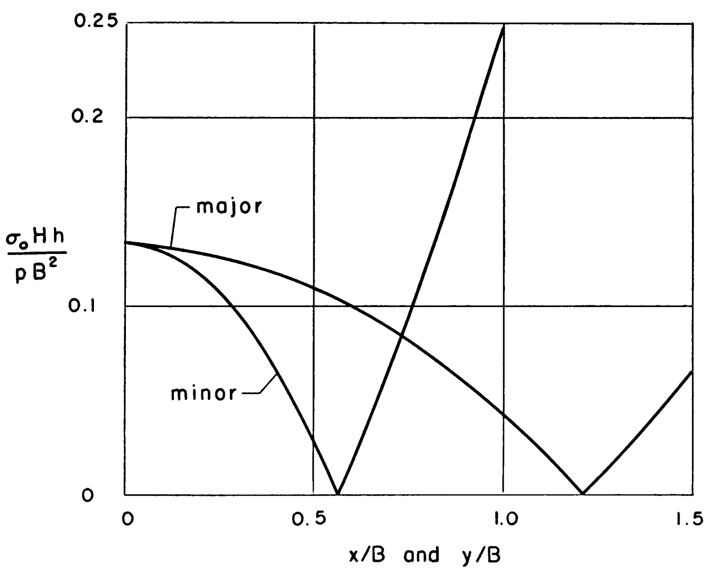

FIG. 7. Variation of thickness across major and minor semi-axes for ellipse under constant pressure ($A = 1.5B$).

In the annular region, $Q_2 = 0$ and Q_1 is given by Eq. (25). The value of Q_n is therefore

$$Q_n = Q_1 \cos(\theta_n - \varphi) = \frac{\partial M_{0C}}{\partial \alpha} \cos(\theta_n - \varphi), \tag{67}$$

evaluated on γ. Equating the two values (66) and (67) of Q_n leads after some reduction to

$$\tfrac{1}{2} p \rho_3 - \frac{D(\varphi)}{\rho_3^2} = p \frac{ab}{a^2 + b^2} \frac{(A^2 b^2 \cos^2 \varphi + B^2 a^2 \sin^2 \varphi)^{1/2}}{(A^2 \cos^2 \varphi + B^2 \sin^2 \varphi)^{1/2}(Ab \cos^2 \varphi + Ba \sin^2 \varphi)}. \tag{68}$$

This equation determines the function $D(\varphi)$ and substitution in Eq. (60) gives the value of M_{0C} in terms of the coordinates (α, φ).

Figure 7 shows the variation of the thickness h along the major and minor axes of the ellipse for the particular case $A = 1.5B$.

The volume of the face sheets in the minimum volume design can be obtained without determining the design explicitly. The plate is at collapse under a mode w such that the rate of dissipation D per unit area of the plate has the value $D = k\sigma_0 H h$. During collapse in the mode w, the pressure p is the only external force which does work and it follows that

$$-\int pw \, dA = \int D \, dA = \tfrac{1}{2} k \sigma_0 H V, \tag{69}$$

where the integrals are taken over the area of the plate and where V is the volume of the face sheets. For a circular plate of radius B under constant pressure, the volume of the face sheets is given by [4]

$$V = 0.117 \frac{\pi p B^4}{\sigma_0 H}. \tag{70}$$

Equation (69) was used to compare the average thicknesses of the face sheets of an elliptical plate of semi-axes A and B and a circular plate of radius B, both plates

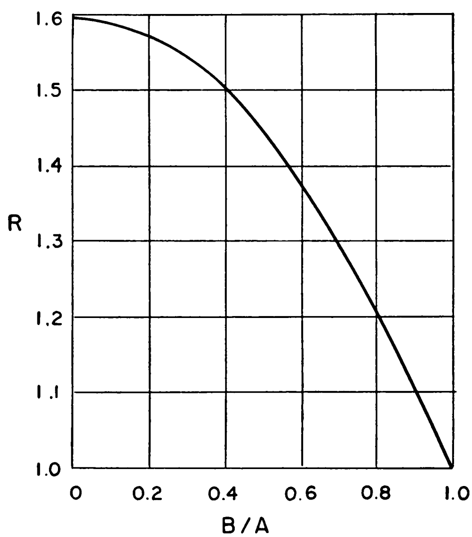

FIG. 8. Ratio of average thicknesses of constant pressure designs for elliptical plate with semi-axes A and B and circular plate of radius B.

being subjected to the same constant pressure p. The ratio R of the average thicknesses is shown in Fig. 8 for the range $0 \leq B/A \leq 1$. The volume of the face sheets for the elliptical plate is given by

$$V = 0.117 \frac{pR}{\sigma_0 H} \pi A B^3. \tag{71}$$

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