THE KRON METHOD OF TEARING AND THE DUAL METHOD OF IDENTIFICATION*

BY

A. I. WEINZWEIG

University of California, Berkeley

1. Introduction. Over the past several years, Gabriel Kron has published a series of papers expounding his method of solving network problems by tearing the network into smaller components, solving the problem on each component, then interconnecting the solutions to obtain a solution to the original problem [3, 4, 5, 6, 7]. We wish to present a precise mathematical formulation of this procedure. This not only establishes the validity of the method but simplifies and extends it, and moreover, leads to a dual method we call the method of identification.

We first formulate a general network problem and establish a necessary and sufficient condition for the existence of a unique solution. This has independent interest for it simplifies and extends the Kron-LeCorbeiller mixed method of solving network problems [2, 8]. Following Weyl and Eckmann [1] an electrical network is considered as a 1-dimensional cell complex and the problem formulated in terms of the chains and cochains of this complex. The solution of the problem is essentially effected by inverting a certain matrix, the matrix of the solution. The method of tearing (identification) transfers the problem to a second network obtained by tearing (making identifications in) the original network. There the solution matrix is inverted by inverting two matrices, the component matrix and the connection matrix ("interconnecting the solutions"). Although the rank of the component matrix is greater than that of the solution matrix, it is strongly diagonal and can be inverted by inverting each of the diagonal submatrices ("solving the problem on each component"). This is actually a special application of a more general procedure developed in Sec. 8 whereby the solution matrix is inverted by inverting two other matrices, the first of rank greater and the second of rank less than the solution matrix. If the inverse of the first is known or for some reason more easily computable (as in the case of tearing and identification) then this leads to a simpler solution. This also furthers Kron's goal of "storing solutions".

For a detailed discussion of the history of the network problem the reader is referred to Roth [10, 12]. For an evaluation of Kron's method of tearing we again refer to Roth [12] in addition to the papers of Kron.

2. The network equations. An electrical network K can be considered as a 1-dimensional cell complex. K is assumed to have lumped design constants with no impedanceless or admittanceless branches. The k-dimensional chains of K with coefficients in the field of complex numbers $C_k(K)$ is then a vector space and the k-cochains $C^k(K)$ with the same coefficients is the dual space. The boundary operator ∂ is a linear transformation with dual the coboundary operator δ. Orienting K, the positively oriented k-cells can be regarded both as elements of $C_k(K)$ and $C^k(K)$ and as such define dual primitive bases.

*Received September 29, 1958; revised manuscript received October 9, 1959.

The current flowing in each branch in the direction of orientation defines a 1-chain i of K. In the same way, the *emf* in each branch, the voltage drop across the passive coil in each branch and the potential difference across each branch define 1-cochains e V and E respectively. To complete the picture, the current flowing out of each node or vertex and the potential of each vertex define respectively a 0-chain I and a 0-cochain P. These quantities are all illustrated on a representative branch ab oriented from a to b (Fig. 1). Clearly $V = E + e$.

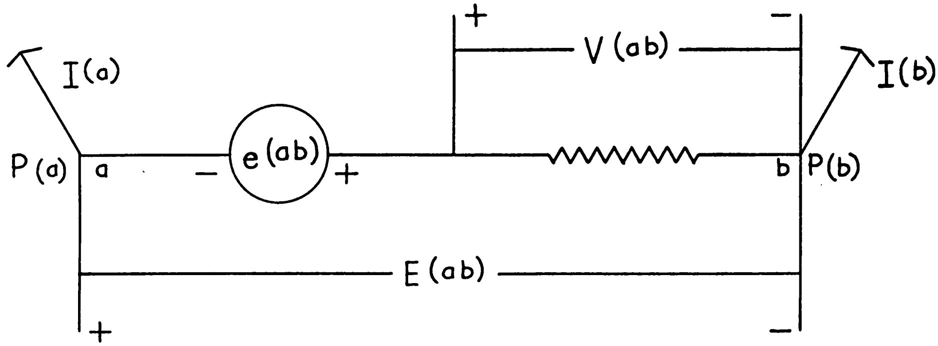

FIG. 1.

The impedance matrix [8, p. 2] $Z = [Z_{i,j}]$ and the admittance matrix $Y = [Y^{i,j}]$ represent, in terms of the primitive bases, linear transformations $\zeta: C_1(K) \to C^1(K)$ and $\eta: C^1(K) \to C_1(K)$ respectively, where under the usual conditions $\zeta^{-1} = \eta$. Kirkhoff's current and voltage laws are expressed by $\partial i = I$, $\delta P = E$ and Ohm's law by $\zeta(i) = V$ or $\eta(V) = i$. These together with $V = E + e$ constitute the *network equations*.

3. Canonical decompositions. Let A_1 be any subspace of the vector space A. There is a (not unique) complement A_2 of A_1 in A such that $A = A_1 \dotplus A_2$ is a direct sum decomposition. This decomposition induces the *dual decomposition* of the dual space $A^* = A_1^* \dotplus A_2^*$, where A_2^* (A_1^*), the annihilator in A^* of $A_1(A_2)$ may be regarded as the dual space of $A_2(A_1)$. We denote by $\sigma(A_1)$: $A_1 \to A$ the inclusion of A_1 in A and $\pi(A_1)$: $A \to A_1$ the projection of A onto A_1. Then $\sigma(A_1)^*$: $A^* \to A_1^*$ is the projection of A^* onto A_1^*, that is $\sigma(A_1)^* = \pi(A_1^*)$.

The linear transformation ξ: $A \to A^*$ is said to be *inherently non-singular* if $\pi(A_1^*)\xi\sigma(A_1)$ is non-singular for every subspace A_1 of A. This implies that ξ is itself non-singular and that ξ^{-1} is also inherently non-singular where we identify A and A^{**} under the usual canonical isomorphism.

Let $Z_1(K)$ and $B_0(K)$ be the null space and image space of ∂. Any decompositions $C_1(K) = Z_1(K) \dotplus R_1(K)$ or $C_0(K) = D_0(K) \dotplus B_0(K)$ are said to be *canonical*. In this case, the restriction of ∂ to $R_1(K)$ defines an isomorphism of $R_1(K)$ onto $B_0(K)$. Observe that $Z_1(K)$ and $B_0(K)$ are canonical. If $C^1(K) = D^1(K) \dotplus B^1(K)$ and $C^0(K) = Z^0(K) \dotplus R^0(K)$ are the corresponding dual decompositions then $Z^0(K)$ and $B^1(K)$ are precisely the null space and image space of δ.

4. The network problem. The data for a general network problem consists of (a) a network K; (b) decompositions $C_1(K) = C_1 \dotplus C_2$, $C_0(K) = D_1 \dotplus D_2$ consistent with ∂, that is $\partial C_i \subset D_i$, $i = 1, 2$; (c) $i_2 \in C_2$, $e_1 \in C_1^*$, $I_1 \in D_1$, $P_2 \in D_2^*$ where $C^1(K) = C_1^* \dotplus C_2^*$, $C^0(K) = D_1^* \dotplus D_2^*$ are dual to those given above; (d) a linear transformation

ζ: $C_1(K) \to C^1(K)$ (the impedance form) or η: $C^1(K) \to C_1(K)$ (the admittance form). A solution to the network problem consists of i, I, E, e, P satisfying the network equations $\partial i = I$, $\delta P = E$, $\eta(E + e) = i$ or $\zeta(i) = E + e$, such that $\pi(C_2)i = i_2$, $\pi(C_1^*)e = e_1$, $\pi(D_1)I = I_2$, $\pi(D_2^*)P = P_2$.

Observe that ζ or η are dependent only on the design constants of the network and not on the network itself. Thus, ζ or η will be the same for any network made up of the same coils.

Theorem. A necessary and sufficient condition for the existence of a unique solution to any network problem is that ζ and hence $\eta = \zeta^{-1}$ be inherently non-singular.

Proof. It follows from the consistency of the given decompositions with ∂ that we can find decompositions $C_1(K) = Z_{11} \dotplus R_{11} \dotplus Z_{12} \dotplus R_{12}$, $C_0(K) = D_{01} \dotplus B_{01} \dotplus D_{02} \dotplus B_{02}$ such that $C_i = Z_{1i} \dotplus R_{1i}$, $D_i = D_{0i} \dotplus B_{0i}$, $i = 1, 2$ and $C_1(K) = (Z_{11} \dotplus Z_{12}) \dotplus (R_{11} \dotplus R_{12})$, $C_0(K) = (D_{01} \dotplus D_{02}) \dotplus (B_{01} \dotplus B_{02})$ are canonical decompositions. Let $C^1(K) = D^{11} \dotplus B^{11} \dotplus D^{12} \dotplus B^{12}$ and $C^0(K) = Z^{01} \dotplus R^{01} \dotplus Z^{02} \dotplus R^{02}$ be the corresponding dual decompositions. In terms of these decompositions the pertinent chains and cochains have representations as $i = j_1 + J_1 + j_2 + J_2$, $I = I_1 + I_2$, $P = P_1 + P_2$, $E = E_1 + E_2$ and $e = e_{11} + e_{12} + e_{21} + e_{22}$, where $i_2 = j_2 + J_2$, $I_j \in B_{0j}$, $E_j \in B^{1j}$, $P_j \in D_j$, $j = 1, 2$, $e_1 = e_{11} + e_{12}$ and $E_2 = \delta P_2$. Then $I_2 = \partial i_2$ and since the restriction of ∂ to $R_{11} \dotplus R_{12} = R_1(K)$ is an isomorphism of $R_1(K)$ onto $B_0(K) = B_{01} \dotplus B_{02}$, $J_1 = \partial^{-1}I - J_2$.

Setting $A_1 = Z_{11}$, $A_2 = R_{11} \dotplus Z_{12} \dotplus R_{12}$ yields a decomposition $C_1(K) = A_1 \dotplus A_2$ wherein the component of i in A_2, namely $i_2 + J_1$ and the component e_{11} of V in $A_1^* = D^{11}$ are known. This is called the *solution decomposition*. The network problem is solved once either i or V is known for then $\zeta(i) = V$ or $V = \eta(i)$ gives the other and $e = \pi(C_2^*)V - E_2 + e_1$, $E = E_2 + \pi(C_1^*)V - e_1$, $P = \delta^{-1}(E)$ and $I = \partial i$ is a solution. Hence the problem reduces to determining j_1 or $V_2 = V - e_{11}$. In the admittance form $V_2 = [\pi(A_2)\eta\sigma(A_2^*)]^{-1} (J_1 + i_2 - \pi(A_2)\eta(e_{11}))$ and in the admittance form $j_1 = [\pi(A_1^*)\zeta\sigma(A_1)]^{-1} (e_{11} - \pi(A_1)\zeta(i_2 + J_1))$. The transformations $\eta(A_2) = \pi(A_2)\eta\sigma(A_2^*)$, $\zeta(A_1) = \pi(A_1^*)\zeta\sigma(A_1)$, the *solution transformations* are non-singular since ζ and hence η are inherently non-singular.

The uniqueness follows readily, for any other solution i', E', I', P', e' can be represented in terms of the decompositions above as $i' = j_1' + J_1' + j_2 + J_2$, $e' = e_{11} + e_{12} + e_{21}' + e_{22}'$, $E = E_1' + E_2'$, $P = P_1' + P_2$, $I = I_1 + I_2'$ whence $I_2' = \partial J_2 = I_2$, $J_1' = \partial^{-1}(I) - J_2 = J_1$, etc. We remark that P is uniquely determined only to within a constant value on each connected component of K. Knowing the potential of one vertex of each component, say it is grounded, P can be uniquely determined.

To establish the necessity of the condition, let C_1 be any subspace of $C_1(K)$ and $i_1 \in C_1$ such that $\pi(C_1^*)\zeta\sigma(C_1)i_1 = 0$. Passing to a new network K' if necessary, C_1 can be taken as a subspace of $Z_1(K)$. Such a network can always be obtained by identifying vertices in K; (if all the vertices are identified, then $Z_1(K^1) = C_1(K'))$. Then let $R_1(K)$ be a complement of $Z_1(K)$ in $C_1(K)$ and Z_{11} a complement of C_1 in $Z_1(K)$. Set $C_2 = Z_{11} \dotplus R_1(K)$. Then $C_1(K) = C_1 \dotplus C_2$, $C_0(K) = D_0(K) \dotplus B_0(K)$ are decompositions consistent with ∂ and together with $i_2 = 0$, $e_1 = 0$, $I_1 = 0$, $P_2 = 0$ and ζ, define a network problem with solution $i = 0$, $e = E = 0$, $P = 0$, $I = 0$. By the unicity of the solution it follows that $i_1 = 0$ so that $\pi(C_1^*)\zeta\sigma(C_1)$ is non-singular and ζ is inherently non-singular.

The solution outlined in the proof above is actually a generalization and simplification of the Kron-LeCorbeiller orthogonal or mixed method [2, 8]. Roth [10] considered a special case of the network problem, where the solution decomposition is canonical and established a necessary condition for the existence of a unique solution. In a subsequent paper [12] he established a necessary and sufficient condition, namely that $\zeta(i)\, i = 0$ only if $i = 0$. If ζ satisfies this condition then ζ is said to be *ohmic*. Actually, ζ is ohmic if and only if ζ is inherently non-singular. Let C_1 be any subspace of($C_1 K$) and $i \in C_1$. Then $\sigma(C_1)^*\zeta\sigma(C_1)i = 0$ only if $\zeta(i)i = 0$ for the annihilator of C_1 can be taken as the complement of C_1^* in $C^1(K)$. Hence if ζ is ohmic, $i = 0$ and ζ is inherently non-singular conversely, let ζ be inherently non-singular and $0 \neq i \in C_1(K)$. Then taking C_1 as the subspace of $C_1(K)$ generated by i, it follows from the fact that $\sigma(C_1)^*\zeta\sigma(C_1)i \neq 0$ that $\zeta(i)i \neq 0$ and ζ is ohmic.

5. A more general formulation. The network problem can be formulated more generally. Let X and Y be two vector spaces, $\xi: X \to Y$ a linear transformation. Then the roles of $C_1(K)$, $C_0(K)$ and ∂ in the network problem as formulated above, are replaced by X, Y and ξ respectively. Thus the data for a network problem consists of spaces X, Y; linear transformations $\xi: X \to Y$, $\lambda: X \to X^*$; decompositions $X = X_1 \dotplus X_2$, $Y = Y_1 \dotplus Y_2$ consistent with ξ; and $x_1^* \in X_1^*$, $x_2 \in X_2$, $y_1 \in Y_1$, $y_2^* \in Y_2^*$, where $X^* = X_1^* \dotplus X_2^*$, $Y^* = Y_1^* \dotplus Y_2^*$ are the corresponding dual decompositions. A solution then consists of $x \in X$, $x^* \in X^*$, $y \in Y$, $y^* \in Y^*$ such that $\xi x = y$, $\lambda x = x^* + \xi^* y^*$ and the projections of x, x^*, y, y^* in X_1^* , X_2 , Y_1 , Y_2^* are x_1^* , x_2 , y_1 , y_2^* respectively. Thus we have a problem in linear algebra.

6. Matrix formulation. Choosing bases in $C_1(K)$, $C^1(K)$, $C_0(K)$, $C^0(K)$, the transformations η, ζ, ∂, δ can then be represented by matrices $[\eta]$, $[\zeta]$, $[\partial]$, $[\delta]$, the chains i, I and the cochains e, E, P by column matrices $[i]$, $[I]$, $[e]$, $[E]$, $[P]$ respectively, where $[\eta]^{-1} = [\zeta]$, since ζ is assumed inherently non-singular. The network equations then become the matrix equations $[e] + [E] = [\zeta][i]$, $[i] = [\eta]([e] + [E])$, $[\partial][i] = [I]$, $[\delta][P] = [E]$. When the bases in $C_1(K)$, $C^1(K)$ and $C_0(K)$, $C^0(K)$ are dual then $[\partial]^* = [\delta]$. If the basis for $C_1(K)$ contains a basis for $Z_1(K)$, the remaining elements span a complement $R_1(K)$ of $Z_1(K)$ so that $C_1(K) = Z_1(K) + R_1(K)$ is a canonical decomposition. The basis is then called a *canonical basis* for $C_1(K)$. Similarly, we define a canonical basis for $C_0(K)$. The isomorphism $\partial \mid R_1(K)$ of $R_1(K)$ and $B_0(K)$ can be represented by a non-singular matrix $[\partial]'$. In this case $[\partial]$ has the form

$$[\partial] = \begin{bmatrix} [0] & [\partial]' \\ [0] & [0] \end{bmatrix},$$

where [0] indicates the appropriate zero matrix. Finally, if the canonical basis for $C_0(K)$ is such that $B_0(K)$ is spanned by the boundaries of those elements of the canonical bases for $C_1(K)$ which span $R_1(K)$, then $[\partial]'$ is just the unit matrix. We say the canonical bases are *compatible*. With compatible canonical bases,

$$[i] = \begin{bmatrix} [j] \\ [J] \end{bmatrix}, \qquad [I] = \begin{bmatrix} [I]_1 \\ 0 \end{bmatrix}, \qquad [E] = \begin{bmatrix} 0 \\ [E]_2 \end{bmatrix}, \qquad [P] = \begin{bmatrix} [P]_1 \\ [P]_2 \end{bmatrix},$$

where $[J] = [I]_1$, $[E]_2 = [P]_2$. In this way, all the information is transferred directly to $C_1(K)$ and $C^1(K)$ which leads to the solution decomposition $C_1(K) = A_1 \dotplus A_2$.

In practice the data for a network problem is given with respect to some bases. The design constants determine both the admittance and impedance matrix which are the matrix representations of η and ζ with respect to dual primitive bases. Let $< b_1, \cdots, b_N >$ denote the primitive basis in terms of which (and its dual) $[\eta] = Y$ and let $< b_1', \cdots, b_N' >$ be the canonical basis which effects the solution decomposition $C_1(K) = A_1 \dot{+} A_2$, that is $\{b_1, \cdots, b_r\}$ spans A_1 and $\{b_{r+1}, \cdots, b_N\}$ spans A_2. The change from coordinates with respect to the primitive basis to coordinates with respect to the canonical basis is given by a non-singular matrix $T = [t_{ij}]$, $i, j = 1, \cdots, N$. Thus if $[i]$ and $[i]'$ are the matrix representations of i in terms of the primitive and canonical bases respectively, then $[i]' = T[i]$. Similarly T^*, the transpose of T represents the change in coordinates in $C^1(K)$ from the dual canonical basis to the dual primitive basis. Let T_1 be the submatrix of T defined by $[t_{ij}]$, $i = 1, \cdots, N$, $j = r + 1, \cdots, N$. T_1 can be regarded as the matrix representation of the projection $\pi(A_2)$ in terms of the primitive basis for $C_1(K)$ and the basis $< b_{r+1}', \cdots, b_N' >$ for A_2. Similarly T_1^* represents the inclusion $\sigma(A_2^*) = \pi(A_2)^*$ in terms of the dual bases. The matrix representation for the solution transformation $\eta(A_2)$ in terms of the basis $< b_{r+1}', \cdots, b_N' >$ for A_2 and the dual basis for A_2^* is then $T_1 Y T_1^*$. This then is the matrix to be inverted in the solution of the network problem.

The computations can often be simplified by the judicious choice of bases leading to simpler matrix representations. This has already been indicated above. In particular, the 1-simplices of a maximal tree in K generate a complement $R_1(K)$ of $Z_1(K)$. Each of the remaining simplices gives rise to a simple circuit and the simple circuits so formed are a basis for $Z_1(K)$. The basis for $C_1(K)$ so constructed is called a *simple canonical* basis. The coordinate transformation T from a primitive to a simple canonical basis has the form

$$T = \begin{bmatrix} [1] & [0] \\ T_2 & [1] \end{bmatrix},$$

where $[1]$ is the unit matrix and the only non-zero entries of T_2 are ± 1. Moreover T^{-1} is obtained from T by replacing the submatric T_2 by $- T_2$.

For a more detailed discussion of this the reader is referred to a forthcoming work of the author [13].

Since the solution of a network problem reduces essentially to inverting the solution matrix, we often speak of the inverse of the solution matrix as the solution.

7. The transformation matrices. For any two networks K, K' made up of the same coils, $C_1(K)$ and $C_1(K')$ are canonically isomorphic and may be identified. However, to each of the networks K, K' corresponds a subspace $Z_1(K)$, $Z_1(K')$ which is determined by the topology of the network.

Consider the set of canonical bases arising from the various networks made up of the same N coils. Observe that a canonical basis for one network need not be a canonical basis for a different network. Let b_n be any such basis and Y^n, Z_n, i^n, e_n, E_n the matrix representations of η, ζ, i, e, E in terms of b_n and its dual. If b_m is another canonical basis the change from coordinates with respect to b_m to those with respect to b_n is given by the non-singular *transformation matrix* C_n^m. Then,

$$i^m = C_n^m i^n, \quad e_n = C_n^{m*} e_m, \quad E_n = C_n^{m*} E_m, \quad Z_n = C_n^{m*} Z_m C_n^m, \quad Y^m = C_n^m Y^n C_n^{m*}.$$

Moreover $C_n^m = (C_m^n)^{-1}$ and if b_P is another canonical basis then $C_n^p = C_m^p C_n^m$. The set of transformation matrices [7] is a subset of the full linear group of $N \times N$ non-singular matrices but they do not, as Kron asserts [7] form a subgroup since the product $C^m_n C^q_p$ is defined as a *transformation matrix* only when $n = q$.

If b_n and b_m are bases related to the networks K_n and K_m, the transformation matrix C^m_n permits the "transference" of the data of a network problem on K_n to one on K_m. If the problem has there been solved, say i^m and V_m have been determined so that $i^m = Y^m V_m$, then $i^n = C^n_m i^m$, $V_n = C^{m*}_n V_m$ is such that

$$Y^n V_n = C^n_m Y^m C^{n*}_m C^{m*}_n V_m = C^n_m Y^m V_m = C^n_m i^m = i^n$$

is a solution on K_n. Thus problems and solutions can be transferred from one network to the other.

8. The use of known solutions. Let $C_1(K) = A_1 \dotplus A_2$ be the solution decomposition for a given network problem and $\eta(A_2) = \pi(A_2)\eta\sigma(A_2^*)$, $\zeta(A_1) = \pi(A_1^*)\,\zeta\sigma(A_1)$ the the admittance and impedance solution transformations respectively. If $\eta^{-1} = \zeta$ is known, the admittance form of the network problem can be transformed into the impedance form and can thus be solved by inverting $\zeta(A_1)$ rather than $\eta(A_2)$. Whenever the rank of the former is less than that of the latter, this will be a simpler problem. In fact, we can write $\eta(A_2)^{-1} i_2 = \zeta(i_2 - \zeta(A_1)^{-1}\,(\pi(A_1^*)\zeta i_2))$. Thus knowing the inverse of η, enabled us to invert $\eta(A_2)$ by inverting instead $\zeta(A_1)$.

More generally, let $C_1(K) = B_1 \dotplus B_2$ with $B_2 \supset A_2$ and let $\eta(B_2) = \pi(B_2)\;\eta\sigma(B_2^*)$: $B_2^* \to B_2$ where $C^1(K) = B_1^* \dotplus B_2^*$ is the dual decomposition. We can find a complement A_3 of A_2 in B_2 so that $B_2 = A_3 \dotplus A_2$. Then, as above, $\eta(A_2)^{-1}(i_2) = \eta(B_2)^{-1}(i_2 - (\pi(A_3^*)\eta(B_2)^{-1}\sigma(A_3))^{-1}\;\pi(A_3^*)\eta(B_2)^{-1} i_2)$ so that, knowing $\eta(B_2)^{-1}$, $\eta(A_2)^{-1}$ can be computed by essentially inverting $\pi(A_3^*)\eta(B_2)^{-1}\sigma(A_3)$, the *connection transformation.* (see Sec. 9 below).

There is an analogous impedance formulation. Thus $\zeta(A_1)^{-1} V_1 = \eta(V_1 - \eta(A_2)^{-1}(\pi(A_2^*)\eta V_1))$ and more generally, let $C_1(K) = B_1' \dotplus B_2'$ with $B_1' \supset A_1$, $\zeta(B_1') = \pi(B_1'^*)\,\zeta\sigma(B_1')$, where $C'(K) = B_1'^* + B_2'^*$ the dual decomposition. Then as before, B_1' can be represented as $B_1' = A_1 \dotplus A_3'$ and

$$\zeta(A_1)^{-1} V_1 = \zeta(B_1')^{-1}(V_1 - (\pi(A_3')\zeta(B_1')^{-1}\sigma(A_3'^*))^{-1}\pi(A_3)\zeta(B_1')^{-1} V_1).$$

To obtain a matrix formulation of the above procedure let $b_n = < b_n^1, \cdots, b_n^N >$ be a canonical basis effecting the solution decomposition $A_1 \dotplus A_2$, that is $\{b_n^1, \cdots, b_n^r\}$ span A_1 and $\{b_n^{r+1}, \cdots, b_n^N\}$ span A_2. In terms of this basis and its dual. η is represented by the matrix Y^n. Partitioning Y^n,

$$Y^n = \begin{bmatrix} Y_1^n & Y_2^n \\ Y_3^n & Y_4^n \end{bmatrix},$$

where Y_4^n is an $N - r \times N - r$ submatrix of Y^n representing the solution transformation $\eta(A_2)$. On the other hand, let $b_m = < b_m^1, \cdots, b_m^N >$ effect the decomposition $C_1(K) = B_1 \dotplus B_2$ where $\{b_m^1, \cdots, b_m^s\}$ span B_1 and $\{b_m^{s+1}, \cdots, b_m^N\}$ span B_2, and let Y^m represent η in terms of the basis b_m and its dual and Y_4^m be the $N - s \times N - s$ submatrix of Y^m which represents $\eta(B_2)$ in terms of the basis $< b_m^{s+1}, \cdots, b_m^N >$ and its dual. The basis b_n is chosen to effect both decompositions, that is $< b_n^{s+1}, \cdots, b_m^N >$ spans B_2 and D_n^m then represents the change in coordinates in B_2. In practice, the computations are carried out in terms of the basis b_n while Y_4^m and its inverse are given in terms of the basis b_m hence $D_n^{m*} Y^{m-1} D_n^m$ gives the matrix representation of $\eta(B_2)^{-1}$ in terms of b_n. Finally, if $D_n^m = [d_{ij}]$, $i, j = s + 1, \cdots, N$ then $E_n^m = [d_{ij}]$, $i = s + 1, \cdots, r$, $j = s + 1, \cdots, N$ gives the matrix representation of $\sigma(A_3)$ in terms of the bases $< b_n^{s+1}, \cdots, b_n^r >$ and $< b_m^{s+1}, \cdots, b_m^N >$ so that the connection matrix to be inverted is $E_n^{m*} Y^m E_n^m$.

9. The method of tearing and the method of identification. In both these methods the topology of the network is utilized to obtain a decomposition $C_1(K) = B_1 \dotplus B_2$ with $\eta(B_2)$ and hence Y_4^m above, strongly diagonal. In this way the matrix representation of $\eta(A_2)$, Y_4^n is inverted by inverting several matrices of smaller rank.

Let K_0 be a network or complex and $K_1, \cdots, K_k$ subcomplexes of K_0 such that every 1-cell of K_0 lies in exactly one of the subcomplexes K_i, $i = 1, \cdots, k$, and no coil of K_i is inductively coupled to a coil of K_j, for $i \neq j$. Then

$$C_1(K_0) = C_1(K_1) \dotplus C_1(K_2) \dotplus \cdots \dotplus C_1(K_k).$$

Moreover, the admittance transformation η respects this decomposition, that is $\eta_i = \pi(C_1(K_i))\ \eta\sigma(C^1(K_i))$ is just the admittance transformation for the network K_i, $i = 1, \cdots, k$. Choosing a (canonical or simple canonical) basis consistent with this decomposition, the matrix representation of η is strongly diagonal.

Let K_{k+1} be the complex consisting of the disconnected subcomplexes K_i and let $C_1(K_i) = Z_1(K_i) \dotplus R_1(K_i)$, $i = 1, \cdots, k$, be canonical decompositions. Then $Z(K_{k+1}) = Z_1(K_1) \dotplus \cdots \dotplus Z_1(K_k)$ and $Z_1(K_{k+1}) \dotplus R_1(K_{k+1})$ is a canonical decomposition for $C_1(K)$ where $R_1(K_{k+1}) = R_1(K_1) \dotplus \cdots \dotplus R_1(K_k)$. Clearly $Z_1(K_{k+1}) \subset Z_1(K_0)$ so that $R_1(K_0)$ can be chosen such that $Z_1(K_0) \dotplus R_1(K_0)$ is a canonical decomposition for $C_1(K_0) = C_1(K_{k+1})$ and $R_1(K_0) \subset R_1(K_{k+1})$. Setting $\eta(K_i) = \pi(R_1(K_i)\ \eta\sigma\ (B^1(K_i))$, $\eta(K_{k+1})$ agrees with $\eta(K_i)$ on $B^1(K_i)$ for $i = 1, \cdots, k$. Hence for the proper choice of basis, the matrix representation of $\eta(K_{k+1})$ is strongly diagonal. Thus, $\eta(K_{k+1})$ can be inverted by inverting each of $\eta(K_i)$, $i = 1, \cdots, k$. If the subcomplexes K_i, $i = 1, \cdots, k$ are chosen to consist of as few distinct complexes as possible, the matrix representations of $\eta(K_i)$ for the "same" complexes, can be made identical thus simplifying the task of inverting $\eta(K_{k+1})$.

If the solution decomposition $C_1(K_0) = A_1 \dotplus A_2$ for a general network problem on K_0 is such that $A_2 \subset R(K_{k+1})$ then the procedure of Sec. 8 can be applied. This is the method of tearing. If dim $A_2 = N - k$ and dim $R_1(K_{k+1}) = N - s$, then the inversion of the $N - r \times N - r$ solution matrix is effected by inverting each of the diagonal submatrices of the strongly diagonal $N - s \times N - s$ matrix and the $r - s \times r - s$ connection matrix.

The problems considered by Kron illustrating his method are all in admittance form with canonical solution decompositions (and hence of the special form considered by Roth [10]). In the method of tearing, the problem is transferred from K_0 to K_{k+1} where it is now of the more general form treated above, for the canonical decomposition for K_0 is *not* a canonical decomposition for K_{k+1}. In this case, however, $A_2 = R_1(K_0) \subset R_1(K_{k+1})$ and the above considerations apply. In general, however, $A_2 \supset R_1(K_0)$ so that the condition that $R_1(K_{k+1}) \supset A_2$ may not be met. The problem can then be cast into the impedance form by inverting the strongly diagonal matrix representation of η. Then $A_1 \subset Z_1(K_{k+1})$ and we can apply the admittance form of the technique of Sec. 8 wherein we invert the solution matrix by inverting the strongly diagonal matrix representation of $\zeta(K_{k+1}) = \pi(D^1(K_{k+1}))\ \zeta\sigma(Z_1(K_{k+1}))$ and an impedance connection matrix.

The greater the number of pieces into which K_0 is torn to form K_{k+1}, the smaller dim $Z_1(K_{k+1})$ and the greater dim $R_1(K_{k+1})$, and hence the rank of the connection matrix. The smaller the individual pieces the smaller the rank of the transformations $\eta(K_i)$. In practice, these two things have to be balanced. However, in order to apply the technique of Sec. 8, $R_1(K_{k+1})$ must contain A_2 .

The method of tearing is not in general suited to problems in the impedance form for in this case, to apply the technique of Sec. 8 we want to shift the problem from K_0 to a new network K_{k+2} where $Z_1(K_{k+2}) \supset A_1$ and the matrix representation of $\zeta(K_{k+2})$ can be chosen strongly diagonal. Since $Z_1(K_0) \supset A_1$, it is sufficient to have $Z_1(K_{k+2}) \supset Z_1(K_0)$. K_{k+2} is obtained from K_0 by identifying in K_i , all those vertices common to two of the subcomplexes K_j , $i, j = 1, \cdots, k$. This process of identification can be modified by selective identification provided only that if K_i' is the subcomplex of K_{k+2} obtained by identifications in K_i then K_{k+2} can be torn into the disjoint subcomplexes K_i' without opening any cycles of K_{k+2} .

BIBLIOGRAPHY

1. B. Eckmann, *Harmonische Funktionen und Randwertaufgaben in einem Komplex*, Comm. Math. Helv. **17**, 240–255 (1944–1945)
2. G. Kron, *Tensor analysis of networks*, John Wiley and Sons, New York, 1939
3. G. Kron, *A set of principles to interconnect the solutions of physical systems*, J. Appl. Phys. **24**, No. 8, 965–980 (August, 1954)
4. G. Kron, *A method to solve very large physical systems in easy stages*, Proc. Inst. Radio Engrs. **42**, No. 4, 680–686 (April, 1954)
5. G. Kron, *Solving highly complex elastic structures in easy stages*, J. Appl. Mechanics, Paper No. 54–A–25
6. G. Kron, *Detailed example of interconnecting piece-wise solutions*, J. Franklin Inst. **259**, No. 4 (April, 1955)
7. G. Kron, *Tearing and interconnecting as a form of transformation*, Quart. Appl. Math. **13**, No. 2, 147–159 (July, 1955)
8. P. LeCorbeiller, *Matrix analysis of electrical networks*, Harvard University Press, 1950
9. J. C. Maxwell, *Electricity and magnetism*, Clarendon Press, Oxford, 1873
10. J. P. Roth, *An application of algebraic topology to numerical analysis: On the existence of a solution to the network problem*, Proc. Natl. Acad. Sci. **41**, No. 7, 518–521 (July, 1955)
11. J. P. Roth, *The validity of Kron's method of tearing*, Proc. Natl. Acad. Sci. **41**, No. 8, 599–600 (August, 1955)
12. J. P. Roth, *An application of algebraic topology: Kron's method of tearing*, Quart. Appl. Math. **17**, No. 2, 1–24 (April, 1959)
13. A. I. Weinzweig, *Topology and network theory*, in preparation